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OSCILLATION CRITERIA AND GROWTH OF NONOSCILLATORY SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS

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Consider the differential equation

$$L_n y(t) + f(t, y(t)) = 0, \tag{E}$$

where $n > 1$ and L_n is an n^{th} order disconjugate operator defined by

$$L_n y(t) = a_n(t)(a_{n-1}(t)(\dots (a_1(t)(a_0(t)\, y(t))')' \dots)')'.$$

The expressions

$$L_0 y(t) = a_0(t)\, y(t),$$

$$L_i y(t) = a_i(t)(L_{i-1} y(t))', \qquad i = 1, 2, \dots, n,$$

are called the "quasi-derivates" of y at the point t. We denote by $D(L_k)$ a set of real valued functions y defined on $\langle t_y, \infty)$ for which $L_k y(t)$ exists for every $t \in \langle t_y, \infty)$.

The specific assumptions that we make are:

(A1) a_i, $i = 0, 1, \dots, n$, are real valued, positive and continuous functions defined on $\langle 0, \infty)$ and

$$\int_0^\infty a_i^{-1}(r)\, \mathrm{d}r = \infty, \qquad i = 1, 2, \dots, n-1,$$

(A2) f is real valued, continuous function defined on $\langle 0, \infty) \times R$ and nondecreasing in y for fixed t, satisfying $yf(t, y) > 0$, for $y \neq 0$.

Define functions

$$I_0(t, s) = 1,$$

$$I_k(t, s, i_k, i_{k-1}, \dots, i_1) = \int_s^t a_{i_k}^{-1}(r) \cdot I_{k-1}(r, s, i_{k-1}, i_{k-2}, \dots, i_1)\, \mathrm{d}r,$$

where $0 \leqq t < \infty$, $0 \leqq s < \infty$ and $k = 1, 2, \dots, n-1$.

For simplicity we put

$$J_i(t, s) = a_0(t) I_i(t, s, 1, 2, \dots, i),$$

$$J_i(t) = J_i(t, 0),$$

$$K_i(t, s) = a_n(t) I_i(t, s, n-1, n-2, \dots, n-i),$$

$$K_i(t) = K_i(t, 0),$$

$$H_{ik}(t, s) = \int_s^t I_{k-i}(t, r, i, i+1, \dots, k-1) a_k^{-1}(r) \cdot$$

$$\cdot I_{n-k-1}(t, r, n-1, n-2, \dots, k+1)\,\mathrm{d}r,$$

$$H_{ik}(t) = H_{ik}(t, 0),$$

where $0 \leqq t < \infty$, $0 \leqq s < \infty$ and $0 \leqq i \leqq k \leqq n$.

It is useful to note that

$$I_k(t, s, i_k, i_{k-1}, \dots, i_1) = (-1)^k I_k(s, t, i_1, i_2, \dots, i_k), \tag{1}$$

from which it follows that

$$I_k(t, s, i_k, i_{k-1}, \dots, i_1) = \int_s^t a_1^{-1}(r) I_{k-1}(t, r, i_k, i_{k-1}, \dots, i_2)\,\mathrm{d}r \tag{2}$$

and the generalized Taylor's formula

$$\begin{aligned} L_i y(t) = {} & \sum_{j=i}^{k} (-1)^{j-i} L_j y(s) I_{j-i}(s, t, j, j-1, \dots, i+1) + \\ & + (-1)^{k-i+1} \int_t^s I_{k-i}(r, t, k, k-1, \dots, i+1) a_{k+1}^{-1}(r) I_{k+1} y(r)\,\mathrm{d}r, \end{aligned} \tag{3}$$

where $0 \leqq t < \infty$, $0 \leqq s < \infty$, $0 \leqq i \leqq k \leqq n-1$ and $y \in D(L_{k+1})$.

To obtain the main results we need following lemmas. The first is adapted from the papers of Čanturija [1], Švec [10] and Elias [2].

