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ON THE ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF FUNCTIONAL DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENT

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Let R^n denote the n — dimensional vector space with norm $|\cdot|$, $R_- = (-\infty, 0]$, $R_+ = [0, \infty)$ and $R = (-\infty, \infty)$. Let C be the Banach space of all continuous and bounded functions g: $R_- \to R^n$ with the sup-norm $\|\cdot\|$. Let ψ: $R_+ \to (0, \infty)$ be a nondecreasing continuous function.

Let D be the vector space of all continuous functions g: $R \to R^n$ which are bounded in R_- and ψ — bounded in R_+ (i.e. $\psi(t)^{-1}|g(t)|$ is bounded on R_+).

Let $\{I_k\}_{k=1}^{\infty}$ be a sequence of compact intervals such that $\bigcup_{k=1}^{\infty} I_k = R_+$, where $I_k = [0, k]$ and for every $k \in N$ we have $I_k \subset I_{k+1} \subset R_+$.

Let F be a Fréchet space of all continuous functions g: $R_+ \to R^n$ with a locally convex topology defined by systems of seminorms

$$p_k(g) = \max_{t \in I_k} \psi^{-1}(t)\,|g(t)|, \qquad k \in N.$$

Let F_1 be a Fréchet space of all continuous functions g: $R_+ \to R$ with a locally convex topology defined by systems of seminorms

$$p_{k_1}(g) = \max_{t \in I_k} |g(t)|_1, \qquad k \in N,$$

where $|\cdot|_1$ means the absolute value in R.

Let ω: $R_+ \to R$ be a continuous function such that $\omega(0) = 0$ and f: $R_+ \times C \to R^n$ be a continuous function.

If $h \in C$, then denote $D_h = \{g \in D: g(t) = h(t), t \in R_-\}$ with topology of locally uniformly convergence.

Denote $a^+ = \max\{a, 0\}$ for any $a \in R$, $\operatorname{sgn} 0 = 0$ and $\operatorname{sgn} a = 1$ for $a > 0$.

If $g \in D$ and $u \in R$, then g_u is the function defined for $s \in R_-$ by $g_u(s) = g(u + s)$.

Let $Q\colon R \to R^{n\times n}$ be a continuous matrix function. Then Q_u denotes the matrix function defined by $Q_u(s) = Q(u + s)$ for $s \in R_-$ and $u \in R$. further, $Q_u g_u$, where $u \in R$, is defined by $Q_u(s)\, g_u(s) = Q(u + s)\, g(u + s)$ for $s \in R_-$. It is evident that for every $u \in R$, $g_u \in C$, $Q_u g_u \in C$. Let $P\colon R_+ \to R^{n\times n}$ be a continuous matrix function.

We consider the following initial problem

$$x'(t) = P(t) f(t, Q_{\omega(t)} x_{\omega(t)}) \tag{1}$$

$$x_0 = h, \tag{2}$$

where $h \in C$ is uniformly continuous in R_-.

By solution of (1), (2) we understand any function $x\colon R \to R^n$ which is continuous on R, $x(t) = h(t)$ for all $t \in R_-$, x is differentiable on R_+ and satisfies (1) everywhere on R_+.

Denote by X the set of all solutions of the initial problem (1), (2).

In the paper [1] an asymptotic properties of solutions of the equation (1) are studied in cases that (1) is a delay differential system. In the paper [2] an initial value problem for the functional differential equation with deviating argument $x'(t) = f(t, x_{\omega(t)})$ in Banach space is studied whereby the function f satisfies the Lipschitz condition. Further, the paper [2] deals with bounded solutions of an integral inequality with deviating argument.

The aim of this paper is to provide sufficient conditions for the existence of a solution $x \in X$ on R_+ having some asymptotic properties. These results generalize the results of [1] and [2] where methods from [2] are applied.

