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ON A VECTOR N-POINT BOUNDARY VALUE PROBLEM

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In the paper an n-point linear boundary value problem for the n-th order nonlinear differential systems is studied. The method of solving this problem is closely related to that in the paper [4] where De la Valèe Poussin problem is investigated. The properties of the Green function for the corresponding scalar problem are here derived in details and it is shown that there exists a path of regularity for this function.

The following vector n-point boundary value problem

$$x^{(n)} = f(t, x, x', \dots, x^{(n-1)}) \tag{1}$$

$$U_i(x) = \sum_{j=1}^{n} \alpha_i^{(j)} x^{(j-1)}(a_i) = a_i, \qquad i = 1, 2, \dots, n \tag{2}$$

will be considered where

$$f \in C(D, R^d), \quad D = \langle a_1, a_n \rangle \times \underbrace{R^d \times \dots \times R^d}_{n\text{-times}}, \quad d \geq 1, \ \alpha_i^{(j)} \in R,$$

$$a_i \in R, \ A_i \in R^d, \quad i = 1, 2, \dots, n, \ j = 1, 2, \dots, n, \quad a_1 < a_2 < \dots < a_n,$$

$$\sum_{j=1}^{n} |\alpha_i^{(j)}| > 0, \qquad i = 1, 2, \dots, n.$$

I The Green function for the scalar problem

First we consider scalar differential equation of the n-th order with homogeneous boundary conditions

$$y^{(n)} = 0 \tag{1''}$$

$$U_i(y) = 0, \qquad i = 1, 2, \dots, n. \tag{2''}$$

For that problem the following lemma is true.

Lemma 1. The homogeneous boundary value problem (1″), (2″) has only the trivial solution iff the determinant

$$\Delta = \left| \sum_{j=1}^{k} \frac{(k-1)!}{(k-j)!} \alpha_i^{(j)} a_i^{k-j} \right|_{\substack{i=1,2,\dots,n \\ k=1,2,\dots,n}}$$

is different from zero.

Proof. If $y(t)$ is a solution of (1″), (2″), then it has the form $y(t) = c_0 + c_1 t + \dots + c_{n-1} t^{n-1}$, where $c_k \in R$, $k = 0, 1, \dots, n-1$, satisfy the system

$$\sum_{j=1}^{n} \alpha_i^{(j)} \left[\sum_{k=0}^{n-1} c_k t^k \right]^{(j-1)} \Bigg|_{t=a_i} = 0, \qquad i = 1, 2, \dots, n. \tag{3}$$

By an easy calculation we get

$$\sum_{k=1}^{n} c_{k-1} \sum_{j=1}^{k} \frac{(k-1)!}{(k-j)!} \alpha_i^{(j)} a_i^{k-j} = 0, \qquad i = 1, 2, \dots, n.$$

Δ is the determinant of system (3) and this system has only the trivial solution iff Δ is different from zero. Hence, the solution $y(t)$ of the problem (1″), (2″) is trivial iff $\Delta \neq 0$.

Remark 1. Troughout the paper the inequality $\Delta \neq 0$ will be assumed.

The following lemma will be derived by using Theorems 1 and 2 from [1], p. 50—52.

Lemma 2. The following statements are true.

a) For $k = 1, 2, \dots, n-1$ there exist particular Green functions $G_k = G_k(t, s)$: $\langle a_1, a_n \rangle x (a_k, a_{k+1}) \to R$ with the following properties:

1. $\dfrac{\partial^i G_k(\,.\,, s)}{\partial t^i}$, $i = 0, 1, \dots, n-2$, $k = 1, 2, \dots, n-1$, are continuous functions of variable t in $\langle a_1, a_n \rangle$ for each fixed $s \in (a_k, a_{k+1})$.

2. $\dfrac{\partial^{n-1} G_k(\,.\,, s)}{\partial t^{n-1}}$, $k = 1, 2, \dots, n-1$, is continuous in the variable t in $\langle a_1, s) \cup (s, a_n \rangle$ and at the point $t = s$, it is discontinuous with a jump of discontinuity which is equal to one, i.e.

$$\lim_{t \to s^+} \frac{\partial^{n-1}}{\partial t^{n-1}} G_k(t, s) - \lim_{t \to s^-} \frac{\partial^{n-1}}{\partial t^{n-1}} G_k(t, s) = 1, \qquad a_k < s < a_{k+1}.$$

3. The functions $G_k(\,.\,, s)$, $k = 1, 2, \dots, n-1$, are solutions of the problem (1″), (2″) in $\langle a_1, s) \cup (s, a_n \rangle$ for each $s \in (a_k, a_{k+1})$.

4. The functions $G_k(t, s)$, $k = 1, 2, \dots, n-1$ are uniquely determined by the properties 1, 2, 3.

b) The functions $\frac{\partial^i}{\partial t^i} G_k(t, s)$, $k = 1, 2, \ldots, n-1$, $i = 0, 1, \ldots, n-2$, are continuous functions of both variables t, s in the rectangle $\langle a_1, a_n \rangle x (a_k, a_{k+1})$, whereas $\frac{\partial^{n-1}}{\partial t^{n-1}} G_k(t, s)$ is continuous in the variables t, s in the triangles $a_1 \leqslant t \leqslant s$, $s \leqslant i \leqslant a_n$ for $s \in (a_k, a_{k+1})$.

c) If $r \in C(\langle a_1, a_n \rangle)$ then the function

$$y(t) = \sum_{k=1}^{n-1} \int_{a_k}^{a_{k+1}} G_k(t, s) r(s) \, \mathrm{d}s, \qquad a_1 \leqslant t \leqslant a_n,$$

is the solution of the problem

$$y^{(n)} = r(t) \tag{1''}$$

$$U_i(y) = 0, \qquad i = 1, 2, \ldots, n. \tag{2''}$$

Proof. Statement a) has been proved in [1] p. 51.

In the proof of b) we make use of the proof of a). Let $y_1, y_2, \ldots, y_n$ be the fundamental system of solutions of the equation (1″). We shall show that the Green function $G_k(t, s)$ is continuous in the variable $s \in (a_k, a_{k+1})$ for a fixed point $t \in \langle a_1, a_n \rangle$.

The function $G_k(t, s)$ has the form

$$G_k(t, s) = \begin{cases} \sum_{j=1}^{n} d_j(s) y_j(t), & a_1 \leqslant t < s \\ \sum_{j=1}^{n} b_j(s) y_j(t), & s < t \leqslant a_n \end{cases} \tag{4}$$

where $d_j(s)$, $b_j(s)$, $j = 1, 2, \ldots, n$ are for fixed $s \in (a_k, a_{k+1})$ convenient constants independent of t.

Denote

$$c_j = -d_j + b_j, \qquad j = 1, 2, \ldots, n. \tag{5}$$

by the properties 1 and 2 of the Green function $G_k(\,.\,, s)$ it follows that c_j, $j = 1, 2, \ldots, n$, are solutions of the system

$$\begin{aligned} &\sum_{j=1}^{n} c_j y_j^{(i)}(s) = 0, \qquad i = 0, 1, \ldots, n-2 \\ &\sum_{j=1}^{n} c_j y_j^{(n-1)}(s) = 1. \end{aligned} \tag{6}$$

Since the determinant of this system is the Wronskian w of the fundamental system of solutions at the point $s \in (a_k, a_{k+1})$ and this is different from zero, the functions $c_j = c_j(s)$, are uniquely determined and they are continuous, because $c_j = \dfrac{w_j(s)}{w(s)}$, where w_j is the determinant which is constructed in a usual way.

The boundary conditions

$$U_i(y) = 0, \qquad i = 1, 2, \ldots, n, \tag{2''}$$

are satisfied by $G_k(t, s)$. After some calculations we obtain the equations

$$\begin{aligned} \sum_{j=1}^{n} b_j U_i(y_j) &= \sum_{j=1}^{n} c_j U_i(y_j), & i &= 1, 2, \ldots, k, \\ \sum_{j=1}^{n} b_j U_i(y_j) &= 0, & i &= k+1, \ldots, n. \end{aligned} \tag{7}$$

Since $y_1, \ldots, y_n$ form a fundamental system of solutions and the problem (1″), (2″) has only the trivial solution, the determinant of the system (7) is different from zero and independent of s.

The coefficients $b_j, j = 1, 2, \ldots, n$ which we obtain from the system (7) depend linearly on $c_j(s), j = 1, 2, \ldots, n$ and therefore the functions $b_j, j = 1, 2, \ldots, n$ are continuous in the variable $s \in (a_k, a_{k+1})$. By the conditions (5) the continuity of coefficients $d_j(s), j = 1, 2, \ldots, n$ in the variable $s \in (a_k, a_{k+1})$ follows.