Lemma 1. Let $y \in D(L_n)$ be such that $y(t) \geqq 0$ and $L_n y(t) \leqq 0$ on $\langle t_y, \infty)$. Then there exists a $T \geqq t_y$ and an integer k, $0 \leqq k < n$, such that

$$n + k \quad \text{is odd,} \tag{4}$$

$$(-1)^{n+i+1} y(t) L_i y(t) > 0, \quad i = k+1, k+2, \dots, n-1, \quad \text{for } t > T, \tag{5}$$

$$\lim_{t \to \infty} L_i y(t) = 0, \qquad i = k+1, k+2, \dots, n-1, \tag{6}$$

$$y(t) L_i y(t) > 0, \qquad i = 0, 1, \dots, k, \quad \text{for} \quad t > T, \tag{7}$$

(8) $$\lim_{t\to\infty} L_i y(t) = \infty, \qquad i = 0, 1, \ldots, k-1,$$

(9) $$\lim_{t\to\infty} L_k y(t) = c, \qquad c \in \langle 0, \infty),$$

(10) $$\frac{L_{i+1} y(t)}{L_{i+1} J_k(t, T)} \leqq \frac{L_i y(t)}{L_i J_k(t, T)}, \qquad i = 0, 1, \ldots, k-1, \quad \text{for} \quad t > T.$$

The following two lemmas are standard comparison theorems cited from [7].

Lemma 2. Let E be an open set in $R \times R$ and g a continuous, real valued function defined on E. Suppose that $\langle t_0, t_0 + a)$ is the largest interval in which the maximal solution r of scalar differential equation with an initial condition

$$u' = g(t, u), \qquad u(t_0) = u_0,$$

exists. Let $m \in C((t_0, t_0 + a), R)$, $(t, m(t)) \in E$ for $t \in \langle t_0, t_0 + a)$, $m(t_0) \leqq u_0$, and for a fixed Dini derivative,

$$Dm(t) \leqq g(t, m(t)),$$

$t \in \langle t_0, t_0 + a) - S$, where S is a set of measure zero. Then,

$$m(t) \leqq r(t), \qquad t \in \langle t_0, t_0 + a).$$

Lemma 3. Let $f \in C(J \times R^n, R^n)$, $f(t, x)$ be monotonic nondecreasing in x for each t and

$$x(t) \geqq x_0 + \int_{t_0}^{t} f(s, x(s))\, \mathrm{d}s,$$

where $x \in C(J, R^n)$. Suppose that r is the minimal solution of

$$u' = f(t, u), \qquad u(t_0) = x_0,$$

existing on $\langle t_0, \infty)$. Then,

$$x(t) \geqq r(t), \quad \text{for} \quad t \geqq t_0.$$

Our results on the behavior of the solution of (E) will involve the first-order scalar differential equations

(11 k) $$u' = H_{1k}(t, a) f(t, u),$$

(12 ik) $$v' = H_{ik}(t, a) f\left(t + t_0, \frac{J_k(t)}{L_{i-1} J_k(t)} v\right),$$

(13 k) $$z' = -f(t, H_{1k}(t, a) z).$$

Throughout this paper $u_k(t, a, u_0)$ will denote the minimal (maximal) solution of (11 k) with $u(a) = u_0$ if $u_0 > 0$ $(u_0 < 0)$. In a similar fashion, $z_k(t, a, z_0)$ will denote maximal (minimal) solution of (13 k) with $z(a) = z_0$ if $z_0 > 0$ $(z_0 < 0)$.

Definition 1. Let $g \in C(R^2, R)$ be such that $yg(t, y) > 0$ when $y \neq 0$ and let $(t_0, y_0) \in R^2$. Let $x_1(t)$ and $x_2(t)$ be the maximal and the minimal solutions of

$$x' = g(t, x), \qquad x(t_0) = y_0, \tag{14}$$

respectively. We say that solution $x_3(t)$ of (14) is of finite escape time, or o.f.e.t. for short, if there exists $t_0 < t_3 < \infty$ such that

$$\lim_{t \to t_3} |x_3(t)| = \infty .$$

If for every $(t_0, y_0) \in R^2$ there exists $t_0 < t_1 \leqq t_2 < \infty$ such that

$$\lim_{t \to t_1} |x_1(t)| = \lim_{t \to t_2} |x_2(t)| = \infty ,$$

then every solution of (14) is o.f.e.t. and we say that (14) is o.f.e.t.

Definition 2. We say that y is a solution of (E) if $y(t)$ satisfies (E) in the interval $\langle t_y, \infty)$ for some $t_y \geqq 0$ and $y \in D(L_n)$.

A solution of (E), not identically zero, is said to be oscillatory if it has arbitrarily large zeros. Otherwise a solution is called nonoscillatory.

The equation (E) is called oscillatory if all solutions of (E) are oscillatory.

A solution of (E) is said to be of order k if there exists an integer k, $1 \leqq k \leqq n - 1$, such that (4), (5), (7) hold.

Remark 1. Without loss of generality we shall assume that functions a_0, a_n are identically equal to one on $\langle 0, \infty)$. Moreover all proofs will be carried out only for positive functions y (solutions of (E)). Proofs for negative ones are similar.

