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ANOTHER VIEW OF DISTANCE SETS AND RATIO SETS

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1 Introduction

If (X, ϱ) is a metric space, then for every $A \subset X$ we can define the set

$$D(A) = \{\varrho(x, y);\ x, y \in A\},$$

which is termed the *distance set* of the set A.

Analogously, for every $A \subset (0, +\infty)$, we define

$$R(A) = \left\{\frac{x}{y};\ x, y \in A\right\}.$$

The set $R(A)$ is called the *ratio set* of A.

Many results concerning distance sets and ratio sets, which have already become classical, can be found e.g. in [1], [2], [3], [4], [5], [7] or in more recent papers [8] and [9].

The present paper deals with the following aspect of the mentioned topic. Let (X, ϱ) be a metric space. Then to every subset A of X there corresponds $D(A) \subset [0, +\infty)$. Thus we may consider the distance set as a value of a set-valued function mapping the subsets of the metric space (X, ϱ) into the subsets of the interval $[0, +\infty)$. Some interesting propositions in this context can be found in [6]:

Theorem 1.1. Let (X, ϱ) be a metric space. There exists exactly one function F defined on 2^X with range in $2^{[0, +\infty)}$ and having the following properties:

(1) For every $\emptyset \neq A \subset X$ we have $0 \in F(A)$ and $F(\emptyset) = \emptyset$.

(2) For every $A, B \subset X$ with $A \subset B$ we have $F(A) \subset F(B)$.

(3) For every $A \subset X$ we have $\mathrm{d}(F(A)) = \mathrm{d}(A)$.

(4) If $\{0, a\} \subset F(A)$, then for some compact set $K \subset A$ we have $F(K) = \{0, a\}$.

(5) For every two-point set $\{x, y\} \subset X$, the set $F(\{x, y\})$ is closed.

The function described by the above properties is exactly $D(A)$.

Remark. In (3), the notation $\mathrm{d}(M)$ means the usually defined diameter of a set M in (X, ϱ).

Proof of this theorem can be found in the above-mentioned paper [6], where it is also shown that the properties (1) through (5) are not independent. Moreover, the following theorem holds.

Theorem 1.2.

a) Let $F(A)$ have the properties (1) through (4). Then it also has the following property, stronger than (5):

(5*) Image of any two-point set is a two-point set.

b) Properties (1) through (4) are independent.

2 Distance sets in quasi-metric spaces

It seems natural to study distance sets also in quasi-metric spaces.

Definition 2.1. Let X be a nonempty set. then a nonnegative function $\varrho: X \times X \to [0, +\infty)$ satisfying

1) $\forall x, y \in X$: $\varrho(x, y) = 0 \Leftrightarrow x = y$

and

2) $\forall x, y, z \in X$: $\varrho(x, z) \leq \varrho(x, y) + \varrho(y, z)$

will be called a quasi-metric and the pair (X, ϱ) will be referred to as a quasi-metric space.

Remark. The notion of a quasi-metric space is not new. As far as the author knows, it was introduced for the first time in [10]. Other papers dealing with quasi-metric spaces are [11], [12] and [13]. Let us mention that the term *oriented metric space* used in [12] and [13] is equivalent with the quasi-metric space. However, none of the above papers deals with distance sets. They treat only some topological properties of quasi-metric spaces, the topology being in some sense derived from the quasi-metric.

In the present paper, topology is used only in connection with compactness. The reader will see that this is really a formal matter. In fact, a two-point set is compact in any topology.

Definition 2.2. Let (X, ϱ) be a quasi-metric space and let $A \subset X$. Then the set

$$D(A) = \{\varrho(x, y);\ x, y \in A\}$$

will be termed the *distance set* of A.

Now, similarly as in the preceding section, let us try to consider $D(A)$ as a set-valued function mapping the subsets of X into the subsets of $[0, +\infty)$.