With respect to (4) the Green function G_k is continuous in variables t, s in the sets $a_1 \leqslant t \leqslant s$, $s \leqslant t \leqslant a_n$ for $s \in (a_k, a_{k+1})$ as the sum of products. With respect to property 1 this implies that G_k is continuous in the set $\langle a_1, a_n \rangle x (a_k, a_{k+1})$, $k = 1, 2, \ldots, n-1$.

The statements for $\dfrac{\partial^i}{\partial t^i} G_k(t, s)$, $i = 1, 2, \ldots, n-1$, $k = 1, \ldots, n-1$ can be proved similarly.

The statement c) has been proved in [1], theorem 2. p. 52.

Remark 2. Let $f \in C(D, R^d)$ and consider the problem (1), (2). Denote

$$\sum_{k=1}^{n-1} \int_{a_k}^{a_{k+1}} G_k(t, s) f(s, x, x', \ldots, x^{(n-1)}) \, ds =$$

$$= \left(\sum_{k=1}^{n-1} \int_{a_k}^{a_{k+1}} G_k(t, s) f_i(s, x, x', \ldots, x^{(n-1)}) \, ds \right) \qquad i = 1, 2, \ldots, d.$$

Let $\varphi(t)$, $\varphi: \langle a_1, a_n \rangle \to R^d$ be the solution of (1), (2) for $f \equiv 0$. Then according to Lemma 2 the problem (1), (2) is equivalent to the problem of finding a solution from C^{n-1} of the equation

$$x(t) = \varphi(t) + \sum_{k=1}^{n-1} \int_{a_k}^{a_{k+1}} G_k(t, s) f(s, x(s), x'(s), \ldots, x^{(n-1)}(s)) \, ds.$$

With respect to the statements 1 to 4 of lemma 2 we shall find the form of the function $G_k(t, s)$, $k = 1, 2, \ldots, n-1$. Denote

$$g_i(s) = \sum_{k=1}^{n} \frac{(-1)^{n+k}}{(n-k)!} \alpha_i^{(k)} (s - a_i)^{n-k}, \qquad i = 1, 2, \ldots, n-1$$

$\Delta_{ij} = (-1)^{i+j} D_{ij}$, where D_{ij} is the subdeterminant of Δ corresponding to the element in the i-th row and the j-th colomn.

Then the following lemma is true.

Lemma 3. The functions $G_k(t, s)$, $k = 1, 2, \ldots, n-1$ have the form

$$G_k(t, s) = \begin{cases} \displaystyle\sum_{i=1}^{k} \sum_{j=1}^{n} \frac{g_i(s)}{\Delta} \Delta_{ij} t^{j-1} - \frac{(t-s)^{n-1}}{(n-1)!}, & a_1 \leqslant t \leqslant s \\ \displaystyle\sum_{i=1}^{k} \sum_{j=1}^{n} \frac{g_i(s)}{\Delta} \Delta_{ij} t^{j-1}, & s \leqslant t \leqslant a_n \end{cases}$$

for $s \in (a_k, a_{k+1})$.

The function $\varphi(t)$ is determined by the relation

$$\varphi(t) = \frac{1}{\Delta} \sum_{i=1}^{n} \sum_{j=1}^{n} A_i \Delta_{ij} t^{j-1}, \qquad t \in \langle a_1, a_n \rangle.$$

Proof. According to (4) and with respect to the form of (1″)

$$G_k(t, s) = \begin{cases} \displaystyle\sum_{j=1}^{n} d_j t^{j-1} & a_1 \leqslant t < s \\ \displaystyle\sum_{j=1}^{n} b_j t^{j-1} & s < t \leqslant a_n \end{cases}$$

$s \in (a_k, a_{k+1})$.

As in (5) we put $c_j = b_j - d_j$, $j = 1, 2, \ldots, n$.

In the system (6) where

$$y_j(t) = t^{j-1}, \qquad j = 1, 2, \ldots, n \tag{8}$$

the Wronskian $w(s)$ of the fundamental system y_j, $j = 1, \ldots, n$ is equal to

$$w(s) = \begin{vmatrix} 1 & s & s^2 & \ldots & s^{n-1} \\ 0 & 1 & 2s & \ldots & (n-1)s^{n-2} \\ & & & \ldots & \\ & & & \ldots & \\ & & & \ldots & \\ 0 & 0 & 0 & & (n-1)! \end{vmatrix} = \prod_{k=0}^{n-1} k! \tag{9}$$

whereas the function

$$w_i(s) = \begin{vmatrix} 1 & s & \dots & s^{i-2} & 0 & s^{i+1} & \dots & s^{n-1} \\ 0 & 1 & \dots & (i-2)s^{i-3} & 0 & (i+1)s^i & \dots & (n-1)s^{n-2} \\ & & \dots & & & & & \\ & & \dots & & & & & \\ & & \dots & & & & & \\ 0 & 0 & \dots & 0 & 1 & 0 & \dots & (n-1)! \end{vmatrix} =$$

$$= (-1)^{i+n} \begin{vmatrix} 1 & s & s^2 & \dots & s^{i-2} & s^i & s^{i+1} & \dots & s^{n-1} \\ & 1 & . & . & . & . & . & . & . \\ & & 2! & . & . & . & . & . & . \\ & & & & . & . & . & . & . \\ & & & & (i-2)! & . & . & . & . \\ & & & & & i!s & . & . & . \\ & & 0 & & & i! & (i+1)!s & . & . \\ & & & & & & (i+1)! & . & . \\ & & & & & & & & (n-1)!s \end{vmatrix} =$$

$$= \frac{(-1)^{i+n}}{(n-i)!} \prod_{k=0}^{i-2} k! \prod_{k=i}^{n-1} k!s = \frac{(-1)^{i+n}}{(n-i)!} s^{n-i} \frac{\prod\limits_{k=0}^{n-1} k!}{(i-1)!}, \qquad i = 1, 2, \dots, n.$$

Then the solution $(c_1, c_2, \dots, c_n)$ of the system (6) in the special case of the equation (1″) is given by relations

$$c_j = \frac{w_j(s)}{w(s)} = \frac{(-1)^{j+n} . s^{n-j}}{(n-j)!(j-1)!}, \qquad j = 1, 2, \dots, n. \tag{10}$$

Consider the system (7).

If we denote

$$g_i = \sum_{j=1}^{n} c_j U_i(y_j)$$

where $y_j(t) = t^{j-1}$, then on the basis of (2) and (10)

$$g_i(s) = \sum_{j=1}^{n} c_j U_i(t^{j-1}) = \sum_{j=1}^{n} c_j \sum_{k=1}^{j} \alpha_i^{(k)} \frac{(j-1)!}{(j-k)!} a_i^{j-k} =$$

$$= \sum_{k=1}^{n} \alpha_i^{(k)} \sum_{j=k}^{n} \frac{(-1)^{j+n} . s^{n-j}}{(n-j)!(j-1)!} \cdot \frac{(j-1)!}{(j-k)!} a_i^{j-k} =$$

$$= \sum_{k=1}^{n} \alpha_i^{(k)} \frac{(-1)^{n+k}}{(n-k)!} \sum_{j=k}^{n} (-1)^{j-k} \frac{(n-k)!}{(n-j)!(j-k)!} s^{n-j} a_i^{j-k} =$$

$$= \sum_{k=1}^{n} \alpha_i^{(k)} \frac{(-1)^{n+k}}{(n-k)!} (s - a_i)^{n-k}. \tag{11}$$

Since $(b_1, \ldots, b_n)$ is the solution of the system (7), with respect to (2) and (8) the i-th component b_i we get as $\frac{\tilde{\Delta}_i}{\Delta}$ where the determinant Δ is determined in Lemma 1 and $\tilde{\Delta}_i$ is the determinant which we get from the determinant Δ replacing the elements of the i-th column by the column $(g_1, g_2, \ldots, g_k, 0, \ldots, 0)^T$. With respect to the meaning of Δ_{ij} it is true that

$$b_i = \sum_{j=1}^{k} g_j(s) \frac{\Delta_{ji}}{\Delta}, \qquad i = 1, 2, \ldots, n.$$

Then for $s < t \leqslant a_n$

$$G_k(t, s) = \sum_{j=1}^{n} b_j t^{j-1} = \sum_{i=1}^{k} \frac{g_i(s)}{\Delta} \sum_{j=1}^{n} \Delta_{ij} t^{j-1}.$$

Since according to (5) $d_j = b_j - c_j$, $j = 1, 2, \ldots, n$, for $a_1 \leq t < s$

$$G_k(t, s) = \sum_{j=1}^{n} b_j t^{j-1} - \sum_{j=1}^{n} c_j t^{j-1}.$$

But (10) implies that

$$\sum_{j=1}^{n} c_j t^{j-1} = \sum_{j=1}^{n} (-1)^{j+n} \frac{t^{j-1} . s^{n-j}}{(n-1)!(j-1)!} =$$

$$= \frac{(-1)^{n+1}}{(n-1)!} \sum_{j=1}^{n} (-1)^{j-1} \frac{(n-1)!}{(n-j)!(j-1)!} s^{n-j} t^{j-1} =$$

$$= \frac{(-1)^{n-1}}{(n-1)!} (s-t)^{n-1} = \frac{(t-s)^{n-1}}{(n-1)!}$$

and hence $G_k(t, s) = \sum_{i=1}^{k} \frac{g_i(s)}{\Delta} \sum_{j=1}^{n} \Delta_{ij} t^{j-1} - \frac{(t-s)^{n-1}}{(n-1)!}$ for $t \in \langle a_1, s)$, $s \in (a_k, a_{k+1})$.

