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ON THE EXPECTED VALUE OF VECTOR LATTICE — VALUED RANDOM VARIABLES

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The purpose of the paper is to generalize some statements which hold for real random variables and to prove these for random elements with values in partially ordered spaces. These statements were generalized for regular vector lattices by Kantorovitch and Potocký respectively (see [2] or [3]). The need for generalization stems from the fact that regular vector lattices form a very special class of vector lattices and their properties are similar to those of real numbers.

It can be shown that a vector lattice E is regular if and only if the following conditions hold:

1. E has σ-property,
2. (o) — convergence in E is equivalent to the convergence with a regulator.

It is obvious that the second mentioned condition is very restrictive. Our aim is to extend results of Kantorovitch for vector lattices, having only the first property. In other words, we are trying to generalize these results by substituing the diagonal property with much weaker assumption, namely with σ-property. Recall that this class contains all Banach vector lattices.

The first part of the paper contains some basic notions, which are used in the text. The other notions (e.g. (o) — convergence, (r) — convergence etc.) can be found in [1] or [6].

In the second part the definition of random variable with values in the vector lattice can be found. In the following part the notion of the expected value is presented, its correctness and some of its properties are proved.

Finally the Lebesgue theorem for random variables with values in a vector lattice is proved.

Definition 1.1. A vector lattice E has the σ-property if for evety sequence $(x_n)_n$ of elements from E there exists $1 > 0$, $1 \in E$, that $|x_n| \leq K_n 1$, $K_n \in N$.

From now on, let (Z, S, P) be a probability space and E be a σ-complete vector lattice with σ-property.

Definition 1.2. A sequence (f_n) of functions from Z to E converges to a function f almost uniformly if for every $\varepsilon > 0$ there exists a set $A \in S$ such that $P(A) < \varepsilon$ and (f_n) convergee relatively uniformly on $Z - A$; i.e. there exists a sequence (a_n) of real numbers converging to 0 and an element $r \in E$ such that $|f_n(z) - f(z)| \leq a_n r$ for each $z \in Z - A$.

In the following part the notion of random variable is presented.

Definition 2.1. A function $f: Z \to E$ is called a simple random variable if there exists a sequence of mutually disjoint sets E_i, $E_i \in S$, $\bigcup E_i = S$ and a sequence $(x_i)_{i=1}^n$ of elements from E, such that $f(z) = x_i$ for every $z \in E_i$.

Consider the class of simple random variables. From this class we can obtain the class of functions, which are the limits of sequenes of simple random variables. The following definition is natural.

Definition 2.2. A non-negative function $f: Z \to E$ is called a random variable if there exists a non-decreasing sequence $(f_n)_n$ of non-negative simple random variables such that (f_n) converges to f almost uniformly.

Definition 2.3. A function $f: Z \to E$ is called a random variable if there exist non-negative random variables f_1 and f_2 such that $f(z) = f_1(z) - f_2(z)$ for each z.

It can be shown that the space of random variables defined in the above manner is a vector lattice, which is closed with respect to the almost uniform convergence (see [4]).

In what follows the notion of the expected value is presented and its properties are discussed.

Definition 3.1. Let (Z, S, P) be a probability space, E be a vector lattice. A simple random variable $f: Z \to E, f = \sum_{i=1}^{n} x_i \chi_{E_i}$ $x_i \in E$, $E_i \in S$; $E_i \cap E_j = \emptyset$, $i \neq j$, $\bigcup E_i = S$ has the expected valued Ef defined as follows:

$$Ef = \sum_{i=1}^{n} x_i P(E_i)$$

It is obvious, that this definition is correct. The integral defined on the set of simple integrable functions can be viewed as a function. It is easy to show that this function is linear and monotone. Besides, the following theorem holds.

Theorem 3.1. Let $(f_n)_n$ be a decreasing sequence of simple random variables having the values in a σ-complete vector lattice E with σ-property such that $f_n \downarrow 0$ almost uniformly. Then

$$(o) - \lim Ef_n = 0$$

Proof. See [5].

Definition 3.2. A non-negative random variable f is said to be integrable if all

f_n have the expected value Ef_n and the sequence (Ef_n) is bounded in E. Then the expected value of the non-negative integrable random variable is defined as follows:

$$Ef = \int f \, \mathrm{d}P = (o) - \lim \int f_n \, \mathrm{d}P$$

The existence of limit follows-directly from the definition. We now prove the uniqueness of the definition. We shall prove that $(o) - \lim \int f_n \, \mathrm{d}P$ does not depend on the choice of the sequence (f_n) which converges to f almost uniformly.

Let (f_n) and (g_n) be two nondecreasing sequences of random variables for which the conditions of the definition 3.2. are satisfied.

