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SUMS OF THE FORM $\frac{1}{N}\sum_{n=1}^{N} f_n(nx)$ AND UNIFORM DISTRIBUTION mod 1

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1 Introduction

In the paper a theorem on the behavior of sums of the form $\frac{1}{N}\sum_{n=1}^{N} f_n(nx)$ (by $N \to \infty$) is proved. This theorem is a generalization of a result from [1] (p. 123—124). Some applications of this generalization are given.

In [2] (p. 96, Exercise 169, Solution on p. 275) the following problem of E. Steinitz is formulated:

Determine for real x the function f,

$$f(x) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} (\cos n\pi x)^{2n}$$

It is shown there that f is the so called Riemann function, i.e. $f(x) = 0$ for x irrational and $f(x) = \frac{1}{q}$ for $x = \frac{p}{q}$ (canonical form of the rational number x). The solution of the mentioned problem comes from G. Pólya.

In [1] (p. 123—124) a general result (see Theorem A in what follows) concerning sums of the mentioned type is introduced. The solution of the mentioned problem of E. Steinitz can be obtained also as a consequence of Theorem A.

Theorem A. Let f_n: $R \to R$ $(n = 1, 2, \ldots)$ be periodic functions with the period 1, let $f_n | [0, 1]$ $(n = 1, 2, \ldots)$ be Riemann integrable functions. Suppose that the following conditions are satisfied:

a) There exists a $M > 0$ such that for each $x \in R$ and each $n = 1, 2, \ldots$ we have $0 \leq f_n(x) \leq M$

b) For each ε, $0 < \varepsilon < \frac{1}{2}$, each $n = 1, 2, \ldots$ and $x \in [\varepsilon, 1 - \varepsilon]$ we have $f_n(x) \leqq f_n(\varepsilon)$ and

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f_n(\varepsilon) = 0$$

Then for each uniformly distributed mod 1 sequence $\omega = (\omega(n))_{n=1}^{\infty}$ we have

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f_n(\omega(n)) = 0$$

In the second part of this paper we shall give a generalization of Theorem A.

2 The main result

The following theorem is a generalization of Theorem A. In what follows $\mu(M)$ denotes the Jordan measure of the set M.

Theorem 1. Let f_n: $R \to R$ ($n = 1, 2, \ldots$) be non-negative periodic functions with the period 1. Suppose that

a) there exists an $M > 0$ such that for each $n = 1, 2, \ldots$ and $x \in R$ we have $f_n(x) \leqq M$;

b) for each $\varepsilon > 0$ there exists a set $H_\varepsilon \subset [0, 1)$ consisting of a finite number of non-overlapping intervals such that $\mu(H_\varepsilon) \leqq \varepsilon$ and

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} M_n^\varepsilon = 0 \tag{1}$$

where

$$M_n^\varepsilon = \sup_{x \in [0, 1) \setminus H_\varepsilon} f_n(x) \qquad (n = 1, 2, \ldots)$$

Then for each uniformly distributed mod 1 sequence $\omega = (\omega(n))_{n=1}^{\infty}$ we have

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f_n(\omega(n)) = 0$$

Proof. Put for brevity

$$S_N = \sum_{n=1}^{N} f_n(\omega(n)) = S_N^{(1)} + S_N^{(2)} \tag{2}$$

where

$$S_N^{(1)} = \sum_{n \leqq N;\, \{\omega(n)\} \in H_\varepsilon} f_n(\omega(n)),$$

$$S_N^{(2)} = \sum_{n \leqq N;\ \{\omega(n)\} \in [0,\,1) \setminus H_\varepsilon} f_n(\omega(n))$$

($\{t\}$ denotes the fractional part of the real number t).

Denote by $A(H; N, \omega)$ the number of all $n \leqq N$ such that $\{\omega(n)\} \in \in H (H \subset [0, 1))$. Let $\varepsilon > 0$. Then a simple estimation gives

(3) $$S_N^{(1)} \leqq MA(H_\varepsilon; N, \omega)$$

Since ω is uniformly distributed mod 1, we have

(4) $$\limsup_{N \to \infty} \frac{A(H_\varepsilon; N, \omega)}{N} \leqq \varepsilon$$

It follows from (3) and (4) that

(5) $$\limsup_{N \to \infty} \frac{S_N^{(1)}}{N} \leqq M\varepsilon$$

Further, we have evidently

$$S_N^{(2)} \leqq \sum_{n=1}^{N} M_n^\varepsilon$$

and therefore on account of (1)

(6) $$\lim_{N \to \infty} \frac{S_N^{(2)}}{N} = 0$$

According to (5), (6) we get from (2)

$$\limsup_{N \to \infty} \frac{S_N}{N} \leqq M\varepsilon$$

Since ε is an arbitrary positive number, the theorem follows.

