UNIVERSITAS COMENIANA
ACTA MATHEMATICA UNIVERSITATIS COMENIANAE
LVI—LVII

CONTINUITY PROPERTIES OF MULTIFUNCTIONS

KATARÍNA SAKÁLOVÁ, Bratislava

The quantity of generalized continuity notions caused that many attempts have been done to find a general approach to generalized continuity. We do not give the review of such studies. The aim of our note is to give some remarks and some extensions of the approach presented in [2]. We involve multivalued mappings in our considerations.

1 Preliminaries

In what follows we deal with multifunctions $F\colon X \to Y$, where $(X, \mathscr{S})$ and $(Y, \mathscr{T})$ are arbitrary topological spaces. First we introduce some useful and important families of sets, which we can define in an arbitrary topological space. In the following table we show, besides symbols and definitions of these systems, also their essential properties. Majority of them are known and others are easily provable.

Remark 1. Let X be an arbitrary topological space and $A \subset X$ any subset of X. In what follows let $\bar{A}^X$ denote the closure of a set A in X and let $\mathrm{Int}_X A$ denote the interior of A in X. If there is no misunderstanding, X will be omitted.

Remark 2. The family of all α-open sets in $(X, \mathscr{S})$ is a topology, so we use also the symbol $\mathscr{S}^\alpha$ instead of $\mathscr{Z}_9$. Similarly we use the notion $\mathscr{S}^\Theta$ instead of $\mathscr{Z}_7$.

2 Continuity properties of multifunctions

Multifunction $F\colon X \to P(Y)$ is a mapping defined on X with the values in the power set of Y, but we write $F\colon X \to Y$ for shortness. Throughout the paper, if nothing else is said, we use the symbols X, Y instead $(X, \mathscr{S})$, $(Y, \mathscr{T})$ respectively for topological spaces which are the domain or the range of F. For the well known notion of upper or lower semicontinuity of F see [6]. One can also define upper and lower semicontinuity by means of inverse images of an open set V (notation $F^+(V)$ and $F^-(V)$): F is upper (resp. lower) semi-

Table 1

Let $(X, \mathscr{S})$ be an arbitrary topological space

Symbol	Used terminology	An open set U belongs to the system if and only if	The system is invariant under
$\mathscr{X}_1$	[3]	U is complement of a compact set	arbitrary unions, finite intersections
$\mathscr{X}_2$	[5]	U is complement of a countably compact set	arbitrary unions, finite intersec.
$\mathscr{X}_3$	[2]	U is complement of a Lindelöf set	arbitrary unions, finite intersec.
$\mathscr{X}_4$	[2]	U is complement of a connected set	finite intersec.
$\mathscr{X}_5$	[2]	there exists an open set $\acute{U}$ such that the complement of U is boundary of $\acute{U}$	arbitrary unions, finite intersec.
$\mathscr{X}_6$	[2] s. of regular open sets	$U = \operatorname{Int} \bar{U}$	finite intersec.
$\mathscr{X}_7$	[2] s. of Θ-open sets	for any $x \in U$ there exists an open V such that: $x \in V \subset \bar{V} \subset U$	arbitrary unions, finite intersec.
$\mathscr{X}_8$	[2]	U is the complement of an almost compact set	finite intersec.
$\mathscr{X}_9$	system of α-open [4] sets	A set A belongs to the system if and only if $A \subset \operatorname{Int} \overline{\operatorname{Int} A}$	arbitrary unions, finite intersec.

continuous if and only if for any open set V in Y we have $F^+(V) = \{x \in X;\ F(x) \subset V\}$ (resp. $F^-(V) = \{x \in X;\ F(x) \cap V \neq \emptyset\}$) is open in X.

Definition 1A. Let X, Y be arbitrary topological spaces and let F: $X \to Y$ be a multifunction. Let P be some property of F. P is called an upper continuity property if there exist topologies $\mathscr{S}^*$ on X and $\mathscr{T}^*$ on Y such that F: $(X, \mathscr{S}^*) \to (Y, \mathscr{T}^*)$ is upper semicontinuous (we omit the prefix "semi" for shortness) if and only if F has a property P.

Definition 1B. The property P is called a lower continuity property, if there exists topologies $\mathscr{S}^*$ on X and $\mathscr{T}^*$ on Y such that F: $(X, \mathscr{S}^*) \to (Y, \mathscr{T}^*)$ is lower continuous if and only if F has the property P.