The function φ is the solution of (1), (2) for $f \equiv 0$. Therefore φ has the form $\varphi = p_1 + p_2 t + \ldots + p_n t^{n-1}$ where the vectors $p_j \in R^d$ satisfy the system

$$\sum_{j=1}^{n} p_j U_i(t^{j-1}) = A_i, \qquad i = 1, 2, \ldots, n.$$

Similarly as in the scalar case we get that $p_j = \frac{\tilde{\tilde{\Delta}}_j}{\Delta}$, where $\tilde{\tilde{\Delta}}_j$ we get from the determinant Δ when $(A_1, A_2, \ldots, A_n)^T$ is in the j-th colomn.

Hence

$$p_i = \frac{1}{\Delta} \sum_{j=1}^{n} A_j \Delta_{ji}$$

and

$$\varphi(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{\Delta} A_j \Delta_{ji} t^{i-1}.$$

II Properties of the Green function

Let the function $G: \langle a_1, a_n \rangle x \bigcup_{k=1}^{n-1} (a_k, a_{k+1}) \to R$ be defined by the relation $G(t, s) = G_k(t, s)$ for $a_1 \leqslant t \leqslant a_n$, $a_k < s < a_{k+1}$. Then the following lemma is true.

Lemma 4. The function $\frac{\partial^p}{\partial t^p} G(t, s)$, $p = 0, 1, \ldots, n-2$, $t \in \langle a_1, a_n \rangle$, $s \in \bigcup_{k=1}^{n-1} (a_k, a_{k+1})$ can be continuously extended on $\langle a_1, a_n \rangle x \langle a_1, a_n \rangle$ iff $\alpha_i^{(n)} = 0$ for all $i = 2, 3, \ldots, n-1$. The function $\frac{\partial^{n-1}}{\partial t^{n-1}} G(t, s)$ can be continuously extended on the triangles $a_1 \leqslant t < s$, $s < t \leqslant a_n$ where $a_1 < s < a_n$ iff $\alpha_i^{(n)} = 0$ or $\Delta_{in} = 0$ for all $i = 2, 3, \ldots, n-1$.

Proof. The case $p = 0$. By Lemma 3 it follows that the functions $G_k(t, s) \in C(\langle a_1, a_n \rangle x (a_k, a_{k+1}))$ and they can be continuously extended on $\langle a_1, a_n \rangle x \langle a_k, a_{k+1} \rangle$, $k = 1, 2, \ldots, n-1$. The continuity of G on the line segment $s = a_i$, $i = 2, \ldots, n-1$, $a_1 \leqslant t \leqslant a_n$ will be shown if it holds

$$\lim_{s \to a_i^-} G_{i-1}(t, s) = \lim_{s \to a_i^+} G_i(t, s)$$

for all $t \in \langle a_1, a_n \rangle$.

The last equality according to Lemma 3 is equivalent to the equalities

$$\begin{aligned} &\sum_{k=1}^{i-1} \frac{g_k(a_i)}{\Delta} \sum_{j=1}^{n} \Delta_{kj} t^{j-1} - \frac{(t-a_i)^{n-1}}{(n-1)!} = \\ &= \sum_{k=1}^{i} \frac{g_k(a_i)}{\Delta} \sum_{j=1}^{n} \Delta_{kj} t^{j-1} - \frac{(t-a_i)^{n-1}}{(n-1)!} \end{aligned} \tag{12}$$

for $t \in \langle a_1, a_i)$,

$$\sum_{k=1}^{i-1} \frac{g_k(a_i)}{\Delta} \sum_{j=1}^{n} \Delta_{kj} t^{j-1} = \sum_{k=1}^{i} \frac{g_k(a_i)}{\Delta} \sum_{j=1}^{n} \Delta_{kj} t^{j-1}, \tag{13}$$

for $t \in (a_i, a_n\rangle$, and

$$\sum_{k=1}^{i-1} \frac{g_k(a_i)}{\Delta} \sum_{j=1}^{n} \Delta_{kj} a_i^{j-1} = \sum_{k=1}^{i} \frac{g_k(a_i)}{\Delta} \sum_{j=1}^{n} \Delta_{kj} a_i^{j-1} - \frac{(a_i - a_i)^{n-1}}{(n-1)!}, \tag{14}$$

for $t = a_i$.

The relations (12), (13), (14) are true iff $g_i(a_i) = \alpha_i^{(n)} = 0$, $i = 2, 3, \ldots, n-1$. Futher statements can be proved in a similar way.

A part of Lemma 1 from [3] will be stated here as Lemma 5. Its statements follow from the analyticity of the function G in the variable t as well as in the variable s.

Lemma 5. Let $p \in \{0, 1, \ldots, n-1\}$, $t_0 \in \langle a_1, a_n \rangle$, $s_0 \in \langle a_1, a_n \rangle$.

a) If $\frac{\partial^p}{\partial t^p} G(t_0, s) = 0$ for all $s \in \langle a, b \rangle \subset \langle a_1, a_n \rangle$, then for each k, $k \in \{1, 2, \ldots, n-1\}$ such that $(a_k, a_{k+1}) \cap (a, b) \neq \emptyset$, and $t_0 \notin (a_k, a_{k+1})$

$$\frac{\partial^p}{\partial t^p} G(t_0, s) = 0 \quad \text{for all} \quad s \in \langle a_k, a_{k+1} \rangle.$$

If $t_0 \in (a_k, a_{k+1})$, $(a, b) \cap (a_k, t_0) \neq \emptyset$, $((a, b) \cap (t_0, a_{k+1}) \neq \emptyset)$, then

$$\frac{\partial^p}{\partial t^p} G(t_0, s) = 0 \quad \text{for all} \quad s \in \langle a_k, t_0 \rangle$$

$$\left(\frac{\partial^p}{\partial t^p} G(t_0, s) = 0 \quad \text{for all} \quad s \in \langle t_0, a_{k+1} \rangle \right).$$

b) If $\frac{\partial^p}{\partial t^p} G(t, s_0) = 0$ for all t from a subinterval $\langle a', b' \rangle \subset \langle a_1, a_n \rangle$, then in case $(a', b') \cap (a_1, s_0) \neq \emptyset$ $((a', b') \cap (s_0, a_n) \neq \emptyset)$

$$\frac{\partial^p}{\partial t^p} G(t, s_0) = 0 \quad \text{for all} \quad t \in \langle a_1, s_0 \rangle$$

$$\left(\frac{\partial^p}{\partial t^p} G(t, s_0) = 0 \quad \text{for all} \quad t \in \langle s_0, a_n \rangle \right).$$

Further properties of the Green function are given in the following lemma.

Lemma 6. For each $p \in \{0, 1, \ldots, n-1\}$ and $i \in \{1, 2, \ldots, n\}$

$$\frac{\partial^p}{\partial t^p} G(a_i, s) \equiv 0 \quad \text{is true for all} \quad s \in \langle a_1, a_n \rangle \tag{15}$$

iff $\alpha_i^{(j)} = 0$ for all $j = 1, 2, \ldots, n$, $j \neq p+1$.