In the following example we shall illustrate the usefulness of Theorem 1.

Example 1. Put

$$f_n(x) = \begin{cases} \{|\operatorname{tg} \pi x|\}^n & \text{if } x \neq \dfrac{2a+1}{2},\ a \in Z \\ 0 & \text{if } x = \dfrac{2a+1}{2} \text{ for a suitable } a \in Z \end{cases}$$

(Z is the set of all integers).

The function f_n $(n = 1, 2, \ldots)$ is evidently non-negative and periodic with the period 1.

We shall show that each function f_n (n = 1, 2, ...) has an infinite number of discontinuity points in the interval [0, 1).

The function g, $g(x) = |\text{tg}\, \pi x| \left(x \in [0, 1) \setminus \left\{ \frac{1}{2} \right\} \right)$ is increasing on $\left[0, \frac{1}{2}\right)$ and

$$\lim_{x \to \frac{1}{2}-} g(x) = +\infty, \qquad \lim_{x \to 0+} g(x) = g(0) = 0$$

Denote by x_k such a point from $\left[0, \frac{1}{2}\right)$ for which $g(x_k) = k$ $(k = 1, 2, \ldots)$. Then we have evidently

$$\text{(7)} \qquad 0 < x_1 < x_2 < \ldots, \quad \lim_{k \to \infty} x_k = \frac{1}{2}$$

Further, $1 - x_k \in \left(\frac{1}{2}, 1 \right)$ and

$$\text{(8)} \qquad 1 - x_1 > 1 - x_2 > \ldots, \quad \lim_{k \to \infty} (1 - x_k) = \frac{1}{2},$$

$g(1 - x_k) = k \qquad (k = 1, 2, \ldots)$.

It is easy to see that the set of discontinuity points of f_n $(n = 1, 2, \ldots)$ in [0, 1) coincides with the set

$$B = \left\{ \frac{1}{2}, x_1, 1 - x_1, x_2, 1 - x_2, \ldots, x_k, 1 - x_k, \ldots \right\}.$$

Let x be an irrational number, $x \in [0, 1)$. Let $\varepsilon > 0$. Choose an $\eta > 0$ such that the numbers $\frac{1}{2} - \frac{\eta}{2}, \frac{1}{2} + \frac{\eta}{2}$ do not belong to B and

$$\eta < \frac{\varepsilon}{2} \tag{9}$$

On account of (7), (8) only a finite number of elements of the set B lie outside the interval $J = \left(\frac{1}{2} - \frac{\eta}{2}, \frac{1}{2} + \frac{\eta}{2} \right)$. Denote by $x_1, x_2, \ldots, x_m, 1 - x_1, 1 - x_2, \ldots, 1 - x_m$ the elements of B lying outside the interval J. Let us construct pairwise disjoint open intervals I_k, I'_k $(k = 1, 2, \ldots, m)$ such that $x_k \in I_k$, $1 - x_k \in I'_k$ $(k = 1, 2, \ldots, m)$, $I_k \cap J = I'_k \cap J = \emptyset$ $(k = 1, 2, \ldots, m)$ and moreover, if η_k and η'_k denotes the lenght of I_k and I'_k respectively, then

$$\text{(10)} \qquad \sum_{k=1}^{m} (\eta_k + \eta'_k) < \frac{\varepsilon}{2}$$

Put

$$H_\varepsilon = J \cup \bigcup_{k=1}^{m} (I_k \cup I'_k)$$

Let us consider that $\lim_{r \to x_1 -} g(x) = 1$. Therefore we can choose the left end-point a_1 of I_1 in such a way that $a_1 < x_1$ and

$$g(a_1) = \sup_{x \in [0,1) \setminus H_\varepsilon} \{g(x)\}$$

But then using the notation of Theorem 1 we get from the definition of f_n

$$M_n^\varepsilon = \sup_{x \in [0,1) \setminus H_\varepsilon} f_n(x) = g^n(a_1) = |\text{tg}\, \pi a_1|^n \to 0 \qquad (n \to \infty)$$

Hence the condition (1) in Theorem 1 is satisfied.

Further, according to (9) and (10) we get

$$\mu(H_\varepsilon) = \eta + \sum_{k=1}^{m} (\eta_k + \eta'_k) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus the assumption b) in Theorem 1 holds. Also the assumption a) holds (it suffices to put $M = 1$).