We shall be employing the hypotheses:

$$\psi^{-1}(t) \int_0^t |P(s) f(s, 0)|\, ds < K_1 < \infty \quad \text{for} \quad t \in R_+, \tag{A_1}$$

$$|P(t)[f(t, Qz_1) - f(t, Qz_2)]| \leqslant \varphi(t, \|z_1 - z_2\|) \quad \text{for} \quad t \in R_+, \tag{A_2}$$

and for $z_1, z_2 \in C$, where $\varphi\colon R_+ \times R_+ \to R_+$ is a locally integrable function on R_+ for each fixed second argument and nondecreasing in the second argument for each fixed $t \in R_+$,

$$\psi^{-1}(t) \int_0^t \varphi(s, c_1 + c_2 \operatorname{sgn} \omega^+(s)\, \psi[\omega^+(s)])\, ds < K_2 < \infty, \tag{A_3}$$

for $t \in R_+$ and any $c_1 \geqslant 0$, $c_2 \geqslant 0$,

$$\int_0^t |P(s) f(s, 0)|\, ds < \bar{K}_1 < \infty \quad \text{for} \quad t \in R_+, \tag{A_4}$$

$$(A_5) \qquad \int_0^t \varphi(s, c_1 + c_2 \operatorname{sgn} \omega^+(s)\, \psi[\omega^+(s)])\, ds < \bar{K}_2 < \infty,$$

for $t \in R_+$ and any $c_1 \geqslant 0$, $c_2 \geqslant 0$.

Remark 1. It is evident that from the assumption (A_4) follows the assumption (A_1) and from (A_5) follows (A_3). If, moreover, the function ψ is bounded on R_+ then from (A_1) follows (A_4) and from (A_3) follows (A_5).

Theorem 1. Let the assumptions (A_1)—(A_3) be satisfied. Then there exists a ψ — bounded solution $x \in X$ defined on R_+.

Proof. Let $\lambda \geqslant \psi^{-1}(0)\,\|h\| + K_1 + K_2$. Denote

$$M = \{g \in D_h \colon \psi^{-1}(t)\,|g(t)| \leqslant \lambda, \quad t \in R_+\}.$$

It is evident that M is a convex closed set in D_h.

Define an operator T on M by

$$(3) \qquad T(g)(t) = \begin{cases} h(t) & \text{for} \quad t \in R_- \\ h(0) + \displaystyle\int_0^t P(s) f(s, Q_{\omega(s)} g_{\omega(s)})\, ds, & \text{for} \quad t \in R_+. \end{cases}$$

We show that $TM \subset M$.

If $t \in R_-$, then $T(g)(t) = h(t)$.

If $t \in R_+$, then with respect to $(A_1) - (A_3)$ we get

$$|T(g)(t)| \leqslant |h(0)| + \int_0^t |P(s) f(s, Q_{\omega(s)} g_{\omega(s)})|\, ds \leqslant |h(0)| +$$

$$+ \int_0^t |P(s) f(s, 0)|\, ds + \int_0^t \varphi(s, \|g_{\omega(s)}\|)\, ds \leqslant |h(0)| +$$

$$+ \int_0^t |P(s) f(s, 0)|\, ds + \int_0^t \varphi(s, \|h\| + \psi^{-1}(s)\,|g(s)| \operatorname{sgn} \omega^+(s)\, \psi[\omega^+(s)])\, ds \leqslant$$

$$\leqslant |h(0)| + \int_0^t |P(s) f(s, 0)|\, ds + \int_0^t \varphi(s, \lambda + \lambda \operatorname{sgn} \omega^+(s)\, \psi[\omega^+(s)])\, ds,$$

from which it follows

$$|\psi^{-1}(t)\, T(g)(t)| \leqslant |\psi^{-1}(t)\, h(0)| + \psi^{-1}(t) \int_0^t |p(s) f(s, 0)|\, ds +$$

$$+ \psi^{-1}(t) \int_0^t \varphi(s, \lambda + \lambda \operatorname{sgn} \omega^+(s)\, \psi[\omega^+(s)])\, ds \leqslant \lambda.$$

Further, we show that T is continuous on M. Let $\{g^n\}_{n=1}^{\infty}$, $g^n \in M$, be a sequence converging uniformly to $g \in \mathrm{M}$ on every compact interval $\mathrm{I}_k \subset \mathrm{R}_+$. It is evident that with regard to the function ω, for every compact interval I_k there exists a compact interval I_l such that if $t \in \mathrm{I}_k$, then $\omega(t) \in \mathrm{I}_l$.