Example 1. F is upper α-continuous if for any $V \in \mathscr{T}$ we have $F^+(V) \in \mathscr{S}^\alpha$ (see [7] and Remark 2). Upper α-continuity defined in this way is an upper continuity property, because upper α-continuity of F is equivalent to upper continuity of F: $(X, \mathscr{S}^*) \to (Y, \mathscr{T}^*)$ where $\mathscr{S}^*$ is identical with the topology $\mathscr{S}^\alpha$ of all α-open sets in $(X, \mathscr{S})$ and $\mathscr{T}^*$ is identical with $\mathscr{T}$.

In many cases the corresponding property may be obtained in such a way that $\mathscr{S}^* = \mathscr{S}$ or $\mathscr{T}^* = \mathscr{T}$. If $\mathscr{S}^* = \mathscr{S}$ we call these continuity properties (because of changing only the topology $\mathscr{T}$) more precisely $\mathscr{T}$ — continuity properties. If $\mathscr{T}^* = \mathscr{T}$ we call the corresponding continuity properties φ-continuity properties.

Example 2. F is faintly upper continuous (see [2]) if for any $V \in \mathscr{T}^{\Theta}$ (see Remark 2) we have $F^+(V) \in \mathscr{S}$. Faintly upper continuity of F is an upper $\mathscr{T}$ — continuity property because faintly upper continuity is upper continuity if $\mathscr{S}$ is topology on X and $\mathscr{T}^*$ is the topology $\mathscr{T}^{\Theta}$ of all Θ-open sets in $(Y, \mathscr{T})$.

Definition 3. Let $(X, \mathscr{S})$, $(Y, \mathscr{T})$ be topological spaces and F a multivalued mapping of X to Y.

3.1. [3] F is upper (resp. lower) c-continuous if and only if for any $V \in \mathscr{T}$ such that the complement of V is compact we have $F^+(V) \in \mathscr{S}$ (resp. $F^-(V) \in \mathscr{S}$).

3.2. [5] F is upper (resp. lower) c^*-continuous if and only if for any $V \in \mathscr{T}$ such that the complement of V is countably compact we have $F^+(V) \in \mathscr{S}$ (resp. $F^-(V) \in \mathscr{S}$).

3.3. [2] F is upper (resp. lower) l-continuous (if and only if for any $V \in \mathscr{T}$ such that the complement of V is Lindelöf we have $F^+(V) \in \mathscr{S}(F^-(V) \in \mathscr{S})$).

3.4. [2] F is upper (resp. lower) s-continuous if and only if for any $V \in \mathscr{T}$ such that the complement of V is connected we have $F^+(V) \in \mathscr{S}$ (resp. $F^-(V) \in \mathscr{S}$).

3.5. [2] F is weak upper (resp. lower) continuous if and only if for any $V \in \mathscr{T}$ for which there exists $V' \in \mathscr{T}$ such that the complement of V is the boundary of V' we have $F^+(V) \in \mathscr{S}$ (resp. $F^-(V) \in \mathscr{S}$).

3.6. [2] F is almost (Singal) upper (resp. lower) continuous if and only if for any $V \in \mathscr{T}$ such that $V = \operatorname{Int} \bar{V}$ we have $F^+(V) \in \mathscr{S}$ $(F^-(V) \in \mathscr{S})$.

3.7. [2] F is faintly upper (resp. lower) continuous if and only if for any $V \in \mathscr{T}^{\Theta}$ we have $F^+(V) \in \mathscr{S}(F^-(V) \in \mathscr{S})$.

3.8. [2] F is upper (resp. lower) h-continuous if and only if for any $V \in \mathscr{T}$ such that the complement of V is almost compact we have $F^+(V) \in \mathscr{S}$ (resp. $F^-(V) \in \mathscr{S}$).

Remark 3. Any of the properties from Definition 3.1, 3.2, 3.3, 3.5, 3.7 are upper and also lower $\mathscr{T}$ — continuity properties. It similarly follows from *Table 1* and Definition 2.

The following theorem gives a sufficient condition for an arbitrary lower property P of F to be a lower $\mathscr{T}$ — continuity property.

Theorem 1. Let P be some property of F and $\mathscr{A} \subset 2^Y$ a family (of nonempty subsets of Y) which is a base such that: F has the property P if and only if for arbitrary $V \in \mathscr{A}$ we have $F^-(V) \in \mathscr{S}$. Then P is a lower $\mathscr{T}$ — continuity property.