Note that, given $M \subset X$, we shall use the notation $\mathrm{d}(M) = \sup\{\varrho(x, y);$

$x, y \in M\}$ for the diameter of M in the quasi-metric space (X, ϱ). The following assertion holds.

Theorem 2.1. Let (X, ϱ) be a quasi-metric space. There exists exactly one function F defined on 2^X with range in $2^{[0, +\infty)}$ and having the following properties:

(1′) For every $\emptyset \neq A \subset X$ we have $0 \in F(A)$ and $F(\emptyset) = \emptyset$.

(2′) For every $A, B \subset X$ with $A \subset B$ we have $F(A) \subset F(B)$.

(3′) For every $A \subset X$ we have $\mathrm{d}(F(A)) = \mathrm{d}(A)$.

(4′) If $\{0, a\} \subset F(A)$, then for some compact set $K \subset A$ and some number b we have $F(K) = \{0, a, b\}$.

(5′) For every two-point set $\{x, y\} \subset X$, the set $F(\{x, y\})$ is closed.

(6′) If $x, y \in A$ satisfy $a = \varrho(x, y) \in F(A)$, then also $\varrho(y, x) \in F(A)$.

The above properties characterize the function D.

Proof. First of all we show that the function D enjoys all the properties (1′) through (6′). As (1′) and (2′) are evident, we show that D has property (3′). If $A = \emptyset$, then $D(A) = \emptyset$ and (3′) holds. Suppose that $A \neq \emptyset$. Since $0 \in D(A)$ and $D(A) \subset [0, +\infty)$, we get $\mathrm{d}(D(A)) = \sup D(A) = \sup \{\varrho(x, y);\ x, y \in A\} = \mathrm{d}(A)$. Now we verify (4′). Let $\{0, a\} \subset D(A)$. Therefore there exists at least one pair of points $x, y \in A$ with $\varrho(x, y) = a$. It suffices to put $b = \varrho(y, x)$ and $K = \{x, y\}$ in order to obtain $D(K) = D(\{x, y\}) = \{0, a, b\}$. The remaining properties (5′) and (6′) follow immediately from the definition of D.

Now we are going to show that every function F with properties (1′) through (6′) equals D.

Suppose that a function F has the properties (1′) through (6′). We show that $D(A) \subset F(A)$ for every $A \subset X$. Let $a \in D(A)$. If $a = 0$, then clearly $a \in F(a)$. Thus it is sufficient to consider $a \neq 0$, $a \in D(A)$. Since $a \in D(A)$, there exist points $x, y \in A$ with $x \neq y$ and $a = \varrho(x, y)$. Examine the two-point set $\{x, y\}$. By (3′), the diameter $\mathrm{d}(F(\{x, y\})) = \mathrm{d}(\{x, y\})$ equals either $\varrho(x, y)$ or $\varrho(y, x)$. By (5′), however, $F(\{x, y\})$ is a closed subset of $[0, +\infty)$ containing 0. Therefore $\mathrm{d}(F(\{x, y\})) = \sup F(\{x, y\}) \in F(\{x, y\})$. Hence $\varrho(x, y)$ or $\varrho(y, x)$ is in $F(\{x, y\})$. Now (6′) implies that both $a = \varrho(x, y)$ and $\varrho(y, x)$ belong to $F(\{x, y\})$.

It remains to prove the converse inclusion, i.e. that $F(A) \subset D(A)$ for every $A \subset X$. Let $a \neq 0$, $a \in F(A)$. Then also $\{0, a\} \subset F(A)$. However, (4′) guarantees the existence of a compact $K \subset A$ and a number b with $F(K) = \{0, a, b\}$. In view of (3′) we have

$$\mathrm{d}(\{0, a, b\}) = \mathrm{d}(F(K)) = \mathrm{d}(K) = \varrho(x, y), \tag{1}$$

for some points x, y from the compact K. On the other hand, the diameter of the set on the right-hand side of (1) is equal to a or to b, therefore $\varrho(x, y) = a$ or $\varrho(x, y) = b$.