Let $\varepsilon > 0$ be an arbitrary number and I_k be a given compact interval. We show that on I_k we have

$$\psi^{-1}(t)\,\mathrm{T}(g^n)(t) \rightrightarrows \psi^{-1}(t)\,\mathrm{T}(g)(t).$$

Denote $m = \max_{t \in \mathrm{I}_k} \psi^{-1}(t) = \psi^{-1}(0)$.

Let $g^n \rightrightarrows g$ for $t \in \mathrm{I}_l$, where I_l is a compact interval corresponding to I_k. Since f is continuous and $g^n \rightrightarrows g$ on I_l, there exists a number $n_0 > 0$ such that for $n > n_0$ we have

$$(4) \qquad |\mathrm{P}(t)[f(t, \mathrm{Q}_{\omega(t)} g^n_{\omega(t)}) - f(t, \mathrm{Q}_{\omega(t)} g_{\omega(t)})]| < \frac{\varepsilon}{m\,.\,k}, \qquad t \in \mathrm{I}_k.$$

Using (3) and (4), for $t \in \mathrm{I}_k$ and $n > n_0$ we obtain

$$|\mathrm{T}(g^n)(t) - \mathrm{T}(g)(t)| \leqslant \int_0^t |\mathrm{P}(s)[f(s, \mathrm{Q}_{\omega(s)} g^n_{\omega(s)}) - f(s, \mathrm{Q}_{\omega(s)} g_{\omega(s)})]|\, \mathrm{d}s <$$

$$< \frac{\varepsilon}{m\,.\,k} \int_0^t \mathrm{d}s < \frac{\varepsilon\,.\,k}{m\,.\,k} = \frac{\varepsilon}{m}.$$

From the last inequality for $t \in \mathrm{I}_k$ it follows

$$p_k[\mathrm{T}(g^n)(t) - \mathrm{T}(g)(t)] < \frac{\varepsilon\,.\,m}{m} = \varepsilon.$$

But this means that T is continuous on M.

We show that $\overline{\mathrm{TM}}$ is a compact set. Now, from the fact that $\mathrm{TM} \subset \mathrm{M}$ it follows that, for the functions of TM, the functions $\psi^{-1}(t)\,\mathrm{T}(g)(t)$ are uniformly bounded on R_+.

If $t_1, t_2 \in \mathrm{R}_+$, $t_1 < t_2$ are two arbitrary numbers, then we have the following estimate for T:

$$|\psi^{-1}(t_2)\,\mathrm{T}(g)(t_2) - \psi^{-1}(t_1)\,\mathrm{T}(g)(t_1)| \leqslant |[\psi^{-1}(t_2) - \psi^{-1}(t_1)]\,h(0)| +$$

$$+ \psi^{-1}(t_1) \int_{t_1}^{t_2} |\mathrm{P}(s) f(s, 0)|\, \mathrm{d}s + \psi^{-1}(t_1) \int_{t_1}^{t_2} \varphi(s, \lambda + \lambda \operatorname{sgn} \omega^+(s)\, \psi[\omega^+(s)])\, \mathrm{d}s.$$

The right hand side of this inequality does not depend on g and therefore TM is a set of equicontinuous functions. Hence $\overline{\mathrm{TM}}$ is a compact set.

By Schauder—Tychonoff fixed point theorem, the operator T has a fixed point $g^* \in \mathrm{M}$ and

$$g^*(t) = \mathrm{T}(g^*)(t) = \begin{cases} h(t) & \text{for} \quad t \in \mathrm{R}_- \\ h(0) + \int_0^t \mathrm{P}(s) f(s, \mathrm{Q}_{\omega(s)} g^*_{\omega(s)})\, \mathrm{d}s & \text{for} \quad t \in \mathrm{R}_+ . \end{cases}$$

Hence $g^* \in X$ and is ψ — bounded. This completes the proof.