Proof. Let $p \in \{0, 1, \dots, n-1\}$, $i \in \{1, 2, \dots, n-1\}$ and (15) be true. (15) is equivalent to the equalities

$$\frac{\partial^p}{\partial t^p} G_k(a_i, s) \equiv 0 \quad \text{for all} \quad k = 1, 2, \dots, i-1, \quad s \in \langle a_k, a_{k+1} \rangle,$$

$$\frac{\partial^p}{\partial t^p} G_k(a_i, s) \equiv 0 \quad \text{for all} \quad k = i, i+1, \dots, n-1, \quad s \in \langle a_k, a_{k+1} \rangle. \tag{16}$$

According to Lemma 3 (16) are true iff

$$\frac{g_k(s)}{\Delta} \sum_{j=p+1}^{n} \frac{(j-1)!}{(j-1-p)!} a_i^{j-1-p} \Delta_{kj} = 0, \tag{17'}$$

$k = 1, 2, \dots, i-1$

$$\frac{g_i(s)}{\Delta} \sum_{j=p+1}^{n} \frac{(j-1)!}{(j-1-p)!} a_i^{j-1-p} \Delta_{ij} = (a_i - s)^{n-1-p} \cdot \frac{(n-1)!}{(n-1-p)!} \tag{18}$$

$$\frac{g_k(s)}{\Delta} \sum_{j=p+1}^{n} \frac{(j-1)!}{(j-1-p)!} a_i^{j-1-p} \Delta_{kj} = 0, \tag{17''}$$

$k = i+1, i+2, \dots, n-1$.

(18) is equivalent to the equality

$$\frac{g_i(s)}{(a_i - s)^{n-1-p}} = \text{const} \neq 0$$

and this holds iff $\alpha_i^{(p+1)} \neq 0$ and $\alpha_i^{(j)} = 0$ for $j = 1, 2, \dots, n, j \neq p+1$.

If $\alpha_i^{(j)} = 0$ for all $j = 1, 2, \dots, n, j \neq p+1$, then according to Lemma 1 the elements c_{ij} of the i-th row in the determinant Δ have the form

$$c_{ij} = \begin{cases} 0, & j = 1, 2, \dots, p \\ \dfrac{(j-1)!}{(j-1-p)!} a_i^{j-1-p} \alpha_i^{(p+1)}, & j = p+1, \dots, n. \end{cases}$$

The remaining elements of the determinant Δ are without change. Consider the determinant $\tilde{\Delta}$ which we get from the determinant Δ replacing the elements of the i-th row. Then its expansion by the k-th row has the form

$$\alpha_i^{(p+1)} \sum_{j=p+1}^{n} \frac{(j-1)!}{(j-1-p)!} a_i^{j-1-p} \Delta_{kj} = \tilde{\Delta}.$$

As the elements in the k-th and i-th row are the same, $\tilde{\Delta} = 0$. This implies that (17′), (17″) hold.

The statement for $j = n$ can be proved in a similar way.

Another approach to solve the problem (1), (2) has been taken up from [4] and is based on the following estimates for scalar functions.

There exist positive constants $C_{n,k} > 0$, $k = 0, 1, \ldots, n-1$ such that

$$|x^{(k)}(t)| \leqslant C_{n,k} \cdot \max_{a_1 \leqslant t \leqslant a_n} |x^{(n)}(t)| \tag{19}$$

for any function $x \in C^n(\langle a_1, a_n \rangle, R)$ which satisfies homogeneous boundary conditions (2″).

The meaning of the constants $C_{n,k}$, $k = 0, 1, \ldots, n-1$, is explained in the following theorem

Theorem 1. Let for all $(t, u_0, u_1, \ldots, u_{n-1})$, $(t, v_0, v_1, \ldots, v_{n-1}) \in D$ the function f satisfy the Lipschitz condition

$$|f(t, u_0, u_1, \ldots, u_{n-1}) - f(t, v_0, v_1, \ldots, v_{n-1})| \leqslant \sum_{k=0}^{n-1} L_k |u_k - v_k| \tag{20}$$

where $L_k \in M_{d \times d}$ are nonnegative matrices.

Let $C_{n,k}$, $k = 0, 1, \ldots, n-1$ be the constants from estimates (19) and let the spectral radius ϱ of the matrix $\sum_{k=0}^{n-1} L_k \cdot C_{n,k}$ satisfy

$$\varrho\left(\sum_{k=0}^{n-1} L_k C_{n,k}\right) < 1.$$

Then there exists a unique solution to (1), (2) for all $A_i \in R^d$, $i = 1, 2, \ldots, n$.

The proof of Theorem 1 can proceed in a similar way as the proof of Theorem 1 in [4].

III Admissible system of functions and associated system of constants

Futher we introduce the notion of an admissible system with respect to the Green function G analogically as in [4].

We remind that if G is the Green function for the scalar problem (1″), (2″), then the functions

$$\Phi_j(t) = \int_{a_1}^{a_n} \left| \frac{\partial^j}{\partial t^j} G(t, s) \right| \mathrm{d}s, \qquad a_1 \leqslant t \leqslant a_n, \quad j = 0, 1, \ldots, n-1 \tag{21}$$

are continuous in $\langle a_1, a_n \rangle$.

Definition 1. The system of nonnegative continuous scalar functions φ_j in $\langle a_1, a_n \rangle$, $j = 0, 1, \ldots, n-1$, is called admissible (with respect to the Green function G) if there exist positive constants k_j, $j = 0, 1, \ldots, n-1$, such that

$$\Phi_j(t) \leqslant k_j \varphi_j(t) \tag{22}$$

$a_1 \leqslant t \leqslant a_n$, $j = 0, 1, \ldots, n-1$.

With respect to the boundedness of the functions φ_j, $j = 0, 1, \ldots, n-1$, there exist positive constants $\tilde{k}_{m,j}$, $m, j = 0, 1, \ldots, n-1$, such that

$$\int_{a_1}^{a_n} \left| \frac{\partial^j}{\partial t^j} G(t, s) \right| \varphi_m(s) \, \mathrm{d}s \leqslant \tilde{k}_{m,j} \varphi_j(t), \tag{23}$$

for $a_1 \leqslant t \leqslant a_n$, $m, j = 0, 1, \ldots, n-1$.

Let $k_{m,j} = \inf_{m,j} \tilde{k}_{m,j}$, $m,j = 0,1, \ldots, n-1$. Then (23) is also true for $k_{m,j}$, $m,j = 0, 1, \ldots, n-1$.

Denote

$$K_m = \max \{k_{m,j}, j = 0, 1, \ldots, n-1\}, \qquad m = 0, 1, \ldots, n-1 \tag{24}$$

Hence

$$\int_{a_1}^{a_n} \left| \frac{\partial^j}{\partial t^j} G(t, s) \right| \varphi_m(s) \, \mathrm{d}s \leqslant K_m \varphi_j(t) \tag{25}$$

$$a_1 \leqslant t \leqslant a_n, \; m,j = 0, 1, \ldots, n-1.$$

By the definition of K_m, for constants $\tilde{K}_m < K_m$ the inequality (25) cannot hold for all $t \in \langle a_1, a_n \rangle$, $j = 0, 1, \ldots, n-1$. The constants K_m, $m = 0, 1, \ldots, n-1$, will be called the associated system of constants to the admissible system φ_j, $j = 0, 1, \ldots, n-1$. Hence, the following definition will be of use.

Definition 2. The system of the smallest constants K_m, $m = 0, 1, \ldots, n-1$, such that (25) are true for all $t \in \langle a_1, a_n \rangle$, $m, j = 0, 1, \ldots, n-1$, will be called the associated system of constants to the admissible system φ_j, $j = 0, \ldots, n-1$.

Its meaning is explained in the following theorem.

Theorem 2. Let φ_j, $j = 0, 1, \ldots, n-1$ be an admissible system and K_j, $j = 0, 1, \ldots, n-1$, the associated system of constants to such system. Let the function f satisfy Lipschitz condition (20) with nonnegative matrices $L_k \in M_{d \times d}$ $k = 0, 1, \ldots, n-1$.

Then for any $A_i \in R^d$, $i = 1, 2, \ldots, n$, there exists a unique solution to (1), (2) provided that

$$\varrho\left(\sum_{k=0}^{n-1} K_k L_k \right) < 1. \tag{26}$$

The proof of this theorem would proceed in a similar way as the proof of Theorem 2 in [4].

Corollary 1. Let the function f satisfy Lipschitz condition (20) with non-negative matrices $L_k \in M_{d \times d}$, $k = 0, 1, \ldots, n-1$. Let

$$c_{n,k} = \max_{a_1 \leqslant t \leqslant a_n} \int_{a_1}^{a_n} \left| \frac{\partial^k}{\partial t^k} G(t, s) \right| \mathrm{d}s, \qquad k = 0, 1, \ldots, n-1. \tag{27}$$

Then for any $A_i \in R^d$, $i = 1, 2, \ldots, n$, there exists a unique sulution of (1), (2) provided that

$$\varrho\left(\sum_{k=0}^{n-1} c_{n,k} L_k \right) < 1.$$

The proof is analogous to the proof of Corollary 1 in [4].