Lemma 4. Let $0 \leqq k \leqq n$ is an integer number and let $y \in D(L_{k+1})$ be such that for every $t \in (a, b)$, $0 \leqq a < b \leqq \infty$, hold:

$$y(t)\, L_i y(t) > 0, \qquad i = 0, 1, \ldots, k, \tag{15}$$

$$y(t)\, L_{k+1} y(t) < 0. \tag{16}$$

Then for every $t \in (a, b)$ is satisfied the inequality

$$|\, y(t)| \leqq \sum_{i=0}^{k} |L_i y(t)|\, J_i(t, a).$$

Proof. Let y be positive on (a, b). It implies $L_{k+1} y$ is negative on (a, b). Let $t \in (a, b)$ be fixed. After integration from a to t we have

$$L_k y(t) < L_k y(a).$$

Now we shall repeat this and after k integration from a to t we get

$$y(t) < \sum_{i=0}^{k} L_i y(a)\, J_i(t, a).$$

Theorem 1. Let y be a function satisfying (E) in (a,b), $0 \leqq a < b \leqq \infty$. Then there exists no finite point $t_1 \in (a, b)$ for which

$$\lim_{t \to t_1} L_i y(t) = \infty \quad \text{or} \quad \lim_{t \to t_1} L_i y(t) = -\infty$$

for any $0 \leqq i \leqq n$.

Proof. Suppose there exists some $0 \leqq i \leqq n$ and $t_1 \in (a, b)$ for which

$$\lim_{t \to t_1} L_i y(t) = \infty. \tag{17}$$

Let $Z_j = \{t,\ a < t < b \text{ and } L_j y(t) = 0\}$. If t_1 is an accumulation point of Z_j then by Role's Theorem t_1 is an accumulation point of Z_{j+1} too, and therefore of Z_n. Because of (E) and the properties of f, t_1 is an accumulation point of Z_0 as well, and so it is an accumulation point for Z_k, for every $0 \leqq k \leqq n$, in particular for Z_i, which is a contradiction to (17). Therefore there are t_2, $a < t_2 < t_1$, and an integer number k, $0 \leqq k \leqq n$, such that (15), (16) hold for every $t \in (t_2, t_1)$. By Lemma 4 we get

$$|y(t)| \leqq \sum_{i=0}^{k} |L_i y(t_2)|\, J_i(t, t_2),$$

for every $t \in (t_2, t_1)$.

Then $|y|$ is bounded from above. On the other hand, by the Mean Value Theorem $\limsup_{t \to t_1} L_{i+1} y(t) = \infty$, and if $i + 1 < n$ then $L_{i+1} y$ is monotonic on (t_2, t_1), which implies $\lim_{t \to t_1} L_{i+1} y(t) = \infty$, and therefore $\limsup_{t \to t_1} L_n y(t) = \infty$. By the continuity of f, $\limsup_{t \to t_1} y(t) = -\infty$ as well, in contradiction to the fact that $|y|$ is bounded from above. If we suppose that $\lim_{t \to t_1} L_i y(t) = -\infty$ we get the contradiction by the same way.

Remark 2. By Lemma 1 every nonoscillatory solution of (E) is of order k for some $0 \leqq k \leqq n - 1$. Let N_k be the set of all nonoscillatory solutions of order k of (E). Then we have

$$N = N_0 \cup N_2 \cup N_4 \cup \ldots \cup N_{n-1}$$

for n odd,

$$N = N_1 \cup N_3 \cup N_5 \cup \ldots \cup N_{n-1}$$

for n even,

where N is the set of all nonoscillatory solutions of (E). It is clear that if $N_i = \emptyset$ for every $i = 0, 1, \ldots, n - 1$, then equation (E) is oscillatory.

Lemma 5. Let $y \in N_k$, $0 < k < n$, and let $T \geqq t_y$ be such that $y(t)$ is positive for every $t \geqq T$. Then

$$(18) \quad L_{i-1}y(t) \geqq L_{i-1}y(t_1) + H_{ik}(t, t_1)(L_{n-1}y(t) - c) - \int_{t_1}^{t} H_{ik}(r, t_1) L_n y(r)\, \mathrm{d}r,$$

for every $t \geqq t_1 \geqq T$, $1 \leqq i \leqq k$, where $c = \lim_{t \to \infty} L_{n-1}y(t)$.