If $\varrho(x, y) = a$, then it immediately follows that $a \in D(A)$. If $\varrho(x, y) = b$, then

$a = \varrho(x, y)$, which again is an element of $D(A)$. The proof of the theorem is complete. ■

It can be seen that axioms (1′) through (6′) stated in the preceding theorem are not independent, but the following theorem is true.

Theorem 2.2. a) Properties (1′), (3′) and (4′) imply (5′) and even its following stronger formulation

(5′*) If $x, y \in X$, then $F(\{x, y\})$ has at most three points.

b) Properties (1′), (2′), (3′), (4′) and (6′) are independent.

Proof. a) Let $x, y \in X$. By (1′) we have $0 \in F(\{x, y\})$. If $x = y$, then (3′) obviously implies that $F(\{x, y\}) = \{0\}$.

Now suppose that $x \neq y$. Hence $d(\{x, y\}) > 0$, which by (3′) implies that there exists a point $a \neq 0$ with $\{0, a\} \subset F(\{x, y\})$. Then by (4′) there exist a number b and a compact $K \subset \{x, y\}$ with $F(K) = \{0, a, b\}$. Since (3′) excludes both $K = \{x\}$ and $K = \{y\}$, we obtain $K = \{x, y\}$ and hence $F(K) = F(\{x, y\}) =$ $= \{0, a, b\}$.

b) The following example demonstrates that (6′) is indpendent of the remaining properties.

Put $X = \{x, y\}$, define the quasimetric ϱ by

$$\varrho(x, x) = \varrho(y, y) = 0, \quad \varrho(x, y) = 1, \quad \varrho(y, x) = 2$$

and the function F by

$$F(\{x\}) = F(\{y\}) = \{0\}, \quad F(\{x, y\}) = \left\{0, \frac{1}{2}, 2\right\}.$$

It is easy to see that F has the properties (1′) through (4′), but not (6′).

To prove the indpendence of the remaining properties (1′) through (4′), it is sufficient to consider the corresponding examples form [6], which show the independence of the respective properties in a metric space. Since every metric space is at the same time a quasi-metric space and (6′) is automatically fulfilled in every metric space, those examples whow the independence of our properties (1′) through (4′) as well. ■

To conclude this section, let us consider another relation between metrics and quasi-metrics.

Theorem 1.1 offers an axiomatic characterization of the distance set in metric spaces. Theorem 2.1 does the same in quasi-metric spaces. When comparing the two theorems, we see that the difference between a metric and a quasi-metric reduces essentially to the difference between the axioms (4) and (4′) in the theorems. We can even state the following assertion.

Theorem 2.3. A quasi-metric ϱ is a metric if and only if the distance set can be unambiguously characterized by properties (1) through (5) from Theorem 1.1.

Proof. One implication in our assertion is equivalent with theorem 1.1. Thus it is sufficient to prove the converse implication.

Let ϱ be a quasi-metric. We are going to show that if (1) through (5) unambiguoously characterize the distance set D, then ϱ is a metric. Suppose the contrary, namely that if F is the function determined by properties (1) through (5) from Theorem 1.1, then $F(A)$ is identical with the distance set $D(A)$ for every set A, but ϱ is not a metric. Hence there exists a pair of distinct points $x, y \in X$ with

$$\varrho(x, y) = a \neq b = \varrho(y, x).$$

Put $A = \{x, y\}$. Then $F(A) = D(A) = \{0, a, b\}$. In paticular, $\{0, a\} \subset F(A)$ and by (4) there is a compact $K \subset A$ with $D(K) = F(K) = \{0, a\}$. Since $K \subset A$, one of the following three cases can arise: $K = \{x\}$, $K = \{y\}$, or $K = \{x, y\}$. On the other hand, $D(\{x\}) = D(\{y\}) = \{0\}$ and $D(\{x, y\}) = \{0, a, b\}$, which contradicts the way K has been chosen. The theorem is proved. ■

3 Some analogies for ratio sets

As we have already mentioned in the introduction, questions concerning ratio sets $R(A)$ of sets $A \subset (0, +\infty)$ were studied simultaneously with those concerning distance sets.