Since the functions Φ_j, determined by (21) form an admissible system of functions, the set of all admissible systems of functions is not empty and in view of Theorem 2 the problem arises what are the best (smallest) constants K_k, $k = 0, 1, \ldots, n-1$, for this set. The answer to this question can be given by applying the theory of positive linear operators. Consider the Banach space $E = C(\langle a_1, a_n \rangle, R)$ with the sup-norm, partially ordered by the relation $x \leqslant y$ iff $x(t) \leqslant y(t)$ for all $t \in \langle a_1, a_n \rangle$. Then $(E, \leqslant)$ is an ordered Banach space with positive cone

$$P = \{x \in E; \quad x(t) \geqslant 0, \; a_1 \leqslant t \leqslant a_n\}.$$

P is normal, i.e. every order interval $\langle x, y \rangle = \{z \in E; \, x \leqslant z \leqslant y\}$ is bounded, and P is reproducing, i.e. $E = P - P$.

Let $k \in \{0, 1, \ldots, n-1\}$ and let G be the Green function for the scalar problem (1″), (2″). Define the operator $A_k \colon E \to E$ by

$$A_k x(t) = \int_{a_1}^{a_n} \left| \frac{\partial^k}{\partial t^k} G(t, s) \right| x(s) \, \mathrm{d}s, \quad a_1 \leqslant t \leqslant a_n, \quad x \in E. \tag{28}$$

A_k is a positive linear operator and by using the Ascolli lemma can be easily prove that it is completely continuous.

Futher some of the properties of the operator A_k will be shown. But first we prove Lemma 7.

Lemma 7. Let $\Delta = \det(a_{ij})$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n$ be different from zero and let $j \in \{1, 2, \ldots, n\}$. Then

$$\Delta_{ij} = 0 \quad \text{for all} \quad i = 1, 2, \ldots, n-1 \tag{29}$$

iff

$$a_{nk} = 0 \quad \text{for all} \quad k = 1, 2, \ldots, n, \quad k \neq j. \tag{30}$$

Proof. It is obvious that the equalities (30) form a sufficient condition for the equalities (29).

By contradiction we show that they are also necessary. Suppose that (29) are valid and at the same time there exists a $k \neq j$ such that $a_{nk} \neq 0$. We will suppose that $k < j$. The case $k > j$ would be proved analogically.

We show that $\Delta_{nj} = 0$ which together with (29) implies that $\Delta = 0$, but this contradicts the assumption. Denote by D_{ik}^{lj} the subdeterminant of the determinant D_{lj} in the i-th row and the k-th column. Then

$$D_{nj} = \sum_{i=1}^{n-1} (-1)^{i+k} a_{ik} D_{ik}^{nj} \tag{31}$$

and for $1 \leqslant i < n$

$$D_{ij} = \sum_{m=1}^{i-1} (-1)^{m+k} a_{mk} D_{mk}^{ij} + \sum_{m=i+1}^{n} (-1)^{m+k-1} a_{mk} D_{mk}^{ij} \tag{32}$$

Choose an arbitrary number i, $1 \leqslant i < n$, and let k have the meaning from above. After multiplying the last row in Δ_{ij} by the number $\dfrac{(-1)^{n+i}}{a_{nk}}$ and the k-th column of that determinant by a_{ik} we obtain $\dfrac{(-1)^{n+i}}{a_{nk}} a_{ik} \Delta_{ij}$. Let us add to that determinant the number $(-1)^{i+k} a_{ik} D_{ik}^{nj}$. Using (32) we obtain

$$\begin{aligned}
&(-1)^{n+i} \frac{a_{ik}}{a_{nk}} \Delta_{ij} + (-1)^{i+k} a_{ik} D_{ik}^{nj} = \\
&= \sum_{m=1}^{i-1} (-1)^{n+i+m+k} \frac{a_{ik}}{a_{nk}} a_{mk} D_{mk}^{ij} + \sum_{m=i+1}^{n} (-1)^{n+i+m+k-1} \frac{a_{ik}}{a_{nk}} \cdot a_{mk} D_{mk}^{ij} + \\
&+ (-1)^{i+k} a_{ik} D_{ik}^{nj} = \sum_{m=1}^{i-1} (-1)^{n+i+m+k} \frac{a_{ik}}{a_{nk}} \cdot a_{mk} D_{mk}^{ij} + \\
&+ \sum_{m=i+1}^{n-1} (-1)^{n+i+m+k-1} \frac{a_{ik}}{a_{nk}} a_{mk} D_{mk}^{ij}
\end{aligned} \tag{33}$$

because $D_{nk}^{ij} = D_{ik}^{nj}$.

Then according to (31), (29) and (33) we obtain

$$\begin{aligned}
D_{nj} &= D_{nj} + \sum_{i=1}^{n-1} (-1)^{n-i} \frac{a_{ik}}{a_{nk}} D_{ij} = \sum_{i=1}^{n-1} \left[(-1)^{i+k} a_{ik} D_{ik}^{nj} + (-1)^{n+i} \frac{a_{ik}}{a_{nk}} D_{ij} \right] = \\
&= \sum_{i=1}^{n-1} \left[\sum_{m=1}^{i-1} (-1)^{n-i-m-k} \frac{a_{ik} a_{mk}}{a_{nk}} D_{mk}^{ij} + \sum_{m=i+1}^{n-1} (-1)^{n+i+m+k-1} \frac{a_{ik} a_{mk}}{a_{nk}} D_{mk}^{ij} \right].
\end{aligned} \tag{34}$$

Let $1 \leqslant p < r \leqslant n-1$. Then in the sum (34) there occurs just one summand if $i = p$, $m = r$, and this is $(-1)^{n+p+r+k-1} \frac{a_{pk} a_{rk}}{a_{nk}} D_{rk}^{pj}$ and just one summand for $i = r$, $m = p$ and this is $(-1)^{n+r+p+k} \frac{a_{rk} a_{pk}}{a_{nk}} D_{pk}^{rj}$. Because $D_{rk}^{pj} = D_{pk}^{rj}$, the sum

$$\frac{a_{pk} a_{rk}}{a_{nk}} [D_{rk}^{pj}(-1)^{n+p+r+k-1} + D_{pk}^{rj}(-1)^{n+r+p+k}] = 0$$

and hence $D_{nj} = 0 = (-1)^{n+j} D_{ni} = \Delta_{ni}$.

Lemma 8. Let $p \in \{0, 1, \ldots, n-1\}$ and let $x \in P$, $x \not\equiv 0$ in $\langle a_1, a_n \rangle$. Let

$$\alpha_i^{(n)} = 0, \qquad i = 1, 2, \ldots, n \tag{35}$$

and for all $k \in \{2, 3, \ldots, n-1\}$ there exist at most $k-1$ mutually different numbers $\{j_1, \ldots, j_{k-1}\} \subset \{1, 2, \ldots, n-1\}$ for which

$$\sum_{r=1}^{j_m} \frac{(j_m - 1)!}{(j_m - r)!} \alpha_i^{(r)} a_i^{j_m - r} = 0 \tag{36}$$

is true for each $m = 1, 2, \ldots, k-1$ and for each $i = k+1, k+2, \ldots, n$. Futher let for each $k \in \{2, 3, \ldots, n-1\}$ there exist at most $k-1$ mutually different numbers $\{j_1, \ldots, j_{k-1}\} \subset \{1, 2, \ldots, n-1\}$ for which (36) is true for each $m = 1, 2, \ldots, k-1$ and for each

$$i = 1, 2, \ldots, k. \tag{37}$$

Then $A_p x(t) \equiv 0$ cannot hold in any subinterval $\langle a, b \rangle \subset \langle a_1, a_n \rangle$.

Proof. Proof is analogous as the proof of Theorem 1 in [3]. To prove this statement let us suppose that there is an interval $\langle a, b \rangle \subset \langle a_1, a_n \rangle$ in which $A_p x(t) \equiv 0$. As the functions $x(s)$, $\left| \frac{\partial^p}{\partial t^p} G(t, s) \right|$ are nonnegative, it follows that for each $t_0 \in \langle a, b \rangle$ $\operatorname{supp} x(s) \subset S_{t_0} = \left\{ s \in \langle a_1, a_n \rangle; \frac{\partial^p}{\partial t^p} G(t_0, s) = 0 \right\}$.

Hence there is an interval $\langle a', b' \rangle$ such that $\frac{\partial^p}{\partial t^p} G(t_0, s) = 0$ for each $t_0 \in \langle a, b \rangle$, $s \in \langle a', b' \rangle$. In virtue of Lemma 5 there exists an interval $\langle a_k, a_{k+1} \rangle$, $k \in \{1, 2, \ldots, n-1\}$ such that

$$\frac{\partial^p}{\partial t^p} G(t, s) = 0 \quad \text{for} \quad a_k \leqslant s \leqslant a_{k+1}, \quad s \leqslant t \leqslant a_n \tag{38}$$

or

$$\frac{\partial^p}{\partial t^p} G(t, s) = 0 \quad \text{for} \quad a_k \leqslant s \leqslant a_{k+1}, \quad a_1 \leqslant t \leqslant s. \tag{39}$$

Consider only the case (38). The other one can be investigated in a similar way.