Theorem 2. Let the assumptions (A_2), (A_4), (A_5) be satisfied. Then the set of ψ — bounded solutions $x \in X$ defined on R_+ is non empty and for every such solution $x \in X$ there exists a constant vector $\alpha \in \mathrm{R}^n$ such that

$$\text{(V)} \qquad \lim_{t \to \infty} \psi^{-1}(t) x(t) = \alpha.$$

Proof. Since the condition (A_4) implies the condition (A_1) and (A_5) implies (A_3), from Theorem 1 we obtain that there exists a ψ — bounded solution $x \in X$ on R_+.

We show that (V) holds. Let $x \in X$ be a ψ — bounded solution defined on R_+. Then from (3) we get

$$(5) \quad \psi^{-1}(t) x(t) = \psi^{-1}(t) h(0) + \psi^{-1}(t) \int_0^t \mathrm{P}(s) f(s, \mathrm{Q}_{\omega(s)} x_{\omega(s)})\, \mathrm{d}s, \qquad t \in \mathrm{R}_+ .$$

Monotonicity of the function ψ on R_+ implies the existence of the limit

$$(6) \qquad \lim_{t \to \infty} \psi^{-1}(t) h(0),$$

and the condition (A_4) implies the existence of the limit

$$(7) \qquad \lim_{t \to \infty} \psi^{-1}(t) \int_0^t \mathrm{P}(s) f(s, 0)\, \mathrm{d}s.$$

Similarly, from the condition (A_5) it follows that there exists the limit

$$(8) \qquad \lim_{t \to \infty} \psi^{-1}(t) \int_0^t \varphi(s, \lambda + \lambda \operatorname{sgn} \omega^+(s)\, \psi[\omega^+(s)])\, \mathrm{d}s.$$

From the existence of limits (6), (7) and (8) it follows the existence of the limit

$$\lim_{t \to \infty} \psi^{-1}(t) \int_0^t \mathrm{P}(s) f(s, \mathrm{Q}_{\omega(s)} x_{\omega(s)})\, \mathrm{d}s.$$

Then from (5) we obtain that there exists $\lim_{t \to \infty} \psi^{-1}(t) x(t)$. Hence there exists $\alpha \in \mathrm{R}^n$ such that (V) is true. Thus the theorem is proved.

Theorem 3. Let $0 < q < 1$ and $p \in R_+$ be arbitrary numbers. Let the function ϱ: $R_+ \times R_+ \to R_+$ be locally integrable in the first argument and nondecreasing in the second argument and let for each $t \in R_+$ and any $r \in R_+$

$$\int_0^t \varrho(s, r)\, ds \leqslant qr + p. \tag{9}$$

Let $c \geqslant 0$ be a real number.

If $u(t) \geqslant 0$ is a real continuous solution of the inequality

$$u(t) \leqslant c + \int_0^t \varrho(s, u[\omega^+(s)])\, ds, \qquad t \in R_+, \tag{10}$$

then there exists a real continuous solution $v = v(t, c)$ of the equation

$$v(t) = c + \int_0^t \varrho(s, v[\omega^+(s)])\, ds, \qquad t \in R_+, \tag{11}$$

such that

$$u(t) \leqslant v(t, c) \tag{12}$$

is true for $t \in R_+$.

Proof. Consider the Fréchet space F_1. Let

$$\{T_1^m u(t)\}_{m=1}^{\infty} \tag{13}$$

be a sequence on F_1, where

$$T_1^m u(t) = c + \int_0^t \varrho(s, T_1^{m-1} u[\omega^+(s)])\, ds, \qquad t \in R_+. \tag{14}$$

