UNIVERSITAS COMENIANA
ACTA MATHEMATICA UNIVERSITATIS COMENIANAE
LVI—LVII

ON THE DIFFERENTIAL EQUATION FOR THE GENERALIZED HYPERGEOMETRIC FUNCTION

JOZEF DANČO jr.—ANTON HUŤA jr., Bratislava

Introduction

As it is known, one of the solutions of the Gauss's differential equation of the form

$$x(1-x)y'' + [\gamma_1 - x(\alpha_1 + \alpha_2 + 1)]y' - \alpha_1\alpha_2 y = 0 \tag{1}$$

is the hypergeometric function

$$y(x) = {}_2F_1(x) = \sum_{k=0}^{\infty} \frac{(\alpha_1 + k - 1)_k (\alpha_2 + k - 1)_k}{(\gamma_1 + k - 1)_k} \frac{x^k}{k!} \tag{2}$$

(cf [1] or [2]), where $(z)_n$ denotes the product $z(z-1)\dots(z-n+1)$ for $n > 0$ and $(z)_0 = 1$ (cf [4]). The hypergeometric function is a very general function with three free parameters and in special cases can represent many elementary and transcendental functions.

The well-known confluent hypergeometric function can be one of the solutions of the differential equation of the form

$$xy'' + (\gamma_1 - x)y' - \alpha_1 y = 0 \tag{3}$$

i.e.

$$y(x) = {}_1F_1(x) = \sum_{k=0}^{\infty} \frac{(\alpha_1 + k - 1)_k}{(\gamma_1 + k - 1)_k} \frac{x^k}{k!} \tag{4}$$

where α_1, γ_1 are two free parameters (cf [1] or [2]).

The generalized hypergeometric function can be written in the form

$${}_pF_q(x) = \sum_{k=0}^{\infty} \frac{(\alpha_1 + k - 1)_k \dots (\alpha_p + k - 1)_k}{(\gamma_1 + k - 1)_k \dots (\gamma_q + k - 1)_k} \frac{x^k}{k!} \tag{5}$$

(cf [1] or [3]). The explicit form of the differential equation corresponding to this function is not known from current available literature.

The purpose of this paper is to derive the differential equation for the generalized hypergeometric function. The results are formulated in three theorems.

1 Special case one

Theorem 1. Let q be a nonnegative integer. The solution of the differential equation of the form

$$\sum_{j=0}^{q} B_j x^j {}_0F_q^{(j+1)}(x) = {}_0F_q(x) \tag{6}$$

where

$$\begin{aligned} B_q &= 1, \\ B_{q-i} &= A_i(\gamma_1, \gamma_2, \ldots, \gamma_q) - \sum_{j=0}^{i-1} s(q-j, q-i) B_{q-j} \\ &\quad \text{for } i = 1, 2, \ldots, q-1 \\ \text{and } B_0 &= A_q(\gamma_1, \gamma_2, \ldots, \gamma_q) \end{aligned} \tag{7}$$

where $A_i(\gamma_1, \ldots, \gamma_q)$ are elementary symmetrical polynomials of variables $\gamma_1, \ldots, \gamma_q$

$$\begin{aligned} A_1(\gamma_1, \gamma_2, \ldots, \gamma_q) &= \gamma_1 + \gamma_2 + \ldots + \gamma_q \\ A_2(\gamma_1, \gamma_2, \ldots, \gamma_q) &= \gamma_1\gamma_2 + \gamma_1\gamma_3 + \ldots + \gamma_1\gamma_q + \ldots + \gamma_{q-1}\gamma_q \\ A_q(\gamma_1, \gamma_2, \ldots, \gamma_q) &= \gamma_1\gamma_2\cdots\gamma_q \end{aligned} \tag{8}$$

and $s(i, j)$ are the Stirling numbers of the first kind (cf [4]), is a special case of the generalized hypergeometric function

$${}_0F_q(x) = \sum_{k=0}^{\infty} \frac{1}{(\gamma_1 + k - 1)_k \ldots (\gamma_q + k - 1)_k} \frac{x^k}{k!} \tag{9}$$

Proof. Let us suppose the differential equation is in the form (6), where $0 < j \leq q$, then for its j-th member we can write

$$B_j x^j {}_0F_q^{(j-1)}(x) = B_j x^j \sum_{i=0}^{\infty} \frac{1}{(\gamma_1 + i + j)_{i+j+1} \cdots (\gamma_q + i + j)_{i+j+1}} \frac{x^i}{i!} = \tag{10}$$

$$= B_j \sum_{i=0}^{\infty} \frac{1}{(\gamma_1 + i + j)_{i+j+1} \dots (\gamma_q + i + j)_{i+j+1}} \frac{x^{i+j}}{i!} =$$

$$= B_j \sum_{i=j}^{\infty} \frac{1}{(\gamma_1 + i)_{i+1} \dots (\gamma_q + i)_{i+1}} \frac{x^i}{i!} (i)_j$$