Let us choose a sufficiently small $\varepsilon > 0$. Two cases may happen. Either 1. $k > 1$ or 2. $k = 1$. We shall consider only the first case, the second one would be treated in a similar way (instead of $\langle a_k, a_k + \varepsilon \rangle$ we would have the interval $\langle a_{k+1} - \varepsilon, a_{k+1} \rangle$). (38) implies that for each function $y \in C^n(\langle a_1, a_n \rangle, R)$ satisfying the boundary conditions (2″) and such that supp $y^{(n)} \subset \langle a_k, a_k + \varepsilon \rangle$

$$|y^{(p)}(t)| \leqslant \int_{a_1}^{a_n} \left| \frac{\partial^p}{\partial t^p} G(t, s) \right| |y^{(n)}(s)| \, ds = \int_{a_k}^{a_k + \varepsilon} \left| \frac{\partial^p}{\partial t^p} G(t, s) \right| |y^{(n)}(s)| \, ds = 0,$$

for all t, $a_k + \varepsilon \leqslant t \leqslant a_n$ and hence,

$$y^{(p)}(t) \equiv 0 \quad \text{for} \quad a_k + \varepsilon \leqslant t \leqslant a_n. \tag{40}$$

Now we shall construct a function $y \in C^n(< a_1, a_n \rangle, R)$, with supp $y^{(n)} \subset \langle a_k, a_k + \varepsilon \rangle$, satisfying (2″) and for which (40) is not true. Let

$$y(t) = \begin{cases} P(t), \quad a_1 \leqslant t \leqslant a_k, \\ y(a_k) + \dfrac{y'(a_k)}{1!}(t - a_k) + \ldots + \dfrac{y^{(n-1)}(a_k)}{(n-1)!}(t - a_k)^{n-1} + \\ + \displaystyle\int_{a_k}^{t} \frac{(t - s)^{n-1}}{(n-1)!} h(s) \, ds, \quad a_k \leqslant t \leqslant a_k + \varepsilon \\ P_{n-1}(t), \quad a_k + \varepsilon \leqslant t \leqslant a_n, \end{cases} \tag{41}$$

where the polynomial $P(t)$ is such that its degree is at most $n - 1$ and $P(t)$ satisfies the boundary conditions (2″) in an interval $\langle a_1, a_k \rangle \cdot P_{n-1}(t)$ is the polynomial of degree $n - 1$ and satisfies the boundary conditions (2″) in an interval $\langle a_{k+1}, a_n \rangle$.

The function $h(t)$ will be determined in such a way that $y \in C^{n-1} \subset \subset \langle a_1, a_n \rangle, R)$. Then

$$y^{(n)}(t) = \begin{cases} 0, & a_1 \leqslant t \leqslant a_k, \\ h(t), & a_k \leqslant t \leqslant a_k + \varepsilon, \\ 0, & a_k + \varepsilon \leqslant t \leqslant a_n, \end{cases}$$

and hence, $y \in C^n(\langle a_1, a_n \rangle, R)$, supp $y^{(n)} \subset \langle a_k, a_k + \varepsilon \rangle$ and y satisfies the boundary conditions (2″). Since $y(t)$ is a polynomial of degree $n-1$ in $\langle a_k+\varepsilon, a_n\rangle$, it cannot satisfy (40) and this contradiction shows that the statement is true.

Continuity of $y(t)$ and its derivatives up to order $n-1$ at the point a_k, $a_k+\varepsilon$ can be checked in a similar way as in Theorem 1 from [3].

Now we shall show that there exists the polynomial $P_{n-1}(t)$, $a_k+\varepsilon \leqslant t \leqslant a_n$. Denote $P_{n-1}(t) = c_1 + c_2 t + \ldots + c_n t^{n-1}$. Because the polynomial P_{n-1} must satisfy homogeneous boundary conditions (2″) in an interval $\langle a_k+\varepsilon, a_n\rangle$, the coefficients c_i, $i=1, \ldots, n$, must satisfy

$$c_m \cdot \sum_{j=1}^{m} \frac{(m-1)!}{(m-j)!} \alpha_i^{(j)} a_i^{m-j} = 0, \qquad m = 1, 2, \ldots, n \qquad i = k+1, \ldots, n. \tag{42}$$

It is necessary to show that there exists a solution $(c_1, \ldots, c_n)$ of system (42) such that $c_n \neq 0$.

On the basis of (35) there exists $\alpha_i(j) \neq 0$ for some $j \in \{1, \ldots, n-1\}$ for each $i = 1, 2, \ldots, n$. The rows of the matrix system (42) are identical with the last $n-k$ rows of determinant Δ. Because $\Delta \neq 0$ any row of matrix system (42) is not zero. If some row, for example i-th, were zero but for the last element, from the form of these elements of this matrix it follows that $\alpha_i^{(j)} = 0$ for each $j = 1, 2, \ldots, n-1$, which together with (35) gives that $\alpha_i^{(j)} = 0$ for $j = 1, 2, \ldots, n$ which contradicts the following assumption $\sum_{j=1}^{n} |\alpha_i^{(j)}| > 0$.

Hence for each k we get the system of $n-k$ equations in n variables. On the basis on (36), in the matrix of the system there are at least $n-k+1$ nonzero columns and hence at least one parameter is optional. We choose c_n arbitrary but different from zero and others c_i, $i = 1, 2, \ldots, n-1$ we get as the solution of system (42) for the chosen c_n.

Assumption (37) is used if we proceed from the statement (39).

By this the proof is completed.

We also need the definition of a u_0-positive operator ([4], p. 85).

If $(E, \leqslant)$ is an ordered Banach space with positive cone P, $A\colon E \to E$ is a linear positive operator, and $u_0 \in P$, $u_0 \neq 0$, then we say that A is u_0-positive, if for each nonzero element $x \in P$ there exist constants $a(x) > 0$, $b(x) > 0$ such that

$$a(x)\, u_0 \leqslant Ax \leqslant b(x)\, u_0.$$

In other words, A is u_0-bounded from below as well as u_0-bounded from above.

The following lemma is true.

Lemma 9. Let the assumptions of Lemma 8 be satisfied. Then the operator A_k defined by (28) is Φ_k positive, where Φ_k are defined by (16).

Proof. The operator A_k is linear, positive and completely continuous and it is Φ_k-bounded from above, because for any $x \in P$, $x \neq 0$, there exists a constant $c = c(x) > 0$ such that

$$A_k x(t) = \int_{a_1}^{a_n} \left| \frac{\partial^k}{\partial t^k} G(t, s) \right| x(s) \,\mathrm{d}s \leqslant \left(\max_{a_1 \leqslant t \leqslant a_n} x(t) \right) \cdot \Phi_k(t).$$

According to [2], p. 59, A_k will be Φ_k bounded from below if to any function $x \in P$, $x \neq 0$, there exists a constant $a(x) > 0$ such that

$$A_k^2 x(t) \leqslant a(x)\, \Phi_k(t), \qquad a_1 \leqslant t \leqslant a_n, \tag{43}$$

where A_k^2 means the second iterate of A_k.

Because the assumptions of Lemma 8 are satisfied, $A_k x(t) \equiv 0$ cannot hold in any subinterval $\langle a', b' \rangle \subset \langle a_1, a_n \rangle$ and hence, putting $A_k x(t) = y(t)$, $a_1 \leqslant t \leqslant a_n$, we get that $y \in P$, $y(t) \neq 0$ on any subinterval $\langle a', b' \rangle \subset \langle a_1, a_n \rangle$ and (43) reduces to the inequality

$$A_k y(t) \leqslant a(x)\, \Phi_k(t), \qquad a_1 \leqslant t \leqslant a_n. \tag{43'}$$

In the proof (43') we utilize the properties of the Green function given in Lemmas 5 and 6. From this it follows that

$$\frac{\partial^k}{\partial t^k} G(t, s) \equiv 0 \quad \text{for all} \quad s \in \langle a_1, a_n \rangle$$

is true only in the points a_i, $i = 1, 2, \dots, n$, when $\alpha_i^{(j)} = 0$ for all $j = 1, 2, \dots, n$, and j different from $k + 1$.