Proof. $y \in N_k$ implies that there exists $T \geqq t_y$ such that (4), (5) and (7) hold. By the generalized Taylor's formula we have

$$(19) \quad L_{i-1}y(t) \geqq L_{i-1}y(t_1) + \int_{t_1}^{t} I_{k-i}(t, r, i, i+1, \ldots, k-1)\, a_k^{-1}(r) L_k y(r)\, \mathrm{d}r,$$

for every $t \geqq t_1 \geqq T$, and

$$L_k y(r) \geqq -\int_r^s I_{n-k-1}(\tau, r, n-1, n-2, \ldots, k+1) L_n y(\tau)\, \mathrm{d}\tau,$$

for every $s \geqq r \geqq t_1$, which yields

$$(20) \quad L_k y(r) \geqq -\int_r^{\infty} I_{n-k-1}(\tau, r, n-1, n-2, \ldots, k+1) L_n y(\tau)\, \mathrm{d}\tau,$$

for every $r \geqq t_1$. Combining (19) with (20) we get

$$L_{i-1}y(t) \geqq L_{i-1}y(t_1) - \int_{t_1}^{t} I_{k-i}(t, r, i, i+1, \ldots, k-1)\, a_k^{-1}(r) \cdot$$

$$\cdot \int_r^{\infty} I_{n-k-1}(\tau, r, n-1, n-2, \ldots, k+1) L_n y(\tau)\, \mathrm{d}\tau = L_{i-1}y(t_1) -$$

$$- \int_{t_1}^{t} \left(\int_{t_1}^{\tau} I_{k-i}(t, r, i, i+1, \ldots, k-1)\, a_k^{-1}(r) I_{n-k-1}(\tau, r, n-1,\right.$$

$$\left. n-2, \ldots, k+1)\, \mathrm{d}r \right) L_n y(\tau)\, \mathrm{d}\tau - \int_t^{\infty} \left(\int_{t_1}^{t} I_{k-i}(t, r, i, i+1, \ldots, k-1)\, a_k^{-1}(r) \cdot\right.$$

$$\left. \cdot I_{n-k-1}(\tau, r, n-1, n-2, \ldots, k+1) L_n y(\tau)\, \mathrm{d}r \right) \mathrm{d}\tau \geqq L_{i-1}y(t_1) -$$

$$- \int_{t_1}^{t} H_{ik}(\tau, t_1) L_n y(\tau)\, \mathrm{d}\tau - H_{ik}(t, t_1) \int_t^{\infty} (L_{n-1}y(\tau))'\, \mathrm{d}\tau =$$

$$= L_{i-1}y(t_1) + H_{ik}(t, t_1)(L_{n-1}y(t) - c) - \int_{t_1}^{t} H_{ik}(\tau, t_1) L_n y(\tau)\, \mathrm{d}\tau,$$

for every $t \geqq t_1 \geqq T$, $1 \leqq i \leqq k$, where $c = \lim_{t \to \infty} L_{n-1} y(t)$ and by Lemma 1 $c \in \langle 0, \infty)$.

Theorem 2. Let $y \in N_k$, where k is an integer $0 < k < n$. Then there exists $t_0 \geqq t_y$ such that

(21) $$|y(t)| \geqq |u_k(t, t_1, y(t_1))|$$

for every $t \geqq t_1 \geqq t_0$,

(22) $$|L_{n-1} y(t)| \leqq |c + z_k(t, t_2, L_{n-1} y(t_2))|$$

for every $t \geqq t_2 \geqq t_0$, where $c = 0$ if $k \neq n - 1$ and $c \geqq 0$ if $k = n - 1$.

Proof. We shall assume that y is positive for all $t \gneqq t_0$. The case when y is eventually negative is similar and will be omitted. By Lemma 5 we have that (18) holds. From (E) we get

(23) $$L_n y(t) = -f(t, y(t)).$$

The assumption (A2) implies that $L_n y(t) < 0$ and therefore

$$H_{1k}(t, t_1)(L_{n-1} y(t) - c) > 0 \quad \text{for every} \quad t \geqq t_1 \geqq t_0,$$

where $c = \lim_{t \to \infty} L_{n-1} y(t)$ and, by Lemma 1, $c = 0$ if $k \neq n - 1$ and $c \geqq 0$ if $k = n - 1$. Now we can write

$$y(t) \geqq y(t_1) + \int_{t_1}^{t} H_{1k}(r, t_1) f(r, y(r)) \, \mathrm{d}r \quad \text{for every} \quad t \geqq t_1 \geqq t_0,$$

and by Lemma 3 we get

$$y(t) \geqq u_k(t, t_1, y(t_1)) \quad \text{for every} \quad t \geqq t_1 \geqq t_0.$$