We show that the operator T_1 is continuous on F_1. Let $\{u^n\}_{n=1}^{\infty}$, $u^n \in F_1$ be a sequence and $v \in F_1$ such that

$$u^n \rightrightarrows v \quad \text{as} \quad n \to \infty, \tag{15}$$

on every compact interval $I_k \subset R_+$. Let $\varepsilon > 0$ be an arbitrary number and I_k be a given compact interval. Let I_l be a compact interval such that if $t \in I_k$ then $\omega(t) \in I_l$. Let (15) hold on I_l. Since ϱ is a continuous function and (15) hold on I_l, there exists a number $n_0 > 0$ such that for any $n > n_0$ we have

$$|\varrho(t, T_1 u^n[\omega^-(t)]) - \varrho(t, T_1 v[\omega^+(t)])|_1 < \frac{\varepsilon}{k}, \qquad t \in I_k.$$

Then for $t \in I_k$ and $n > n_0$ we obtain

$$|T_1 u^n(t) - T_1 v(t)|_1 \leqslant \int_0^t [|\varrho(s, T_1 u^n[\omega^-(s)]) - \varrho(s, T_1 v[\omega^+(s)])|_1]\, ds < \frac{\varepsilon . k}{k} = \varepsilon.$$

From the last inequality for $t \in \mathrm{I}_k$ it follows

$$p_{k_1}[\mathrm{T}_1 u^n(t) - \mathrm{T}_1 v(t)] \to 0 \quad \text{as} \quad n \to \infty .$$

Hence the operator T_1 is continuous on F_1.

Let I_1 be a compact interval coresponding to the interval I_k in the sense as above. Since u is a continuous function on R_+, there exists a number $c_0 \in \mathrm{R}_+$ such that on I_l the inequality

$$|u[\omega^+(t)]|_1 \leqslant c_0$$

is true.

Then from (14) with regard to (9) we get

$$|\mathrm{T}_1^m u(t)|_1 \leqslant c + \int_0^t \varrho(s, |\mathrm{T}_1^{m-1} u[\omega^+(s)]|_1)\, \mathrm{d}s \leqslant c + \int_0^t \varrho(s, c_{m-1})\, \mathrm{d}s \leqslant$$

$$\leqslant c_m = a + q c_{m-1}$$

for any $t \in \mathrm{I}_k$ and any $m \in \mathrm{N}$, where $a = c + p$.

By using the principle of mathematical induction we can prove that

(16) $$c_m = a + aq + aq^2 + \dots + aq^{m-1} + c_0 q^m .$$

From (16) one can see that

$$s_m = a + aq + aq^2 + \dots + aq^{m-1}$$

is the m-th partial sum of the geometric series

(17) $$\sum_{m=1}^{\infty} aq^{m-1}$$

with the quotient $0 < q < 1$. Hence the series (17) converges to the sum $\dfrac{a}{1-q}$.

Then from (16) we get the following estimate

$$|c_m|_1 \leqslant \frac{a}{1-q} + c_0,$$

which implies that

$$|\mathrm{T}_1^m u(t)|_1 \leqslant \frac{a}{1-q} + c_0 .$$

But this means that the sequence (13) is uniformly bounded on every compact interval I_k.

Let $t_1, t_2 \in \mathrm{R}_+$, $t_1 < t_2$ be two arbitrary numbers. Then we obtain the following estimate

$$|\mathrm{T}_1^m u(t_2) - \mathrm{T}_1^m u(t_1)|_1 \leqslant \int_{t_1}^{t_2} \varrho(s, |\mathrm{T}_1^{m-1} u[\omega^+(s)]|_1) \, \mathrm{d}s \leqslant \int_{t_1}^{t_2} \varrho(s, \mathrm{K}) \, \mathrm{d}s,$$

where $\mathrm{K} = \dfrac{a}{1-q} + c_0$.

The right hand side of this inequality does not depend on u and therefore the sequence (13) is a set of equicontinuous functions on I_k. Hence the sequence (13) is a compact set on every compact interval I_k.