It follows from this that $\dfrac{\partial^k}{\partial t^k} G(t, s) = 0$ for all $s \in \langle a_1, a_n \rangle$ $i = 1, 2, \dots, n$ cannot hold simultaneously for two or more k, $k \in \{0, 1, \dots, n - 1\}$, because then $\alpha_i^{(j)} = 0$ for all $j = 1, 2, \dots, n$ and for $i \in \{1, 2, \dots, n\}$. In view of that for arbitrary $s \in \langle a_1, a_n \rangle$ there exists $\tau(s)$, $a_i < \tau(s) < t$ such that

$$\frac{\partial^k}{\partial t^k} G(t, s) = \frac{\partial^k}{\partial t^k} G(a_i, s) + \frac{\partial^{k+1}}{\partial t^{k+1}} G(a_i, s) \frac{(t - a_i)}{1!} + \dots +$$
$$+ \dots + \frac{\partial^{n-1}}{\partial t^{n-1}} G(\tau(s), s) \frac{(t - a_i)^{n-1}}{(n-1)!} \neq 0$$

for $k \in \{0, 1, \dots, n - 1\}$ and for all a_i, $i \in \{1, 2, \dots, n\}$.

It follows from this that for a_i, $i \in \{1, 2, \dots, n\}$

$$\lim_{t \to a_i^+} A_k y(t) / \Phi_k(t) =$$

$$= \lim_{t \to a_i^+} \int_{a_1}^{a_n} \left| \frac{\partial^k}{\partial t^k} G(t, s) \right| y(s) \, ds \Big/ \int_{a_1}^{a_n} \left| \frac{\partial^k}{\partial t^k} G(t, s) \right| ds$$

exists and it is different from zero. Similarly it would be shown that $\lim_{t \to a_i^-} A_k y(t)/ /\Phi_k(t)$ there exists and it is different from zero which, in view of (43′), proves the statement of the lemma.

Futher we define a path of regularity as in 9.10 from [2] p. 72.

Definition 3. A finite sequence of points $(t_1, t_2), (t_2, t_3), \ldots, (t_{k-1}, t_k), (t_k, t_1)$ where $t_1, t_2, \ldots, t_k$ are the inner points of Q will be called a path of regularity for the kernel $H(t, s)$: $Q \times Q \to R$ of the operator A

$$Ax(t) = \int_Q H(t, s) x(s) \, ds \tag{44}$$

where Q is the closure of a bounded region in a finitely dimensional Euclidean space, if the kernel $H(t, s)$ at all of these points is continuous and different from zero.

Remark 3. The path of regularity certainly exists if the function $H(t, s)$ is continuous at the point (t_0, t_0) and $H(t_0, t_0) \neq 0$, $t_0 \in Q$, or if it is continuous at the points (t_0, s_0), (s_0, t_0) and $H(t_0, s_0) \neq 0 \neq H(s_0, t_0)$.

Lemma 10 will give the existence of the path of regularity to the kernel of the integral operator A_k which is defined by (28).

Lemma 10. Let A_k, $k = 0, 1, \ldots, n-1$ be the operator defined by (28). The following statements are true.

a) For the kernel $\frac{\partial^k}{\partial t^k} G(t, s)$, $k = 0, 1, \ldots, n-2$, there exists a path of regularity.

b) Let

$$\Delta_{1n} \neq 0 \tag{45}$$

and let there exist a

$$j \in \{1, 2, \ldots, n-1\} \quad \text{such that} \quad \alpha_1^{(j)} \neq 0. \tag{46}$$

Then there exists a two-point path of regularity for kernel $\frac{\partial^{n-1}}{\partial t^{n-1}} G(t, s)$.

Proof. a) Let $k \in \{0, 1, \ldots, n-2\}$ be fixed number. Since $\frac{\partial^k}{\partial t^k} G(t, s)$, $k = 0, 1, \ldots, n-2$, is a continuous function on $\langle a_1, a_n \rangle x \langle a_1, a_n \rangle$ with exeption of points of the set $\{(t, a_{r+1});\ t \in \langle a_1, a_n \rangle,\ r = 1, 2, \ldots, n-2\}$, it suffices to show that

$\frac{\partial^k}{\partial t^k} G(t, s) \equiv 0$ cannot hold for all $t \in \bigcup_{r=1}^{n-1} (a_r, a_{r+1})$. On the basis of Lemma 3 $\frac{\partial^k}{\partial t^k} G(t, t) \equiv 0$ for $t \in \bigcup_{r=1}^{n-1} (a_r, a_{r+1})$ iff $\frac{g_i(t)}{\Delta} \sum_{j=k+1}^{n} \frac{(j-1)!}{(j-1-k)!} t^{j-1-k} \Delta_{ij} = 0$ for $i = 1, 2, \ldots, n-1$ and $t \in \bigcup_{r=1}^{n-1} (a_r, a_{r+1})$.

Since $g_i(t) = 0$ for all $t \in \bigcup_{r=1}^{n-1} (a_r, a_{r+1})$ cannot arise because it is equivalent to $\Delta_{ij} = 0$ for $j = k+1, \ldots, n$, $j = 1, 2, \ldots, n-1$. But this implies, according to Lemma 7, that $\Delta = 0$ and this contradicts the assumption. So $\frac{\partial^k}{\partial t^k} G(t, t) = 0$ cannot hold for all $t \in \bigcup_{r=1}^{n-1} (a_r, a_{r+1})$.

b) Consider $\frac{\partial^{n-1}}{\partial t^{n-1}} G(t, s)$ for $s \in (a_1, a_2)$. By Lemma 3 and 4 we obtain

$$\frac{\partial^{n-1}}{\partial t^{n-1}} G(t, s) = \frac{\partial^{n-1}}{\partial t^{n-1}} G_1(t, s) = \begin{cases} \frac{g_1(s)}{\Delta}(n-1)!\, \Delta_{1n} - 1, \\ \qquad \text{for} \quad a_1 \leqslant t < s \\ \frac{g_1(s)}{\Delta}(n-1)!\, \Delta_{1n}, \\ \qquad \text{for} \quad s < t \leqslant a_n, \end{cases}$$

for $s \in (a_1, a_2)$.

According to (45) $\frac{\partial^{n-1}}{\partial t^{n-1}} G(t, s) \equiv 0$ cannot hold for all $s \in (a_1, a_2)$, $a_1 < t < s$ and with respect to (46) and to the definition of $g_1(s)$, $g_1(s) \not\equiv$ const for $s \in (a_1, a_2)$ and thus, so $\frac{\partial^{n-1}}{\partial t^{n-1}} G(t, s) \equiv 0$ cannot hold for all $s \in (a_1, a_2)$, $a_1 < t < s$. Denote

$$M = \{s_1;\, g_1(s_1) = 0\} \cup \left\{s_2;\, \frac{g_1(s_2)}{\Delta}(n-1)!\, \Delta_{1n} - 1 = 0\right\}.$$

Then for arbitrary two points $p, q \in (a_1, a_2) - M$ the couple (p, q), (q, p) is a two-point path of regularity for the kernel $\frac{\partial^{n-1}}{\partial t^{n-1}} G(t, s)$.

Example 1. Consider the scalar problem

$$y^{(n)} = 0 \tag{47}$$

$$y(a_i) = 0, \qquad i = 1, 2, \ldots, n, \tag{48}$$

$a_i \in R$, $i = 1, 2, \ldots, n$, $a_1 < a_2 < \ldots < a_n$.

Then there exists a path of regularity for the kernel $\dfrac{\partial}{\partial t^{n-1}} G(t, s)$, where G is the Green function of the problem (47), (48).

Proof. The conditions (48) imply that $\alpha_i^{(1)} = 1$, $i = 1, 2, \ldots, n$ and

$$\Delta_{1n} = \begin{vmatrix} 1 & a_2 & a_2^2 & \ldots & (n-1)! & a_2^{n-2} \\ \vdots & & & & & \vdots \\ 1 & a_n & a_n^2 & \ldots & (n-1)! & a_n^{n-2} \end{vmatrix} \neq 0,$$

because $a_1 < a_2 < \ldots < a_n$. On basis of Lemma 10 there exists a path of regularity for the kernel $\dfrac{\partial^{n-1}}{\partial t^{n-1}} G(t, s)$.

Example 2. Consider the scalar problem

$$y^{(n)} = 0 \tag{47}$$

$$\begin{gathered} \sum_{j=1}^{n} \alpha_k^{(j)} y^{(j-1)}(a_k) = 0, \; k = 1, 2, \ldots, n, \; k \neq i, \; \sum_{j=1}^{n} |\alpha_k^{(j)}| > 0, \\ y^{(n-1)}(a_i) = 0. \end{gathered} \tag{49}$$

Then there does not exist a path of regularity for the kernel $\dfrac{\partial^{n-1}}{\partial t^{n-1}} G(t, s)$.