Consider the random variable f_i for a fixed i. Simice $f_i \leq f$ we have $f_i - (f_i \wedge g_n) = f_i - 1/2(f_i + g_n - |f_i - g_n|)$ for each n. From this it follows that the sequence $f_i - (f_i \wedge g_n)$ almost uniformly converges to 0 as $n \to \infty$ and consequently, by Theorem 3.1. that $Ef_i = (o) - \lim E(f_i \wedge g_n) \leq (o) - \lim Eg_n$ for each i.

Definition 3.3. Let (Z, S, P) be a probability space, E be a σ-complete vector lattice with σ-property. A random variable $f: Z \to E$ is said to be integrable if there exist non-negative random variables f_1 and f_2 with expected values Ef_1 and Ef_2 such that $f(z) = f_1(z) - f_2(z)$ for each z. Then the expected value of the integrable random variable is defined as follows:

$$Ef = Ef_1 - Ef_2$$

Correctness of this definition follows from the above construction.

Remark. It is clear from the above mentioned construction that for each random variable f there exists a sequence (f_n) of simple random variables such that (f_n) converges to f almost uniformly.

It can be verified easily from the definition that if f and g are integrable random variables and c, d are any real numbers then also $cf + dg$ is an integrable random variable and

$$\int (cf + dg) \, \mathrm{d}P = c \int f \, \mathrm{d}P + d \int g \, \mathrm{d}P$$

If f and g are integrable random variables such that $f(z) \leq g(z)$, then

$$\int f \, \mathrm{d}P \leq \int g \, \mathrm{d}P$$

Theorem 3.2. If f is a random variable and if g is an integrable random variable such that $|f(z)| \leq g(z)$, then f is an integrable random variable.

Proof. Follows from Definition 3.2.

Our final aim is to prove Lebesgue theorem.

Theorem 3.3. Let $(f_n)_n$ be a sequence of integrable random variables which converges almost uniformly to a random variable f. Let g be such an integrable random variable, that for each n $|f_n| \leq g$.

Then the random variable f is integrable and

$$\int f \, \mathrm{d}P = (o) - \lim \int f_n \, \mathrm{d}P$$

Proof. The integrability of f follows from Theorem 3.2. Now, the only thing left is to show that

$$\int f \, \mathrm{d}P = (o) - \lim \int f_n \, \mathrm{d}P$$

The existence of $(o) - \lim \int f_n \, \mathrm{d}P$, where $(f_n)_n$ is the sequence of integrable functions can be shown by repeating the proof of the existence of $(o) - \lim \int f_n \, \mathrm{d}P$, where f_n were simple random variables.

We have

$$\left|\int f_n \, \mathrm{d}P - \int f \, \mathrm{d}P\right| = \left|\int_{A_j} f_n \, \mathrm{d}P - \int_{A_j} f \, \mathrm{d}P\right| + \left|\int_{A_j^c} f_n \, \mathrm{d}P - \int_{A_j^c} f \, \mathrm{d}P\right| \leqq$$

$$\leqq \int_{A_j} |f_n - f| \, \mathrm{d}P + \left|\int_{A_j^c} f_n \, \mathrm{d}P - \int_{A_j^c} f \, \mathrm{d}P\right|$$

where (A_j) is a sequence of sets from S such that $P(A_j^c) \downarrow 0$ and $(f_n)_n$ is a sequence of integrable random variables converging relatively uniformly to f on every A_j. From the above inequality it follows that

$$\int_{A_j} |f_n - f| \, \mathrm{d}P \leq a_n r P(A_j)$$

where $a_n \downarrow 0$, $r \in E$ i.e. $\int_{A_j} |f_n - f| \, \mathrm{d}P$ converges to 0.

Since

$$\left|\int_{A_j^c} (f_n - f) \, \mathrm{d}P\right| \leq \int_{A_j^c} |f_n - f| \, \mathrm{d}P \leq 2 \int_{A_j^c} g \, \mathrm{d}P$$

where g is an integrable random variable and $P(A_j^c) \downarrow 0$,

$$\left|\int_{A_j^c} (f_n - f) \, \mathrm{d}P\right|$$

converges to 0 too. Consequently

$$\int f \, \mathrm{d}P = (o) - \lim \int f_n \, \mathrm{d}P$$

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SÚHRN

STREDNÁ HODNOTA NÁHODNEJ PREMENNEJ S HODNOTAMI VO VEKTOROVOM ZVÄZE

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Článok zovšeobecňuje Kantorovičove výsledky na prípad vektorových zväzov, ktoré majú iba σ-vlastnosť.

РЕЗЮМЕ

МАТЕМАТИЧЕСКОЕ ОЖИДАНИЕ ДЛЯ СЛУЧАЙНЫХ ВЕЛИЧИН СО ЗНАЧЕНИЯМИ В ВЕКТОРНОЙ РЕШЕТКЕ

МАРТА КЕЛЕМЕНОВА, Братислава

Статья обошает результаты Канторовича на случайные величины со значениями в векторной решетке у которой только σ-свойство.