The expression

$$-\int_{t_2}^{t} H_{1k}(r, t_2) L_n y(r) \, \mathrm{d}r \quad \text{for every} \quad t \geqq t_2 \geqq t_0,$$

is positive too. From (18), (A2) and (23) we have

$$L_n y(t) = (L_{n-1} y(t) - c)' \leqq -f(t, H_{1k}(t, t_2)(L_{n-1} y(t) - c))$$

for every $t \geqq t_2 \geqq t_0$. By Lemma 2 we get

$$L_{n-1} y(t) \leqq c + z_k(t, t_2, L_{n-1} y(t_2)) \quad \text{for every} \quad t \geqq t_2 \geqq t_0.$$

Theorem 3. Let $y \in N_k$, with k an integer, $0 < k < n$, and let Y_k and Q_k be nonnegative functions defined on $\langle 0, \infty)$ with the property that

(24) $$u_0 > Y_k(a) \text{ implies } u_k(t, a, u_0) \text{ has finite escape time,}$$

(25) $\qquad z_0 < Q_k(a)$ implies there exists $t \geqq a$ so that $z_k(t, a, z_0) \leqq 0$.

Then there exists $t_0 \geqq t_y$ such that

$$y(t) \leqq Y_k(t), \tag{26}$$

$$L_{n-1}y(t) \geqq Q_k(t), \tag{27}$$

for every $t \geqq t_0$.

Proof. By Theorem 2 there exists $t_0 \geqq t_y$ so that (21), (22) hold. Suppose there exists $s \geqq t_0$ such that $y(s) > Y_k(s)$. Then by (21), (24) y is o.f.e.t. which is a contradiction to Theorem 1.

Now if, for some $s > t_0$, $L_{n-1}y(s) < Q_k(s)$, by (25) we must have $L_{n-1}y(s_1) \leqq c$, for some $s_1 \geqq s$. $L_{n-1}y$ is, however, a decreasing function and $\lim_{t \to \infty} L_{n-1}y(t) = c$. Therefore $L_{n-1}y(s_1) > c$. This contradiction completes the proof.

Theorem 4. Let k be an integer, $0 < k < n$, $n + k$ be odd. Then $N_k = \emptyset$ if any one of the following condition is valid:

(i) for each $a > 0$ (11 k) is o.e.e.t.,

(ii) for each $a > 0$ and $z_0 \neq 0$ there exists $t \geqq a$ so that $z_k(t, a, z_0) = 0$,

(iii) for each $a > 0$ and $u_0 > 0$, $u_k(t, a, \pm u_0)$ exists on $\langle a, \infty)$ and for some $\varepsilon > 0$ it holds

$$\int_\varepsilon^\infty f(t, u_k(t, a, \pm u_0))\, dt = \pm\infty .$$

Proof. Suppose $y \in N_k$ is positive.

(i) By Theorem 2 we obtain that there exists $t_0 \geqq t_y$ for which

$$y(t) \geqq u_k(t, t_1, y(t_1)) \tag{28}$$

for each $t \geqq t_1 \geqq t_0$.

Therefore y is o.f.e.t. which is a contradiction to Theorem 1.

(ii) From Definition 2 follows that there exists $t_0 \geqq t_y$ such that

$$L_{n-1}y(t) > 0 \tag{29}$$

for each $t \geqq t_0$.

Denote $Q_k(t) = L_{n-1}y(t) + 1$. From Theorem 3 we have

$$L_{n-1}y(t) \geqq L_{n-1}y(t) + 1$$

for each $t \geqq t_0$, which is a contradiction.

(iii) We showed that there exists $t_0 \geqq t_y$ for which (28), (29) hold. From (E), (28) and (A2) we obtain

$$L_n y(t) = (L_{n-1} y(t))' \leqq -f(t, u_k(t, t_0, y(t_0))$$

for each $t \geqq t_0$. After an integration from t_0 to t we get

$$L_{n-1} y(t) \leqq L_{n-1} y(t_0) - \int_{t_0}^{t} f(s, u_k(s, t_0, y(t_0)) \, ds.$$

It implies

$$\lim_{t \to \infty} L_{n-1} y(t) = -\infty,$$

which is a contradiction to (28).

Now we shall suppose

$$f(t, x) = p(t) x$$

and we shall consider an equation

$$L_n y(t) + p(t) y(t) = 0,$$

where $n \geqq 2$.

Theorem 5. Let p be positive and continuous on $\langle 0, \infty)$ and k be an integer, $0 < k < n$, $n + k$ odd.