The coefficients on the right-hand side are known, but those on the left-hand side are not. From the equation (10) we see that the coefficients are to satisfy

$$(11) \qquad (\gamma_1 + i)(\gamma_2 + i) \dots (\gamma_q + i) = i^q + A_1(\gamma_1, \dots, \gamma_q)\, i^{q-1} + \\ + A_2(\gamma_1, \dots, \gamma_q)\, i^{q-2} + \dots + A_q(\gamma_1, \dots, \gamma_q)$$

and the following identity is valid

$$(12) \qquad (i)_w = \sum_{l=0}^{w} s(w, l)\, i^l$$

Comparing the coefficients with equal powers of i and substituting $w = q$ we obtain $B_q = 1$ and repeatedly using the identity (12) for $w = q, q-1, \dots, 1$ we obtain (7). [To clarify the method of obtaining the coefficients B_i *Table 1* in Appendix was constructed.]

2 Special case two

For another special case, when $q = 0$ and p is an integer, we can formulate the following theorem.

Theorem 2. Let p be a nonnegative integer. Then the solution of the differential equation

$$(13) \qquad \sum_{j=0}^{p} D_j x^j\, {}_pF_0^{(j)}(x) = {}_pF_0'(x)$$

where

$$(14) \qquad \begin{aligned} &D_p = 1, \\ &D_{p-i} = A_i(\alpha_1, \dots, \alpha_p) - \sum_{j=0}^{i-1} s(p-j, p-i)\, D_{p-j} \\ &\qquad \text{for } i = 1, 2, \dots, p-1 \\ &\text{and } D_0 = A_p(\alpha_1, \dots, \alpha_p) \end{aligned}$$

is a special case of the generalized hypergeometric function

$$(15) \qquad {}_pF_0(x) = \sum_{k=0}^{\infty} (\alpha_1 + k - 1)_k \dots (\alpha_p + k - 1)_k \frac{x^k}{k!}$$

Proof. The proof is analogous to the proof of the Theorem 1. Let us consider that

$$(16)\quad D_l x^l {}_pF_0^{(l)}(x) = D_l x^l \sum_{i=0}^{\infty} (\alpha_1 + i + l - 1)_{i+l} \dots (\alpha_p + i + l - 1)_{i+l} \frac{x^i}{i!} =$$

$$= D_l \sum_{i=l}^{\infty} (\alpha_1 + i - 1)_i \dots (\alpha_p + i - 1)_i \frac{x^i}{i!} (i)_l$$

and repeatedly using the identity (12) for $w = p, p - 1, \dots, 1$ the coefficients D_i can be obtained in the same way as in the Theorem 1.

3 The general case

The next Theorem 3 will be a generalization of the Theorem 1 and Theorem 2.

Theorem 3. Let p and q be nonnegative integers. then the generalized hypergeometric function (5) is one of the solutions of the differential equation of the form

$$(17)\qquad \sum_{i=0}^{p} D_i x^i {}_pF_q^{(i)}(x) - \sum_{i=0}^{q} B_i x^i {}_pF_q^{(i+1)}(x) = 0$$

where B_i and D_i are coefficients defined in Theorem 1 and Theorem 2.

Proof. The proof can be realized by substituting the generalized hypergeometric function (5) into the left-hand side of the equation (17).

Corollary. Two interesting cases of the generalized hypergeometric function are the cases when $p = q + 1$ and $p = q$. The differential equation in these special cases is of the form

$$(18)\quad (B_q - xD_{q+1}) x^q {}_{q+1}F_q^{(q+1)}(x) + (B_{q-1} - xD_q) x^{q-1} {}_{q+1}F_q^{(q)}(x) +$$

$$+ \dots + (B_0 - xD_1) {}_{q+1}F_q'(x) - D_0 {}_{q+1}F_q(x) = 0$$

and

$$(19)\qquad B_p x^p {}_pF_p^{(p+1)}(x) + (B_{p-1} - xD_p) x^{p-1} {}_pF_p^{(p)}(x) +$$

$$+ (B_{p-2} - xD_{p-1}) x^{p-2} {}_pF_p^{(p-1)}(x) + \dots + (B_0 - xD_1) {}_pF_p'(x) - D_0 {}_pF_p(x) = 0$$

Only in these two cases for $q = 0$ the differential equation is of the first order. The *Table 2* in Appendix gives the survey of the order of the differential equation (17) in dependence on parameters p and q.