Let us have a look at the operation R from another viewpoint. Again, $R(A)$ can be understood as a value of a set-valued function defined on the family of all subsets of the interval $(0, +\infty)$, whose range is a family of subsets of $(0, +\infty)$. Therefore, we shall try to characterize the function R by some of its properties. The following assertion is true.

Theorem 3.1. Let $X = (0, +\infty)$. There exists exactly one function $F\colon 2^X \to 2^X$ with the following properties:

V_1: For every $\emptyset \neq A \subset X$ we have $1 \in F(A)$.

V_2: $F(\emptyset) = \emptyset$. If $A \subset B$, then $F(A) \subset F(B)$.

V_3: Let $A \subset X$, $m = \sup A < +\infty$ and $n = \inf A > 0$. Then $\mathrm{d}(F(A)) = \left(\frac{1}{m} + \frac{1}{n}\right)\mathrm{d}(A)$.

V_4: If $a \in F(A)$, then $\frac{1}{a} \in F(a)$.

V_5: If $\left\{\frac{1}{a}, 1, a\right\} \in F(A)$, then there is a subset K of A with $F(K) = \left\{\frac{1}{a}, 1, a\right\}$.

V_6: If $\{x, y\} \subset X$, then $F(\{x, y\})$ is a compact set. The function described by these properties is exactly the function R.

Proof. It is easy to see that R has all the properties V_1 through V_6. Only V_3

is worth verifying in more detail. From the assumptions of V_3 it follows that $d(A) = m - n$. On the other hand. $d(R(A)) = \frac{m}{n} - \frac{n}{m}$. This immediately implies that V_3 holds for R.

Conversely, suppose that a function $F\colon 2^X \to 2^X$ has the properties V_1 through V_6. First, we show that $R(A) \subset F(A)$ for every $A \subset X$. Consider any $a \in R(A)$. If $a = 1$, then by V_1 we have $a = 1 \in F(A)$. Now let $a \neq 1$. Without loss in generality we may assume that $a > 1$. (For $a < 1$ we would proceed analogously.) Since $a \in R(A)$, there exist $x, y \in A$ with $a = \frac{x}{y}$, $x > y$. By V_2 we have $F(\{x, y\}) \subset F(A)$, and V_6 implies that $F(\{x, y\})$ is a compact set. Therefore we may denote $x_1 = \min F(\{x, y\})$ and $y_1 = \max F(\{x, y\})$. From V_3 we infer that $0 < d(F(\{x, y\})) = \left(\frac{1}{x} + \frac{1}{y}\right)(x - y) = a - \frac{1}{a}$. On the other hand, $d(F(\{x, y\})) = y_1 - x_1$. Therefore

$$y_1 - x_1 = a - \frac{1}{a} > 0. \tag{2}$$

From V_4 and from the definition of x_1 and y_1 it follows that $x_1 = \frac{1}{y_1}$, which after substitution in (2) yields

$$y_1 - \frac{1}{y_1} = a - \frac{1}{a} \tag{3}$$

The last equation reduces to

$$y_1^2 - \left(a - \frac{1}{a}\right)y_1 - 1 = 0.$$

From (2), (3) and from the positivity of the last equation's discriminant it follows that (3) has two distinct real solutions, only one of them being positive. It is clear from the form of the equation (3) that the positive solution is $y_1 = a$. Therefore $a \in F(\{x, y\})$ and hence $a \in F(A)$.