Since the sequence $(nx)_{n=1}^{\infty}$ is for x irrational uniformly distributed mod 1 ([1], p. 10), we get according to Theorem 1 (for x irrational)

$$F(x) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f_n(nx) = 0$$

Now let x be a rational number, $x \in [0, 1)$, $x = \frac{p}{q}$ (the canonical form of x, i.e. $q > 0$, $(p, q) = 1$). Then for each positive integer n there exist integers m, r such that

$$np = mq + r, \qquad 0 \leqq r < q$$

Using a simple estimation we get

(11)
$$S_N = \sum_{n=1}^{N} f_n\left(n\frac{p}{q}\right) = \sum_{r=1}^{q-1} \sum_{n \leqq N;\, np \equiv r (\text{mod}\, q)} f_n\left(\frac{r}{q}\right) \leqq$$
$$\leqq (q-1) \sum_{n=1}^{\infty} \max\left\{f_n\left(\frac{1}{q}\right), f_n\left(\frac{2}{q}\right), \ldots, f_n\left(\frac{q-1}{q}\right)\right\}$$

It follows from the definition of the function f_n that there exists a $d \in [0, 1)$ such that for each $n = 1, 2, \ldots$ we have

$$\max\left\{f_n\left(\frac{1}{q}\right), f_n\left(\frac{2}{q}\right), \ldots, f_n\left(\frac{q-1}{q}\right)\right\} = \{g(d)\}^n$$

Since $0 \leqq \{g(d)\} < 1$, the series on the right — hand side of (11) converges and so

$$\lim_{N\to\infty} \frac{S_N}{N} = 0,$$

$$F\left(\frac{p}{q}\right) = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f_n\left(n\frac{p}{q}\right) = \lim_{N\to\infty} \frac{S_N}{N} = 0$$

Hence the sequence $(F_N)_{N=1}^{\infty}$,

$$F_N(x) = \frac{1}{N}\sum_{n=1}^{N} f_n(nx) \qquad (N = 1, 2, \ldots),$$

pointwise converges to the function F which is identically equal to 0.

3 An analogue of the problem of E. Steinitz

In connection with the problem of E. Steinitz mentioned at the beginning of this paper the following question arises:

Determine the behavior of sums

$$\frac{1}{N}\sum_{n=1}^{N} (\sin n\pi x)^{2n} \qquad (N = 1, 2, \ldots)$$

by $N \to \infty$.

The following theorem gives the answer to the foregoing question.

Theorem 2. The sequence $(F_N)_{N=1}^{\infty}$,

$$F_N(x) = \frac{1}{N}\sum_{n=1}^{N} (\sin n\pi x)^{2n} \qquad (x \in R,\ N = 1, 2, \ldots)$$

converges pointwise on R to the function F, where $F(x) = 0$ if x is irrational or x is a rational number, $x = \frac{p}{q}$ (the canonical form), where q is an odd number, further $F(x) = \frac{1}{q}$, if x is a rational number, $x = \frac{p}{q}$ (the canonical form), where q is an even number.

Proof. Let $0 < \varepsilon < \frac{1}{2}$. By the notation used in Theorem 1 we put

$H_\varepsilon = \left(\frac{1}{2} - \frac{\varepsilon}{2}, \frac{1}{2} + \frac{\varepsilon}{2}\right)$. Then $\mu(H_\varepsilon) = \varepsilon$ and

$$M_n^\varepsilon = \sup_{x \in [0, 1) \setminus H_\varepsilon} (\sin \pi x)^{2n} = \left(\sin \pi \left(\frac{1}{2} - \frac{\varepsilon}{2}\right)\right)^{2n} \qquad (n = 1, 2, \ldots).$$

Hence $\lim_{N \to \infty} M_n^\varepsilon = 0$ and (1) holds. Thus the condition b) in Theorem 1 is satisfied.

Also the condition a) is satisfied (it suffices to put $M = 1$).

Since the sequence $(nx)_{n=1}^\infty$ for x irrational is uniformly distributed mod 1, according to Theorem 1 we have

$$F(x) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} (\sin n\pi x)^{2n} = 0$$

for x irrational.

Let x be a rational number, $x \in [0, 1)$, $x = \frac{p}{q}$ (the canonical form) then for each $n = 1, 2, \ldots$ there exist integers m, r such that

$$np = mq + r, \qquad 0 \leqq r < q.$$

Then we have

$$\left(\sin n\pi \frac{p}{q}\right)^{2n} = \left(\sin \pi \frac{r}{q}\right)^{2n}$$

and so we get

$$S_N = \sum_{n=1}^{N} \left(\sin n\pi \frac{p}{q}\right)^{2n} = \sum_{r=1}^{q-1} \sum_{n \leqq N;\, np \equiv r (\mathrm{mod}\, q)} \left(\sin \pi \frac{r}{q}\right)^{2n}$$

We have the following two possibilities:

(a) The number q (denominator of x) is even,

(b) The number q is odd.