Remark. A brief survey concerning the properties of the discussed differential equation cf. e.g. [5].

Table 1

i^q	i^{q-1}	i^{q-2}	i^{q-3}	i^{q-4}
$B_q = 1$	$s(q, q-1)$	$s(q, q-2)$	$s(q, q-3)$	$s(q, q-4)$
	$-s(q, q-1)$	$B_{q-1}s(q-1, q-2)$	$B_{q-1}s(q-1, q-3)$	$B_{q-1}s(q-1, q-4)$
	$A_1(\gamma_1, \ldots, \gamma_q)$	B_{q-2}	$B_{q-2}s(q-2, q-3)$	$B_{q-2}s(q-2, q-4)$
		$A_2(\gamma_1, \ldots, \gamma_q)$	B_{q-3}	$B_{q-3}s(q-3, q-4)$
			$A_3(\gamma_1, \ldots, \gamma_q)$	B_{q-4}
				$A_4(\gamma_1, \ldots, \gamma_q)$

Table 2

q \ p	0	1	2	3	4	...	p	...
0	1	1	2	3	4	...	p	...
1	2	2	2	3	4	...	p	...
2	3	3	3	3	4	...	p	...
3	4	4	4	4	4	...	p	...
4	5	5	5	5	5	...	p	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	
q	$q+1$	$q+1$	$q+1$	$q+1$	$q+1$	...	r^*	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\ddots$

(the order "r" of the differential equation for ${}_pF_q(x)$)

*) if $p \geqq q$ then $r = q + 1$ else $r = p$.

REFERENCES

1. Bateman, H.—Erdélyi, A.: Higher transcendental functions, Volume I—III. Mac Graw-Hill Book Company Inc., New York, Toronto, London 1953—1955.
2. Lebedyev, N. N.: Special functions and their applications (Russian), Fizmatgiz, Moskva—Leningrad 1963.
3. Ryshik, I. M.—Gradstein, I. S.: Tables of Series, Products and Integrals, VEB Deutscher Verlag der Wissenschaften, Berlin 1957.
4. Riordan, J.: An introduction to combinatorial analysis, John Wiley and Sons Inc., New York, London 1958.

5. **Luke**, Y. L.: Mathematical functions and their Approximations, Academic Press Inc., New York, San Francisco, London 1975.

Author's address:

Jozef Dančo jr.
Anton Huťa jr.
KNOM MFF UK
Matematický pavilón
Mlynská dolina
842 15 Bratislava ČSR

Received: 25. 5. 1988

SÚHRN

O DIFERENCIÁLNEJ ROVNICI PRE ZOVŠEOBECNENÚ HYPERGEOMETRICKÚ FUNKCIU

DANČO JOZEF, ml.—HUŤA ANTON, ml., Bratislava

Z literatúry je známa Gaussova diferenciálna rovnica (1), riešením ktorej je hypergeometrická funkcia (2) a diferenciálna rovnica (3), riešením ktorej je konfluentná hypergeometrická funkcia (4).

V práci je odvodený tvar diferenciálnej rovnice (17) a jej koeficientov, riešením ktorej je zovšeobecnená hypergeometrická funkcia (5). Pritom sú odvodené špeciálne prípady pre $p = 0$ ((6) a (8)) a $q = 0$ ((13) a (15)) a tiež uvedené špeciálne prípady pre $p = q + 1$ (18) a $p = q$ (19).

РЕЗЮМЕ

О ДИФФЕРЕНЦИАЛЬНОМ УРАВНЕНИИ ДЛЯ ОБОБЩЕННОЙ ГИПЕРГЕОМЕТРИЧЕСКОЙ ФУНКЦИИ

ДАНЧО ЙОЗЕФ, мл.—ХУТЯ АНТОН, мл., Братислава

Из литературы известно дифференциальное уравнение Гаусса (1), решением которого является гипергеометрическая функция (2) и дифференциальное уравнение (3), решением которого является вырожденная гипергеометрическая функция (4).

В работе произведен вид дифференциального уравнения (17) и его коэффициентов, решением которого является обобщенная гипергеометрическая функция (5). Притом произведены специальные случаи для $p = 0$ ((6) и (8)) и $q = 0$ ((13) и (15)), а также приведены специальные случаи для $p = q + 1$ (18) и $p = q$ (19).