Now we prove the converse implication, namely that $F(A) \subset R(A)$ for all $A \subset X$. Assume $a \in F(A)$. If $a = 1$, then evidently $a \in R(A)$. If $a \neq 1$ (clearly $a > 0$), V_4 implies that also $\frac{1}{a} \in F(A)$. By V_1 we infer that $1 \in F(A)$ and by V_5 there exists a nonempty set $K \subset A$ with $F(K) = \left\{\frac{1}{a}, 1, a\right\}$. Since $K \neq \emptyset$, from the definition of the ratio set we get that either K is a one-point set (in that case $R(K) = \{1\}$) or K contains at least two distinct points $x \neq y$, that is, $\{x, y\} \subset K$. Observe that, in virtue of V_3. K cannot be a one-point set. In fact, $K = \{x\}$ would imply $d(K) = 0$. but $d(F(K)) = \left|a - \frac{1}{a}\right| > 0$. Therefore there exist two distinct points $x \neq y$ with $\{x, y\} \subset K$. However, in this case from the definition of the ratio set we infer that $R(K) = \left\{1, \frac{x}{y}, \frac{y}{x}\right\}$. From what has already been proved we know that $R(A) \subset F(A)$ for every $A \subset X$. Hence $\left\{1, \frac{x}{y}, \frac{y}{x}\right\} \subset R(K) \subset$ $\subset F(K) \subset \left\{\frac{1}{a}, 1, a\right\}$, and this implies $a = \frac{x}{y}$ or $a = \frac{y}{x}$. Thus in either case we have $a \in R(K)$ and hence $a \in R(A)$. ■

Properties V_1 through V_6 are not independent. We have listed them all just for the sake of simplification of the theorem's proof. Properties V_2 through V_5 are sufficient to characterize the function R. In fact, the following assertion is true.

Theorem 3.2. Suppose that a function F: $2^{(0, +\infty)} \rightarrow 2^{(0, +\infty)}$ has the properties V_2 through V_5. Then

a) F has also property V_1;

b) F has property V_6 in the following stronger form: Image of any two-point set is a three-point set;

c) Properties V_2 through V_5 are independent.

Proof.

a) Let $A \neq \emptyset$. Hence there exists some $x \in A$. We show that $F(\{x\}) = \{a\}$. If it were $a = 1$, by V_4 we would have also $\frac{1}{a} \in F(\{x\})$, and hence $d(F(\{x\})) > 0$, which is impossible. Therefore $a = 1$. However, by V_2 we have $F(A) \supset F(\{x\}) =$ $= \{1\}$.

b) Let $\{x, y\} \subset (0, +\infty)$, $x \neq y$. Then $d(\{x, y\}) > 0$. On the other hand, V_3 implies that $d(F(\{x, y\})) > 0$. Therefore $F(\{x\})$ contains, besides the point 1 ($1 \in F(\{x, y\})$ by a)), at least one point $a \neq 1$. However, by V_4 we have $\frac{1}{a} \in F(\{x, y\})$, hence $F(\{x, y\}) \supset \left\{\frac{1}{a}, 1, a\right\}$. To complete the proof, it is sufficient to show that these two sets are equal.

Suppose the contrary. Let there exist $b \in F(\{x, y\})$ with $b \notin \left\{\frac{1}{a}, 1, a\right\}$. By V_4 we have also $\frac{1}{b} \in F(\{x, y\})$. On the other hand, since $\left\{\frac{1}{b}, 1, b\right\} \subset F(\{x, y\})$,

V_5 implies that there is $M \subset \{x, y\}$ with $F(M) = \left\{\frac{1}{b}, 1, b\right\}$. However, such a set M cannot exist, because $\{x, y\}$ has only four subsets, namely $\emptyset$, $\{x\}$, $\{y\}$, and $\{x, y\}$. their images are $F(\emptyset) = \emptyset$; $F(\{x\}) = F(\{y\}) = \{1\}$ (owing to a)) and $F(\{x, y\}) \supset \left\{\frac{1}{a}, 1, a\right\}$. None of them equals $\left\{\frac{1}{b}, 1, b\right\}$, which proves that $F(\{x, y\}) = \left\{\frac{1}{a}, 1, a\right\}$.

c) Mutual independence of properties V_2 through V_5 will be shown by the following examples.