Proof. Since the topology generated by the base $\mathscr{A}$ (notation $\mathscr{T}(\mathscr{A})$) satisfies conditions for topology $\mathscr{T}^*$ from Definition 2, it remains to show that the following equivalence is fulfilled: $F\colon (X, \mathscr{S}) \to (Y, \mathscr{T}(\mathscr{A}))$ is lower semicontinuous if and only if for arbitrary $V \in \mathscr{A}$ we have $F^-(V) \in \mathscr{S}$. It suffices to show (the converse implication is obvious) that if for arbitrary $V \in \mathscr{A}$ we have $F^-(V) \in \mathscr{S}$ then for arbitrary $V \in \mathscr{T}(\mathscr{A})$ we also have $F^-(V) \in \mathscr{S}$. Let V be arbitrary $V \in \mathscr{T}(\mathscr{A})$. Then there exists family of sets $\{V_i\}_{i \in I}$ from the base $\mathscr{A}$ such that: $\bigcup_{i \in I} V_i = V$. But for arbitrary V_i from this family we have $F^-(V_i) \in \mathscr{S}$. Since $F^-\left(\bigcup_{i \in I} V_i\right) = \bigcup_{i \in I} F^-(V_i)$, we have: $F^-(V) = \bigcup_{i \in I} F^-(V_i) \in \mathscr{S}$.

Then using *Table 1* we have the following Corollary.

Corollary. Any of lower properties from Definition 3.4, 3.6, 3.8 are lower $\mathscr{T}$ — continuity properties.

Similar theorem to Theorem 1 for upper $\mathscr{T}$ — continuity properties does not hold as the following example shows.

Example 3. Let $X = Y = \{1, 2, 3\}$ and $\mathscr{S} = \{\{1\}, \{1, 3\}, X, \emptyset\}$ be the topology on X and $\mathscr{T} = \{\{1\}, \{2\}, \{1, 2\}, Y, \emptyset\}$ be the topology on Y. Let F be defined as: $F(1) = \{3\}$, $F(2) = \{1, 2\}$, $F(3) = \{3\}$. Then F is almost upper continuous (see Definition 3.6). Indeed. The family $\mathscr{A} = \{\{1\}, \{2\}, Y, \emptyset\}$ is the family of all regular open sets in Y (see *Table 1*). For any $V \in \mathscr{A}$ we have $F^+(V) \in \mathscr{S}$. But the topology $\mathscr{T}(\mathscr{A})$ generated by the base $\mathscr{A}$ contains also the set $\{1, 2\}$. Then from the definition of F it follows that the equivalence from Definition 1 does not hold: For any $V \in \mathscr{T}$ we have $F^+(V) \in \mathscr{S}$ but at the same time there exists $V_0 = \{1, 2\} \in \mathscr{T}(\mathscr{A})$ such that $F^+(V_0) = \{2\} \notin \mathscr{S}$.

Now let us introduce some $\mathscr{S}$ — continuity properties:

Definition 4. Let P be some property of $F\colon X \to Y$. P is called a upper (resp. lower) $\mathscr{S}$ — continuity property, if there exists a topology $\mathscr{S}^*$ on X such that $F\colon (X, \mathscr{S}^*) \to (Y, \mathscr{T})$ is upper (resp. lower) continuous if and only if F has the property P.

Definition 5.1. F is upper (resp. lower) strong Θ-continuous if and only if for any $V \in \mathscr{T}$ we have $F^+(V)$ (resp. $F^-(V)) \in \mathscr{S}^\Theta$.

5.2. F is upper (resp. lower) α-continuous if for any $V \in \mathscr{T}$ we have $F^+(V)$ (resp. $F^-(V)) \in \mathscr{S}^\alpha$.

Remark 4. It follows directly from Definition 5 and *Table 1* that upper (resp. lower) strong Θ-continuity (see [2]) and upper (resp. lower) α-continuity (see [7]) are upper (resp. lower) $\mathscr{S}$ — continuity properties.

3 Non-continuity properties

Note that in case of a single valued function $f\colon X \to Y$ both the upper and lower continuity properties are identical. So we introduce the following definition.

Definition 6. Let $f: X \to Y$ be a single valued function. A property P is called a continuity property of f if there exist topologies $\mathscr{S}^*$ on X and $\mathscr{T}^*$ on Y such that $f: (X, \mathscr{S}^*) \to (Y, \mathscr{T}^*)$ is continuous if and only if f has a property P.

In this section we shall discuss some non-continuity properties i.e. the properties which are not continuity properties. To show that a property P of a multifunction F is not a continuity property, it suffices to prove only that a property P of a single valued function f is a non-continuity property.

Definition 7.1. A function $F: X \to Y$ is simply continuous if for any $V \in \mathscr{T}$ there exist $U \in \mathscr{S}$ and $R \subset X$, R nowhere dense $(\operatorname{Int} \bar{R} = \emptyset)$ such that $f^{-1}(V) = U \cup R$. (see [2]).