In view of monotonicity of T_1 on F_1, the inequality (10) implies that

$$u(t) \leqslant \mathrm{T}_1 u(t) \leqslant \mathrm{T}_1^2 u(t) \leqslant \ldots \leqslant \mathrm{T}_1^m u(t) \leqslant \ldots.$$

The compactness and monotonicity of the sequence (13) implies that the sequence (13) converges uniformly on every compact subinterval of R_+ to some continuous function $v(t)$ and hence converges in the space F_1 to the element $v \in \mathrm{F}_1$. From the continuity of the operator T_1 it follows that the function $v(t) = v(t, c)$ is a solution of the equation (11). hence (12) is true. This completes the proof.

If

$$\varrho(t, s) = n(t)\, v(s) \tag{18}$$

then Theorem 3 implies the following corollary.

Corollary 1. Let $0 < q < 1$ and $p \in \mathrm{R}_+$ be arbitrary numbers. Let $n\colon \mathrm{R}_+ \to \mathrm{R}_+$ be a locally integrable function on R_+ such that the function

$$\int_0^t n(s) \, \mathrm{d}s$$

is bounded on R_+. Let $v\colon \mathrm{R}_+ \to \mathrm{R}_+$ be a continuous and nondecreasing function such that for $r \in \mathrm{R}_+$

$$v(r) \leqslant qr + p.$$

Let $c \geqslant 0$ be a real number.

If $u(t) \geqslant 0$ is a real continuous solution of the inequality (10) (where ϱ is given in (18)), then there exists a real continuous solution $v = v(t, c)$ of the equation (11) on R_+ such that (12) is true.

Remark 2. If $\varrho(t, s) = n(t)s$ then Theorem 3 is a special case of Lemma 3 in [2].

Denote $x \in X$ by $x(., h)$.

Theorem 4. Assume that for the function $\varrho\colon \mathrm{R}_+ \times \mathrm{R}_+ \to \mathrm{R}_+$ the conditions

(A_2), (9) hold. Then for any $h_1, h_2 \in C$, h_1, h_2 are uniformly continuous on R_- and $h_1(0) = h_2(0) = 0$ and for the solution x is true

$$\|x_t(\cdot, h_2) - x_t(\cdot, h_1)\| \leqslant v(t, \|h_2 - h_1\|) \quad \text{for} \quad t \in R_+, \tag{19}$$

where $v(t, \|h_2 - h_1\|)$ is the real continuous solution of (11) for $c = \|h_2 - h_1\|$.

Proof. Denote

$$u(t) = \sup_{-\infty < s \leqslant t} |x(s, h_2) - x(s, h_1)| = \|x_t(\cdot, h_2) - x_t(\cdot, h_1)\|, \qquad t \in R_+.$$

By (A_2) it follows that

$$|x(t, h_2) - x(t, h_1)| \leqslant |h_2(0) - h_1(0)| + \int_0^t \varrho(s, \|x_{\omega(s)}(\cdot, h_2) - x_{\omega(s)}(\cdot, h_1)\|) \, ds,$$

for $t \in R_+$, which we have

$$u(t) \leqslant \|h_2 - h_1\| + \int_0^t \varrho(s, u[\omega^+(s)]) \, ds, \qquad t \in R_+.$$

Thus $u(t)$ satisfies (10) with $c = \|h_2 - h_1\|$. It is evident that $u(t)$ is a continuous function. The inequality (19) follows from Theorem 3. This completes the proof.

Next we shall consider the initial problem

$$y'(t) = A(t) y(t) + f(t, y_{\omega(t)}) \tag{20}$$

$$y_+ = h, \tag{21}$$

where $A: R_+ \to R^{n \times n}$ is a continuous matrix function, and f, h have the same meaning as above.

Let Y be the set of all solutions of the initial problem (20), (21). Let $U(t)$ be a fundamental matrix for the system

$$u'(t) = A(t) u(t) \tag{22}$$

such that $U(0) = I$ denotes the identity matrix and $U^{-1}(t)$ is the inverse matrix to $U(t)$ on R_+.