(a) In this case the number r (depending on n) takes on the value $\frac{q}{2}$, too. For $r = \frac{q}{2}$ we have $\sin \pi \frac{r}{q} = 1$.

Put

$$S_N^{(1)} = \sum_{n \leqq N;\, np \equiv \frac{q}{2} (\mathrm{mod}\, q)} \left(\sin \pi \frac{\left(\frac{q}{2}\right)}{q}\right)^{2n} = \sum_{n \leqq N;\, np \equiv \frac{q}{2} (\mathrm{mod}\, q)} 1,$$

$$S_N^{(2)} = \sum_{n \leqq N;\, np \equiv r (\mathrm{mod}\, q)} \left(\sin \pi \frac{r}{q}\right)^{2n}, \qquad r \neq \frac{q}{2},\ 0 \leqq r < q$$

Then we have

(12) $$S_N = S_N^{(1)} + S_N^{(2)}$$

All positive integers np, $1 \leqq n \leqq N$, can be partitioned into blocks of the form

(13) $$(1 + kq)p,\ (2 + kq)p,\ \ldots,\ (q + kq)p$$

(the last block need not be complete), here k is a non-negative integer. Since $(p, q) = 1$, the block (13) is a complete residue system (mod q). Therefore in each block of the form (13) (k is fixed) lies exactly one number which is congruent to $\frac{q}{2}$ (mod q). The number of all blocks (13) is equal to $k_0 + 1$, where k_0 is the greatest integer with

$$(q + k_0 q)p \leqq Np$$

Hence $1 + k_0 = \left[\frac{N}{q}\right]$. Since the last block need not be complete, we have

$$S_N^{(1)} = \left[\frac{N}{q}\right] \quad \text{or} \quad S_N^{(1)} = \left[\frac{N}{q}\right] + 1$$

In both cases we get

(14) $$\lim_{N \to \infty} \frac{S_N^{(1)}}{N} = \frac{1}{q}$$

Further, a simple estimation yields

$$S_N^{(2)} = \sum_{\substack{1 \leqq r \leqq q-1 \\ r \neq \frac{q}{2}}} \sum_{n \leqq N;\, np \equiv r (\mathrm{mod}\, q)} \left(\sin \pi \frac{r}{q}\right)^{2n} \leqq (q-2) \sum_{n=1}^{\infty} \left(\sin \pi \frac{q-1}{2q}\right)^{2n} \qquad r \neq \frac{q}{2}$$

The geometric series on the right — hand side converges because of

$$0 \leqq \left(\sin \pi \frac{q-1}{2q}\right)^2 < 1$$

Therefore

(15) $$\lim_{N \to \infty} \frac{S_N^{(2)}}{N} = 0$$

On account of (14), (15) we get from (12)

$$F\left(\frac{p}{q}\right) = \lim_{N\to\infty} \frac{S_N}{N} = \frac{1}{q}$$

(b) A simple estimation yields

$$\left(\sin \pi\frac{r}{q}\right)^{2n} \leqq \left(\sin \pi\frac{q+1}{2q}\right)^{2n}$$

for each $r = 1, 2, \ldots, q-1$. Using the previous notation we get

$$S_N \leqq (q-1) \sum_{n=1}^{\infty} \left(\sin \pi\frac{q+1}{2q}\right)^{2n} = 0(1),$$

since

$$\left(\sin \pi\frac{q+1}{2q}\right)^2 < 1.$$

Thus we get

$$F\left(\frac{p}{q}\right) = \lim_{N\to\infty} \frac{S_N}{N} = 0$$

The proof is finished.

Remark 1. The solution of the problem of Steinitz led to the Riemann function while the solution of an analogous problem given in Theorem 2 led to a function that is similar to the Riemann function. Each of these functions has an infinite number of discontinuity points. The functions F_N,

$$F_N(x) = \frac{1}{N} \sum_{n=1}^{N} f_n(nx) \qquad (N = 1, 2, \ldots)$$

$$(f_n(x) = (\cos \pi x)^{2n} \quad \text{or} \quad f_n(x) = (\sin \pi x)^{2n} \quad (n = 1, 2, \ldots))$$

are continuous on R and therefore the limit function $F = \lim_{N\to\infty} F_N$ is a function in the first Baire class. It is well — known that the set of discontinuity points of a function in the first Baire class is a set of the first Baire category (cf. [3], p. 182). Hence the set of continuity points of such function is dense in R. If we omit the assumption of continuity of the functions f_n $(n = 1, 2, \ldots)$, then the limit function

$$F = \lim_{N\to\infty} f_N$$

$\left(F_N(x) = \frac{1}{N} \sum_{n=1}^{N} f_n(nx),\ N = 1, 2, \ldots\right)$ can be discontinuous everywhere. This is shown in the following example.