Proof. Since $\alpha_i^{(j)} = 0, j = 1, 2, \ldots, n-1$, it follows that $\Delta_{kn} = 0$ for all $k = 1, 2, \ldots, n$, $k \neq i$. Then $0 \neq \Delta = \alpha_i^{(n)}(n-1)!\,\Delta_{in}$. Consider the form of the function $\dfrac{\partial^{n-1}}{\partial t^{n-1}} G(t, s)$.

On the basis of Lemmas 3 and 4 and from the definition of $g_i(s)$, we get

$$\frac{\partial^{n-1}}{\partial t^{n-1}} G(t, s) = \begin{cases} \dfrac{g_i(s)}{\Delta} \Delta_{in}(n-1)! - 1 = 0, & a_1 \leqslant t < s, \\[2ex] \dfrac{g_i(s)}{\Delta} \Delta_{in}(n-1)! = \dfrac{\alpha_i^{(n)}}{\Delta} \Delta_{in}(n-1)! = 1, & s < t \leqslant a_n \end{cases}$$

for $s \in \bigcup\limits_{k=i}^{n-1} (a_k, a_{k+1})$.

When $s \in \bigcup\limits_{k=1}^{i-1} (a_k, a_{k+1})$, then

$$\frac{\partial^{n-1}}{\partial t^{n-1}} G(t, s) = \begin{cases} -1, & a_1 \leqslant t < s \\ 0, & s < t \leqslant a_n. \end{cases}$$

We suppose that there exists a path of regularity for the kernel $\frac{\partial^{n-1}}{\partial t^{n-1}} G(t, s)$ in form of a finite sequence of points $(t_1, t_2), (t_2, t_3), \ldots, (t_{k-1}, t_k), (t_k, t_1)$ in which $\frac{\partial^{n-1}}{\partial t^{n-1}} G(t, s)$ is continuous and different from zero. From the form of $\frac{\partial^{n-1}}{\partial t^{n-1}} G(t, s)$ it follows that all points of this sequence must be from the set $s < t \leqslant a_n$, $s \in \bigcup_{k=i}^{n-1} (a_k, a_{k+1})$, or from the set $a_1 \leqslant t < s$, $s \in \bigcup_{k=1}^{i-1} (a_k, a_{k+1})$ where $\frac{\partial^{n-1}}{\partial t^{n-1}} G(t, s)$ is continuous and different from zero.

First let $t_1 \in \bigcup_{k=1}^{i-1} (a_k, a_{k+1})$ then $t_1 < t_2 < a_i$, $t_2 \in \bigcup_{k=1}^{i-1} (a_k, a_{k+1})$. The point (t_1, t_2) is from the set where $\frac{\partial^{n-1}}{\partial t^{n-1}} G(t, s) = -1$. By way of analogy we get the points $(t_2, t_3), \ldots, (t_{k-1}, t_k)$ where $t_1 < t_2 < t_3 \ldots t_k < a_i$. Hence the point (t_k, t_1) is from the set $s < t < a_i$, $s \in \bigcup_{k=1}^{i-1} (a_k, a_{k+1})$ where $\frac{\partial^{n-1}}{\partial t^{n-1}} G(t, s)$ is identically equal to zero.

Similarly, in the second case we get a sequence $t_1 > t_2 > \ldots > t_k > a_i$, $t_i \in \bigcup_{k=i}^{n-1} (a_k, a_{k+1})$ and the point (t_k, t_1) from the set $t < s < a_n$, $t \in \bigcup_{k=i}^{n-1} (a_k, a_{k+1})$ where $\frac{\partial^{n-1}}{\partial t^{n-1}} G(t, s)$ is equal zero.

It follows from this that there does not exist a path of regularity for the kernel $\frac{\partial^{n-1}}{\partial t^{n-1}} G(t, s)$ for the problem (47), (49).

Futher Theorem 9.9 from the paper [2] p. 72 is stated here as Lemma 11.

Lemma 11. Let the integral operator A defined by (44) be completely continuous in the space $C = C(Q)$ (or in any $L_p = L_p(Q)$, $1 \leqslant p < \infty$). Let the kernel $H(t, s)$ of this operator be nonnegative and there exists a path of regularity. Then the operator A has a nonnegative eigenfunction which corresponds to a positive eigenvalue $\varrho(A)$.

Remark 4. By a similar way as in Example 2 we can show that in the case that $\alpha_1^{(j)} = 0, j = 1, 2, \ldots, n-1$, the operator

$$A_{n-1} x(t) = \int_{a_1}^{a_n} \left| \frac{\partial^{n-1}}{\partial t^{n-1}} G(t, s) \right| x(s)\, ds = \int_{a_1}^{t} \left| \frac{\partial^{n-1}}{\partial t^{n-1}} G(t, s) \right| x(s)\, ds = \int_{a_1}^{t} x(s)\, ds$$

and this operator has no eigenfunction in E according to the remark in [2], 9.1, p. 66.

On the basis of Lemmas 10 and 11 the following statement is true.

Lemma 12. Let the assumptions (45) and (46) be satisfied. Then the operator A_k, $k = 0, 1, \ldots, n-1$, defined by (28) has a nonegative eigenfunction which corresponds to $\varrho(A_k) > 0$.

On the basis of these properties of the operator A_k the following theorem is true.

Theorem 3. Let the assumptions of Lemmas 8 and 10 be satisfied and φ_k be a nonnegative eigenfunction of the operator A_k. Then the functions

$$\varphi_j(t) = \frac{1}{\varrho(A_k)} \int_{a_1}^{a_n} \left| \frac{\partial^j}{\partial t^j} G(t, s) \right| \varphi_k(s) \, ds, \tag{50}$$

$a_1 \leqslant t \leqslant a_n, \quad j = 0, 1, \ldots, n-1,$

form an admissible system of functions with respect to G such that for the associated system of constants K_j, $j = 0, 1, \ldots, n-1$,

$$K_k = \varrho(A_k) \tag{51}$$

is true.

Proof. The proof is analogous to proof of Theorem 3 in [4] p. 87—88.

The functions φ_j determined by (50) are all continuous and nonnegative in $\langle a_1, a_n \rangle$. Clearly φ_k satisfies (50) for $j = k$. First we show that the functions φ_j, $j = 0, 1, \ldots, n-1$, form an admissible system of functions with repect to G. In agreement with (28), we define the operator $A_j \colon E \to E$ by

$$A_j x(t) = \int_{a_1}^{a_n} \left| \frac{\partial^j}{\partial t^j} G(t, s) \right| x(s) \, ds, \qquad a_1 \leqslant t \leqslant a_n, \ x \in E \tag{52}$$

$j = 0, 1, \ldots, n-1.$

By (21), (22), (50), (52) we have to find such constants $k_j > 0, j = 0, 1, \ldots, n-1$, that

$$\Phi_j(t) = (A_j 1)(t) \leqslant \frac{k_j}{\varrho(A_k)} A_j \varphi_k(t), \qquad a_1 \leqslant t \leqslant a_n. \tag{53}$$

But the proof of (53) runs in the same way as the proof of (43′). Hence we can assert that the existence of $k_j > 0, j = 0, 1, \ldots, n-1$, with property (53) is guaranteed.

Finally we prove (51). In virtue of (25), (24) and (50)

$$K_k = \max \{k_{k,0}, k_{k,1}, \ldots, k_{k,n-1}\} \leqslant \varrho(A_k).$$

On the other hand, Lemma 3 from [4] p. 84 gives an opposite inequality and hence, (51) is true.

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SÚHRN

VEKTOROVÁ n-BODOVÁ OKRAJOVÁ ÚLOHA

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V práci sa skúma n-bodová okrajová úloha pre nelineárne diferenciálne systémy n-tého rádu s lineárnymi okrajovými podmienkami. Metóda riešenia tejto úlohy je podobná ako v práci [4], kde sa uvažuje De la Valèe Poussinova úloha, ktorá je špeciálnym prípadom úlohy (1), (2). Používa sa pritom Krasnoseľského teória kladných operátorov.

V práci sú podrobne preskúmané vlastnosti príslušnej skalárnej úlohy a sú dané podmienky, ktoré zaručujú existenciu cesty regularity tejto funkcie.

РЕЗЮМЕ

ВЕКТОРНАЯ n-ТОЧНАЯ КРАЕВАЯ ЗАДАЧА

ЭВА ШИМАЛЧИКОВА, Братислава

В работе изучается n-точная краевая задача n-того порядка для нелинейных систем с линейными краевыми условиями. Метод решения этой задачи подобный методу в работе [4], в которой решается задача Делавале Пусена, которая является специальным случаем задачи (1), (2). При этом используется Красносельского теория позитивных операторов.

В работе подробно анализированы свойства функции Грина соответствующей скалярной задачи. Здесь также показаны условия, выполнение которых обеспечивает существование дорожки невырожденности этой функции.