(i) If $y \in N_k$ there exists $a \geqq t_y$ such that

$$|y(t)| \geqq |y(a)| \exp \int_a^t H_{1k}(r, a) \, p(r) \, dr$$

for each $t \geqq a$.

(ii) $N_k = \emptyset$ if for each $a > 0$ it is

$$\int_a^\infty p(t) \exp \int_a^t H_{1k}(r, a) p(r) \, dr \, dt = \infty.$$

Proof. The equation (11 k) is

$$u' = H_{1k}(t, a) p(t, u), \qquad u(a) = u_0.$$

The solution of this equation exists on $\langle a, \infty)$ and holds

$$|u_k(t)| = |u_0| \exp \int_a^t H_{1k}(s, a) p(s) \, ds$$

and the results follow immediately from Theorem 2 and Theorem 4.

Theorem 6. Consider the equation

B(k, i) $$v' = H_{ik}(t) f\left(t + t_0, \frac{J_k(t)}{L_{i-1} J_k(t)} v\right).$$

(i) Let $n \geqq 2$ be even, $1 \leqq k \leqq n-1$, $n+k$ be odd, and let i vary one the odd integers between 1 and k inclusive. If at least on of the $(k+1)/2$ equations B(k, i) is o.f.e.t. for every $t_0 \geqq 0$, then $N_k = \emptyset$.

(ii) Let $n \geqq 3$ be odd, $2 \leqq k \leqq n-1$, $n+k$ be odd, and let i vary one the even integers between 2 and k inclusive. If at least one of the $k/2$ equations B(k, i) is o.f.e.t. for every $t_0 \geqq 0$, then $N_k = \emptyset$.

Proof. Let $y \in N_k$ be positive for every $t \geqq t_0 \geqq t_y$. Denote $s = t - t_0$ and $x(s) = y(t)$, and conclude that $x(s)$ satisfies

$$L_n x(s) + f(s + t_0, x(s)) = 0.$$

Rewriting t and y for s and x, with $1 \leqq i \leqq k$ being such that $i + n$ is odd, by Lemma 5 we obtain

$$L_{i-1} y(t) \geqq L_{i-1} y(0) + \int_0^t H_{ik}(r) f(r + t_0, y(r))\, dr.$$

From Lemma 1 we get

$$y(t) \geqq \frac{L_{i-1} y(t)}{L_{i-1} J_k(t)} J_k(t), \quad \text{for every} \quad t \geqq 0.$$

It follows that

$$L_{i-1} y(t) \geqq L_{i-1} y(0) + \int_0^t H_{ik}(r) f\left(r + t_0, \frac{J_k(r)}{L_{i-1} J_k(r)} L_{i-1} y(r)\right) dr,$$

for every $t \geqq 0$. By Lemma 3 we have

$$L_{i-1} y(t) \geqq s(t),$$

where s is a minimal solution of B(k, i) with the initial condition

$$v_0 = L_{i-1} y(0).$$

Let B(k, i) is o.f.e.t. for every $t_0 \geqq 0$. There exists $t_1 \in (0, \infty)$ such that

$$\lim_{t \to t_1} L_{i-1} y(t) = \infty$$

and it is a contradiction with Theorem 1.

Now we shall define a property (A) and a minimal and a maximal solution in N_k of (E). We shall form conditions which ensure that (E) has property (A). We shall need the following lemmas, which appear in [3], and give conditions which are necessary and sufficient for the existence of the maximal and the minimal solutions of order k for (E).

Definition 3. We say that (E) has property (A) iff

(i) $N_i = \emptyset$ for every i, $1 \leqq i \leqq n$,

(ii) $y \in N_0$ implies $\lim_{t \to \infty} y(t) = 0$.

Remark 3. If n is even then (E) is oscillatory iff (E) has the property (A).

Definition 4. Consider N_k for $0 < k < n$. For any $y \in N_k$ the limits

$$\lim_{t \to \infty} L_k y(t) = c_k \qquad \text{(finite)}$$

$$\lim_{t \to \infty} L_{k-1} y(t) = c_{k-1} \qquad \text{(finite or infinite but not zero)}$$

exist (Lemma 1). A $y \in N_k$ is called a maximal solution in N_k if c_k is nonzero and a minimal solution in N_k if c_{k-1} is finite. A maximal solution y in N_0 is a solution in N_0 such that

$$\lim_{t \to \infty} L_0 y(t) = c_0 \qquad (\neq 0, \text{ finite}).$$

Lemma 6. Let $0 \leqq k < n$. A necessary and sufficient condition for (E) to have a maximal solution in N_k is that

$$\int_a^\infty K_{n-k-1}(t) \, |f(t, cA_k(t))| \, \mathrm{d}t < \infty$$

for some a and $c \neq 0$.