Example 2. Let

(16) $$r_1, r_2, \ldots, r_n, \ldots$$

be a one — to — one sequence of all rational numbers of the interval [0, 1). Define the function f_n: $[0, 1) \to R$ ($n = 1, 2, \ldots$) in the following way:

$$f_n(x) = 0 \quad \text{for} \quad x \text{ irrational,}$$
$$f_n(r_k) = 1 \quad \text{for} \quad k \leqq n,$$
$$f_n(r_k) = 0 \quad \text{for} \quad k > n$$

We can extend the function f_n periodically (with the period 1) onto whole real line. Then f_n ($n = 1, 2, \ldots$) has only a finite number of discontinuity points in [0, 1) (it is discontinuous only at $r_1, \ldots, r_n$).
Put for $x \in R$:

$$F_N(x) = \frac{1}{N} \sum_{n=1}^{N} f_n(nx) \qquad (N = 1, 2, \ldots).$$

It follows from the foregoing that F_N has only a finite number of discontinuity points in [0, 1).

For x irrational we have evidently

$$F(x) = \lim_{N \to \infty} F_N(x) = 0$$

Let x be a rational number, $x \in [0, 1)$, $x = \frac{p}{q}$ (the canonical form). Then for each $n = 1, 2, \ldots$ we have $np = mq + r$ with integers m, r, $0 \leqq r < q$. So we get

$$f_n(nx) = f_n\left(m + \frac{r}{q}\right) = f_n\left(\frac{r}{q}\right)$$

The rational numbers

$$\frac{0}{q}, \frac{1}{q}, \ldots, \frac{q-1}{q}$$

are situated in the sequence (16) with the indices (say) $m_0, m_1, \ldots, m_{q-1}$. Put

$$m = \max\{m_0, m_1, \ldots, m_{q-1}\}$$

Then according to the definition of f_n for $n > m$ we have $f_n\left(\frac{r}{q}\right) = 1$ for each $r \in \{0, 1, \ldots, q-1\}$. But then $N - m$ summands in the sum $\sum_{n=1}^{N} f_n\left(\frac{r}{q}\right)$ are equal to 1 provided that $N > m$. Therefore

$$\frac{1}{N}\sum_{n=1}^{N} f_n\left(n\frac{p}{q}\right) \to 1 \qquad (N \to \infty)$$

and so

$$F\left(\frac{p}{q}\right) = \lim_{N\to\infty} F_N\left(\frac{p}{q}\right) = 1$$

Hence the limit function F is the well-known Dirichlet function which is discontinuous everywhere.

REFERENCES

1. Hlavka, E.: Theorie der Gleichverteilung. Bibliographisches Institut Mannheim Wien Zürich, 1979.
2. Pólya, G.—Szegő, G.: Aufgaben und Lehrsätze aus der Analysis I. (Russian translation). Nauka, Moskva 1978.
3. Sikorski, R.: Funkcje rzeczywiste I. PWN, Warszawa 1958.

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SÚHRN

SÚČTY TVARU $\frac{1}{N}\sum_{n=1}^{N} f_n(nx)$ A ROVNOMERNÉ ROZDELENIE mod 1

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V práci je dokázané jedno zovšeobecnené tvrdenie o súčtoch tvaru $\frac{1}{N}\sum_{n=1}^{N} f_n(nx)$, na základe ktorého je určená funkcia $F(x) = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f_n(nx)$, pre niektoré funkcie f_n a sú zostrojené nespojité funkcie f_n, pre ktoré F je všade nespojitá.

РЕЗЮМЕ

СУММЫ ВИДА $\frac{1}{N}\sum_{n=1}^{N} f_n(nx)$ И РАВНОМЕРНОЕ РАСПРЕДЕЛЕНИЕ mod 1

ВОЙТЕХ ЛАСЛО, Нитра—ТИБОР ШАЛАТ, Братислава

В работе содержится доказательство одного обобщенного утверждения о суммах вида $\frac{1}{N}\sum_{n=1}^{N} f_n(nx)$ на основании которого определена функция $F(x) = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f_n(nx)$ для некоторых функций f_n и построены разрывные функции f_n, для которых F всюду разрывная функция.