7.2. A function $f: X \to Y$ is rarely continuous if for any $x \in X$ and any $V \in \mathscr{T}$ with $f(x) \in V$ there are $B \subset Y$ and $U \in \mathscr{S}$ with $x \in U$ such that B is somewhere dense $(\operatorname{Int} B = \emptyset)$, $V \cap B = \emptyset$ and $f(U) \subset V \cup B$. (see [8]).

Neither simple continuity nore rare continuity are continuity properties. We show this statement by contradiction.

First we suppose that simple continuity and rare continuity are continuity properties.

i) In both cases we find some family of functions $\{f_i\}_{i \in I}$ and a function f such that we have: If for any topologies $\mathscr{S}$ on X and $\mathscr{T}$ on Y, all functions in $\{f_i\}_{i \in I}$ are continuous, then also f is continuous for these topologies.

ii) Then we find the topologies $\mathscr{S}_0$ on X and $\mathscr{T}_0$ on Y, such that all functions from $\{f_i\}_{i \in I}$ $(f_i: (X, \mathscr{S}_0) \to (Y, \mathscr{T}_0))$ are simply (resp. rarely) continuous but $f: (X, \mathscr{S}_0) \to (Y, \mathscr{T}_0)$ is not simply (resp. rarely) continuous.

From Definition 6 we obtain then topologies $\mathscr{S}^*$ on X and $\mathscr{T}^*$ on Y such that for any $i \in I$ $f_i: (X, \mathscr{S}^*) \to (Y, \mathscr{T}^*)$ is continuous if and only if $f_i: (X, \mathscr{S}_0) \to (Y, \mathscr{T}_0)$ is simply (resp. rarely) continuous. However from the above definition of $\{f_i\}_{i \in I}$ and f we have that also $f: (X, \mathscr{S}^*) \to (Y, \mathscr{T}^*)$ must be continuous and this is equivalent (Definition 6) to the fact that $f: (X, \mathscr{S}_0) \to (Y, \mathscr{T}_0)$ is simply (resp. rarely) continuous. Contradiction.

Example 4. Simple continuity:

Let $X = N$ and $Y = \{1, 2\}$. Let family $\{f_i\}_{i \in I}$ and f be defined as follows. For any $i \in N$: $f_{2i-1}(n) = 1$ if $n = 2i - 1$ and $f_{2i-1}(n) = 2$ if $n \neq 2i - 1$, $f_{2i}(n) = 1$ if $n \neq 2i$ and $f_{2i}(n) = 2$ if $n = 2i$, $f(n) = 1$ if $n = 2i - 1$ and $f(n) = 2$ for $n = 2i$ for $i = 1, 2, \ldots$

i) If $\{f_i\}_{i \in I}$ are continuous for any topology $\mathscr{S}$ on X and any topology $\mathscr{T}$ on Y, then f is continuous under the same topologies. Inded: If V is arbitrary subset of Y, then we have four posibilities:

1. $1 \notin V$ and $2 \notin V$ then $f^{-1}(\emptyset) = \emptyset \in \mathscr{S}$.
2. $1 \in V$ and $2 \in V$ then $f^{-1}(V) = N \in \mathscr{S}$.
3. $1 \in V$ and $2 \notin V$ then $f^{-1}(V)$ is the family of all odd natural numbers that is $f^{-1}(V) = \bigcup_{i=1}^{\infty} f_{2i-1}^{-1}(V)$.

4. $1 \notin V$ and $2 \in V$ then $f^{-1}(V) = \bigcup_{i=1}^{\infty} f_{2i}^{-1}(V)$ that is the family of all even natural numbers.

(ii) Let $\mathscr{S}_0$ be the family consisting of $\emptyset$ and of all subsets of N that have finite complements. Let $\mathscr{T}_0 = \{X, \emptyset, \{1\}\}$. It remains to show that all elements of family $\{f_i\}_{i \in I}$ are simply continuous functions of $(X, \mathscr{S}_0)$ into $(Y, \mathscr{T}_0)$, but $f\colon (X, \mathscr{S}_0) \to (Y, \mathscr{T}_0)$ is not simply continuous. Let us note that for arbitrary $i \in N$:

$f_{2i-1}^{-1}(\{1\}) = \{2i - 1\}$, where $\operatorname{Int}\{2i - 1\} = \emptyset$,
$f_{2i}^{-1}(\{1\}) = N - \{2i\}$, where $N - \{2i\} \in \mathscr{S}_0$

But $f^{-1}(\{1\})$ is the family of all odd natural numbers which cannot be written in the form $U \cup R = D$ where U is open (because $\operatorname{Int} D = \emptyset$) and R is nowhere dense (because $\operatorname{Int} \bar{D} = X$).