Lemma 7. Let $0 < k < n$. A necessary and sufficient condition for (E) to have a minimal solution in N_k is that

$$\int_a^\infty K_{n-k}(t) \, |f(t, cA_{k-1}(t))| \, \mathrm{d}t < \infty$$

for some a and $c \neq 0$.

Theorem 7. Let the equations

$$\text{(30 k)} \qquad v' = H_{1k}(t) f(t + t_0, v),$$

$$\text{(31)} \qquad v' = H_{11}(t) f(t + t_0, v),$$

are o.f.e.t. for every $t_0 \geqq 0$ and $1 < k < n$, $n + k$ odd. Then (E) has the property (A).

Proof. Let n be an even integer and let $0 < k < n$ be odd. The equation $B(k, 1)$ is equivalent to (30 k), therefore B(k, 1) is o.f.e.t. for every $t_0 \geqq 0$. By Theorem 6 we get that (E) has the property (A).

Let n be an odd integer and let $0 < k < n$ be even. Let $y \in N_k$ be positive. By Lemma 5 we have

$$y(t) > y(0) + \int_0^t H_{1k}(r) f(r + t_0, y(r))\, dr$$

and by Lemma 3 we get that y has finite escape time which is in contradiction to Theorem 1.

Let $k = 0$. We now wish to show that $\lim_{t \to \infty} y(t) = 0$ for every $y \in N_0$. Assume that $y \in N_0$ is positive and $\lim_{t \to \infty} y(t) \neq 0$. It implies y is a maximal solution in N_0. By Lemma 6 there exists some a and $c \neq 0$ such that

$$\int_a^\infty K_{n-1}(t) |f(t + t_0, c)|\, dt < \infty.$$

We now claim that for every $c \neq 0$

$$(\operatorname{sgn} c) \int_0^\infty K_{n-1}(t) f(t + t_0, c)\, dt = \infty. \tag{32}$$

Let us suppose there is a $c > 0$ satisfying

$$\int_0^\infty K_{n-1}(t) f(t + t_0, c)\, dt < \infty$$

and consider an equation

$$u(t) = c - \int_t^\infty K_{n-1}(r) f(r + t_0, u(r))\, dr.$$

Let us choose $a > 0$ such that

$$\int_a^\infty K_{n-1}(t) f(t + t_0, c)\, dt < c.$$

Define the sequence

$$u_1(t) = c,$$

$$u_{m+1}(t) = c - \int_t^\infty K_{n-1}(r) f(r + t_0, u_m(r))\, dr$$

for every $t \geqq a$. The sequence is well defined, since it follows by induction that $0 \leqq u_m(t) \leqq c$ for every $m \geqq 1$ and $t \geqq a$. It is possible to show that there exists two functions w_1 and w_2, such that

$$w_1(t) = \lim_{n \to \infty} u_{2n}(t),$$

$$w_2(t) = \lim_{n \to \infty} u_{2n+1}(t),$$

where $w_2(t) \geqq w_1(t)$ for every $t \geqq a$ and both w_1 and w_2 are monotonic nondecreasing and bounded above by c. By the Monotonic Convergence Theorem it follows that

$$\begin{aligned} w_1(t) &= c - \int_t^\infty K_{n-1}(r) f(r + t_0, w_2(r))\, \mathrm{d}r, \\ w_2(t) &= c - \int_t^\infty K_{n-1}(r) f(r + t_0, w_1(r))\, \mathrm{d}r. \end{aligned} \tag{33}$$

By differentiating the first equation of (33) we obtain

$$w_1'(t) \geqq K_{n-1}(t) f(t + t_0, w_1(t)).$$

$K_{n-1}(t) = H_{11}(t)$ for every $t \geqq 0$. Thus w_1 is bounded from below by a function which is o.f.e.t., being a solution of (33). This is a contradiction to the boundedness of w_1. The existence of a $c < 0$ which contradicts (32) leads to a similar contradiction. Thus we have proved that any solution of order $k = 0$ tends monotonically to zero and it implies (E) has property (A).

Theorem 8. Let g be a positive and continuous function defined on $R \setminus \{0\}$ such that

(i) $g(x) f(t, x)$ is monotonic nondecreasing for any $x \neq 0$,

(ii) $\int_\varepsilon^\infty g(x)\, \mathrm{d}x < \infty$ and $\int_{-\infty}^{-\varepsilon} g(x)\, \mathrm{d}x < \infty$ for some $\varepsilon > 0$.