Denote

$$P(t) = U^{-1}(t) \quad \text{and} \quad Q(t) = \begin{cases} U(t) & \text{for} \quad t \in R_+, \\ I & \text{for} \quad t \in R_-. \end{cases}$$

Theorem 5. Assume that for the matrices P, Q and the function f from (20) the hypotheses of Theorem 2 hold. Then the set of the solutions $y \in Y$ defined on R_+ is non empty and for every such solution $y \in Y$ there exists a constant vector $\alpha \in R^n$ such that

$$y(t) = Q(t) x(t) \quad \text{and} \quad \lim_{t \to \infty} \psi^{-1}(t) x(t) = \alpha. \tag{23}$$

Proof. By the substitution

$$(24) \qquad y(t) = \mathrm{Q}(t)\,x(t)$$

we can transform every initial problem (20), (21) to the initial problem (1), (2), where $y_0 = x_0$. Relationship between sets X, Y is determined by (24).

From Theorem 1 and from (24) it follows that there exists a solution $y \in Y$ defined on R_+. From Theorem 2 moreover it follows that there exists a constant vector $\alpha \in \mathrm{R}^n$ such that (23) holds.

The above assertions imply the following corollaries.

Corollary 2. Assume that the hyptheses of Theorem 5 are satisfied and, furthermore, let $\lim_{t \to \infty} \mathrm{U}(t)$ be a constant matrix. Then the set of ψ — bounded solutions $y \in Y$ defined on R_+ is non empty and for every such solution $y \in Y$ it is true

$$(\mathrm{V}') \qquad \lim_{t \to \infty} \psi^{-1}(t)\,y(t) = \bar{a}, \qquad \bar{a} \in \mathrm{R}^n.$$

Corollary 3. Assume that the hypotheses of Theorem 5 are satisfied and let all solutions of the system (22) be bounded on R_+. Let the function ψ be bounded on R_-. Then for each solution $y \in Y$ defined on R_+ there exists a solution $u(t)$ of (22) on R_+ such that

$$(25) \qquad |y(t) - u(t)| \to 0 \quad \text{as} \quad t \to \infty.$$

Proof. Since all solutions $u(t)$ of (22) are bounded on R_+, there exists a number $r > 0$ such that

$$|\mathrm{U}(t)| \leqslant r \quad \text{for} \quad t \in \mathrm{R}_+.$$

We know that if $a \in \mathrm{R}^n$ is a constant vector then the function $u(t) = \mathrm{U}(t)\,a$ is a solution of (22).

Let $y \in Y$ be a solution defined on R_+. Theorem 5 implies that for the solution y there exists a constant vector $\alpha \in \mathrm{R}^n$ such that for $t \in \mathrm{R}_+$ (23) holds. Consider the solution $u(t)$ of (22) in the form $u(t) = \mathrm{U}(t)\,\alpha$. Then we get

$$|y(t) - u(t)| = |\mathrm{U}(t)\,x(t) - \mathrm{U}(t)\,\alpha| \leqslant |\mathrm{U}(t)|\,|x(t) - \alpha| \leqslant r|x(t) - \alpha| \to 0,$$

as $t \to \infty$, which is (25).

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SÚHRN

O ASYMPTOTICKOM CHOVANÍ RIEŠENÍ FUNKCIONÁLNO-DIFERENCIÁLNYCH ROVNÍC S POSUNUTÝM ARGUMENTOM

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V práci sú uvedené postačujúce podmienky pre to, aby existovalo ψ — ohraničené riešenie funkcionálno-diferenciálnej rovnice s posunutým argumentom tvaru $x'(t) = P(t)f(t, Q_{\omega(t)}x_{\omega(t)})$ a naviac, aby toto riešenie malo isté asymptotické vlastnosti.

РЕЗЮМЕ

ОБ АСИМПТОТИЧЕСКОМ ПОВЕДЕНИИ РЕШЕНИЙ ФУНКЦИОНАЛЬНО-ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ С ОТКЛОНЯЮЩИМСЯ АРГУМЕНТОМ

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В работе приведены достаточные условия для того, чтобы существовало ψ — ограниченное решение функционально-дифференциального уравнения с отклоняющимся аргументом $x'(t) = P(t)f(t, Q_{\omega(t)}x_{\omega(t)})$ и, кроме того, чтобы это решение имело какие-то асимптотические свойства.