Example 5. Rare continuity.

Let $X = \{a, b\}$ and $Y = \{a, b, c\}$. Let family $\{f_i\}_{i \in I}$ consist of two functions f_1, f_2 which are defined as: $f_1(a) = a, f_1(b) = c; f_2(a) = (b), f_2(b) = a$ and f is defined as: $f(a) = b, f(b) = c$.

i) For any topologies on X and Y for which the functions f_1 and f_2 are continuous, the function f is also continuous. Indeed: $f^{-1}(\{a\}) = \emptyset$, $f^{-1}(\{b\}) = f_2^{-1}(\{b\}), f^{-1}(\{c\}) = f_1^{-1}(\{c\})$.

ii) Functions f_1, f_2 are rarely continuous with respect to the topology $\mathscr{S}_0 = \{\emptyset, X\}$ on X and $\mathscr{T}_0 = \{\emptyset, Y, \{b\}, \{c\}, \{b, c\}\}$ on Y, but $f\colon (X, \mathscr{S}_0) \to (Y, \mathscr{T}_0)$ is not rarely continuous.

First let us take f_1: Because the only nonempty open set U in X is $U = X$, it suffices to show that $f_1(U) \subset V \cup B$, where V is an arbitrary open set in Y such that $V \cap f_1(X) \neq \emptyset$ and B is somewhere dense in X. There are two possibilities: If $V = \{c\}$, then $f_1(U) = \{a, c\} = \{a\} \cup V$. However $\{a\}$ is nowhere dense in Y. If $V = \{b, c\}$, then again $f_1(U) \subset \{a\} \cup \{b, c\} = \{a\} \cup V$. Further let us take f_2: Making the same consideration as for f_1 it remains to show: If $V = \{b\}$, then $f_2(U) = \{a, b\} = \{a\} \cup V$. If $V = \{b, c\}$, then $f_2(U) \subset \{a\} \cup \{b, c\} = \{a\} \cup V$.

Let us take now f and prove that f is not rarely continuous at the point a. Because $f(a) = b$, let $V = \{b\}$ contain $f(a)$. Then for the only nonempty open $U = X$ in X we have $f(U) = \{b, c\} = \{b\} \cup \{c\} = V \cup \{c\}$. But $\{c\}$ is not a somewhere dense set in Y.

REFERENCES

1. Gauld, D. B.—Mrševic, M.—Reilly, J. L.—Vamanamurthy, M. K.: Colindelöf topologies and l-continuous functions. Univ. of Auckland, Report Series No. 188, 1982.
2. Gauld, D. B.—Mrševic, M.—Reilly, J. K.—Vamanamurthy, M. K.: Continuity properties of functions. Univ. of Auckland, Report Series No. 203, 1983.

3. Gentry, K. R.—Hoyle, H. B.: *C*-continuous functions. Yokohama Math. J. 18, 1970, 71—76.
4. Njåsted, O.: On some classes of nearly open sets. Pacific J. of Math. 15, 1965, 961—970.
5. Neubrunn, T.: *C*-continuity and closed graphs. Čas. pro pěst. mat. 110, 1985, 172—178.
6. Neubrunn, T.—Náther, O.: On characterization of quasicontinuous multifunctions, Čas. Pro pěst. mat. 107, 1982.
7. Noiri, T.: A function which preserves connected sets. Čas. pro pěst. mat. 107, 1982, 393—396.
8. Popa, V.: Sur certaine decomposition de la continuite dans les espaces topologique. Glasnik Mat. 14, 1979, 359—362.

Author's address:

Received: 20. 4. 1988

Katarína Sakálová
Katedra matematiky
Fakulta riadenia VŠE
Odbojárov 10
832 20 Bratislava

SÚHRN

SPOJITÉ VLASTNOSTI MULTIFUNKCIÍ

KATARÍNA SAKÁLOVÁ, Bratislava

Cieľom článku je uvedenie niektorých zovšeobecnení a komentár k jednému z mnohých prístupov pre zovšeobecnenú spojitosť funkcií a multifunkcií.

РЕЗЮМЕ

НЕПРЕРЫВНЫЕ СВОЙСТВА МУЛЬТИФУНКЦИЙ

КАТАРИНА САКАЛОВА, Братислава

Целью публикуемой статьи является приведение некоторых обобщений и комментария по одному из многих подходов к обобщенной непрерывности функций и мультифункций.