Then:

1. Let $n \geqq 2$ be an integer, then (E) has property (A) if

$$(\operatorname{sgn} c) \int_0^\infty H_{1k}(t) f(t, c)\, \mathrm{d}t = \infty \tag{34}$$

for every $c \neq 0$, $0 < k < n$, $n + k$ odd and for $k = 1$.

2. The condition (34) holds for every $c \neq 0$, $k = 1$, if (E) has property (A).

Proof.

1. Let (34) be satisfied for every $c \neq 0$, $0 < k < n$, $n + k$ odd and for $k = 1$. We show that equations (30 k), (31) are o.f.e.t. for every $t_0 \geqq 0$, $0 < k < n$, $n + k$ odd and for $k = 1$, and the results follow by Theorem 7. In order to show this, we multiply (30 k), (31) by v, and recalling the properties of f we conclude that $|v(t)| \geqq |v(0)|$ for every $t \geqq 0$. Define

$$G(v) = 1 + \int_{v(0)}^{v} g(t)\, \mathrm{d}t,$$

hence, G is monotonic increasing and $\lim_{v \to \infty} G(v) = G(\infty) < \infty$. Assuming $v(0) > 0$, multiplying (30 k), (31) by g, integrating from 0 to t, and using the properties of g yields

$$G(v(t)) \geqq 1 + g(v(0)) \int_0^t H_{1k}(r) f(r + t_0, v(0))\, \mathrm{d}r. \tag{35}$$

As $t \to \infty$, the right hand side of (35) diverges to $+\infty$, so there exists $0 \leqq t_1 < \infty$ for which $G(v(t_1)) = G(\infty)$. Since G is monotonic, it follows that $\lim_{t \to t_1} v(t) = \infty$. A similar conclusion follows by assuming $v(0) < 0$. Now consider the case $v(0) = 0$. If v is not identically zero, then there exists $t_2 > 0$ such that $v(t_2) \neq 0$. After the transformation $s = t - t_2$, it is seen that v is bounded from below by a solution with nonzero initial values, which has finite escape time. It implies that v has finite escape time.

2. Let there exists $c \neq 0$ such that (34) be not satisfied for $k = 1$. From Lemma 7 it follows that there exists a minimal solution in N_1, therefore (E) has not the property (A) and that completes the proof.

Now we shall assume that $a_i(t) = 1$, $0 \leqq i \leqq n$, $t \geqq 0$. Then equation B(k, i) is

$$v' = c_{ik} t^{n-i} f(t + t_0, t^{i-1} v),$$

where

$$c_{ik} = \frac{(k - i + 1)}{k\,!\,(n - k - 1)!\,(n - i)}.$$

Theorem 9. Let B(n − 1, 1) be o.f.e.t. Then the equation

$$y^{(n)}(t) + f(t, y(t)) = 0 \tag{E1}$$

has property (A).

Proof. Let i be an integer, $0 < i < n$. Then

$$c_{1n-1} \leqq c_{1i}.$$

Therefore B(i, 1) is o.f.e.t. for every $t_0 \geqq 0$. Theorem 6 implies that (E1) has property (A).

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SÚHRN

KRITÉRIÁ OSCILÁCIE A RAST NEOSCILATORICKÝCH RIEŠENÍ NELINEÁRNYCH DIFERENCIÁLNYCH ROVNÍC

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V práci sa skúma nelineárna, homogénna diferenciálna rovnica typu $L_n y + f(t, y) = 0$, kde L_n je diskonjugovaný lineárny operátor na intervale $\langle 0, \infty)$ a $yf(t, y) > 0$. K danej rovnici priradíme systém obyčajných diferenciálnych rovníc prvého rádu a z vlastností tohto systému určujeme správanie sa riešení danej rovnice.

РЕЗЮМЕ

КРИТЕРИЯ ДЛЯ ОСЦИЛЯЦИИ И РОСТ НЕОСЦИЛЯЦИОННЫХ РЕШЕНИЙ НЕЛИНЕЙНЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

Д. СУКЕНИК, Братислава

В работе исследуется нелинейное дифференциальное однородное уравнение вида $L_n y + f(t, y) = 0$, где L_n неосцилляционный линейный оператор на $\langle 0, \infty)$ и $yf(t, y) > 0$. К этому уравнению сопоставляется система обыкновенных дифференциальных уравнений первого порядка и из поведения этой системы мы определяем поведение решений данного уравнения.