Example 3.1. Let $A \subset (0, +\infty)$, $A \neq \emptyset$. Put $m = \sup A$, $n = \inf A$. Define

$$F(A) = \begin{cases} \left\{\frac{n}{m}, 1, \frac{m}{n}\right\} & \text{if } A \text{ is a nonempty compact set} \\ R(A) & \text{otherwise.} \end{cases}$$

Property V_2 is not satisfied, but V_3, V_4 and V_5 hold. Since V_4 and V_5 are evident, let us verify V_3 only. In fact, $\mathrm{d}(F(A)) = \frac{m}{n} - \frac{n}{m} = \left(\frac{1}{n} + \frac{1}{m}\right)(m - n) =$ $= \left(\frac{1}{n} + \frac{1}{m}\right)\mathrm{d}(A)$. Hence V_3 is true.

Example 3.2. $F(A) = \{1\}$ for all nonempty $A \subset X$ and $F(\emptyset) = \emptyset$. (We always put $X = (0, +\infty)$.)

It is easy to verify that the mapping F defined above has the properties V_2, V_4 and V_5, but not V_3.

Example 3.3. Let $A \subset X$. Put $F(\emptyset) = \emptyset$. If $A \neq \emptyset$, denote $m = \sup A$, $n = \inf A$ and define

$$F(A) = \begin{cases} \left[1, \left(\frac{1}{n} + \frac{1}{m}\right)\mathrm{d}(A) + 1\right] & \text{if } m < +\infty \text{ and } n > 0 \\ [1, +\infty) & \text{if } m = +\infty \text{ or } n = 0. \end{cases}$$

For the function thus defined, the properties V_2, V_3 and V_5 are fulfilled, but not V_4.

Example 3.4. Let m, n have the same meaning as above. Put $F(\emptyset) = \emptyset$ and, for nonempty A, define

$$F(A) = \begin{cases} \left[\frac{n}{m}, \frac{m}{n}\right], & \text{if } m < +\infty \text{ and } n > 0 \\ (0, +\infty), & \text{if } m = +\infty \text{ or } n = 0. \end{cases}$$

This function F has the properties V_2, V_3 nd V_4, but V_5 is not fulfilled. In fact, it is sufficient to put $A = \{2, 4\}$, which gives $F(A) = \left[\frac{1}{2}, 2\right]$. Then $\left\{\frac{3}{4}, 1, \frac{4}{3}\right\} \subset$ $\subset F(A)$, but no set $M \subset A$ satisfies $F(M) = \left\{\frac{3}{4}, 1, \frac{4}{3}\right\}$.

The proof of the theorem is complete. ■

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SÚHRN

INÝ POHĽAD NA MNOŽINY VZDIALENOSTÍ A PODIELOVÉ MNOŽINY

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V práci je študovaná problematika množín vzdialeností v kvázimetrických priestoroch. Je tu ukázané, že množinu vzdialeností $D(A)$ možno zaviesť axiomaticky. Druhá časť tejto práce je venovaná možnosti axiomatického zavedenia podielovej množiny $R(A)$, ako množinovej funkcie, definovanej na intervale $(0, +\infty)$.

РЕЗЮМЕ

ДРУГОЙ ВЗГЛЯД НА МНОЖЕСТВА РАССТОЯНИЙ И МНОЖЕСТВА ДРОБЕЙ

ЯРОСЛАВ ЧЕРВЕНАНСКИ, Братислава

В работе изучаются множества расстояний в квазиметрических пространствах. Показано, что множество расстояний $D(A)$ можно ввести аксиоматически. Во второй части работы изучается возможность аксиоматического введения множества дробей $R(A) = \left\{\frac{x}{y};\ x, y \in A\right\}$, $A \subset (0, +\infty)$, как функции множества, определенной на интервале $(0, +\infty)$.