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ON THE CONDITIONAL MEAN VALUE OF VECTOR LATTICE VALUED RANDOM VARIABLES

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Inner regular measure spaces and dyadic tree condition for vector lattices were defined in [3]. It was shown that the monotone limit convergence theorem holds for integrals of simple functions defined on an inner regular measure space with values in a vector lattice V if V satisfies a dyadic tree condition.

This paper generalizes results of [3] and it gives a proof of the monotone limit convergence theorem for the conditional mean value of vector lattice valued functions under the same assumptions.

The monotone limit convergence theorem enables to apply Matthes—Wright theorem on the conditional mean value when a vector lattice V is weakly σ-distributive (see [1], [4], [5]).

Preliminaries and notations

Symbol V denotes a vector lattice, i.e. a real linear space V which has a partial ordering $\leq$ such that:

(i) V is a lattice with respect to $\leq$,

(ii) $\forall a, b, c \in V$: $a \leq b \Rightarrow a + c \leq b + c$,

(iii) $\forall a, b \in V \quad \forall \lambda \in \boldsymbol{R}$: $a \leq b,\ 0 \leq \lambda \Rightarrow \lambda a \leq \lambda b$.

V is said to be σ-complete if every upper bounded sequence has a least upper bound. If $\{a_n\}_{n=1}^{\infty}$ is a sequence of elements of V, then $a_n \searrow 0$ means that sequence $\{a_n\}_{n=1}^{\infty}$ is nonincreasing and 0 is its greatest lower bound. We say that $\{a_n\}_{n=1}^{\infty}$ decreases to 0.

A double sequence $\{a(n, k)\}_{n=0,}^{\infty}{}_{k=1}^{2^n}$ of elements of V is called a dyadic tree.

A sequence $\{b_n\}_{n=0}^{\infty}$ is called a chain of a dyadic tree $\{a(n, k)\}_{n=0,}^{\infty}{}_{k=1}^{2^n}$ if there is a sequence $\{k_n\}_{n=0}^{\infty}$ of integers such that:

$$k_0 = 1$$

$$\forall n\colon k_{n+1} \in \{2k_n, 2k_n - 1\}$$

$$\forall n\colon b_n = a(n, k_n).$$

A dyadic tree $\{a(n, k)\}_{n=0}^{\infty}, {}_{k=1}^{2^n}$ is said to be decreasing to 0 if all its chains decrease to 0.

We say that vector lattice V satisfies the dyadic tree condition (briefly DTC) if $\left(2^{-n} \cdot \sum_{k=1}^{2^n} a(n, k)\right) \searrow 0$ whenever $\{a(n, k)\}_{n=0}^{\infty}, {}_{k=1}^{2^n}$ decreases to 0.

In the sequel (X, S, P) denotes a probability measure space (X is a set, S is a σ-algebra on X, and P is a probability measure).

A measure space (X, S, P) is called inner regular if there is a system $\mathscr{K} \subset S$ such that:

(i) $\forall \{K_n\}_{n=1}^{\infty}: (\forall n: K_n \in \mathscr{K}, K_{n+1} \subset K_n, K_n \neq \emptyset) \Rightarrow \bigcap_{n=1}^{\infty} K_n \neq \emptyset$,

(ii) $\forall A \in S: P(A) = \sup\{P(K): K \subset A, K \in \mathscr{K}\}$.

A function $f: X \to V$ is said to be simple if there are sets $A_1, \ldots, A_l \in S$ and elements $a_1, \ldots, a_l \in V$ such that $f = \sum_{i=1}^{l} \chi_{A_i} a_i$, where χ_{A_i} denotes a characteristic function of a set A_i. We may always assume that sets $A_1, \ldots, A_l$ form a partition of X, i.e.:

$$X = \sum_{i=1}^{l} A_i \quad \text{and} \quad A_i \cap A_j = \emptyset \quad \text{for} \quad i \neq j.$$

Moreover, if we have a sequence $\{f_m\}_{m=1}^{\infty}$ of simple functions $f_m: X \to V$, we may assume that there are systems $\{l_m\}_{m=1}^{\infty}$, $\{p_{m,j}\}_{m=1}^{\infty}, {}_{j=1}^{l_m}$, $\{q_{m,j}\}_{m=1}^{\infty}, {}_{j=1}^{l_m}$ of integers, a system $\{A_{m,j}\}_{m=1}^{\infty}, {}_{j=1}^{l_m} \subset S$ and a system $\{a_{m,j}\}_{m=1}^{\infty}, {}_{j=1}^{l_m} \subset V$ such that

(1) $$f_m = \sum_{j=1}^{l_m} \chi_{A_{m,j}} a_{m,j} \quad \text{for all } m$$

(2) $$X = \bigcup_{j=1}^{l_m} A_{m,j} \quad \text{and} \quad A_{m,i} \cap A_{m,j} = \emptyset \quad \text{for} \quad i \neq j$$

(3) $$1 = p_{m,1} < p_{m,2} < \ldots < p_{m,l_m}$$

(4) $$q_{m,1} < q_{m,2} < \ldots < q_{m,l_m} = l_{m+1}$$

(5) $$p_{m,j+1} = q_{m,j} + 1$$

(6) $$A_{m,j} = \bigcup_{i=p_{m,j}}^{q_{m,j}} A_{m+1,i}$$

Let S_0 be a σ-subalgebra of S and $\varphi: X \to \boldsymbol{R}$ be a S-measurable P-integrable function. Radon—Nikodym theorem implies that there is a unique (almost everywhere) S_0-measurable P-integrable function ψ: $X \to \boldsymbol{R}$ such that

$$\int_A \varphi \, dP = \int_A \psi \, dP \quad \text{for any} \quad A \in S_0.$$

The function ψ is called a conditional mean value of φ with respect to S_0 and it is denoted as $E(\varphi | S_0)$.

Now, we may define $E(f|S_0)$ for V-valued simple function. If $f = \sum_{i=1}^{l} \chi_{A_i} \cdot a_i$, $A_i \in S$, $a_i \in V$, put

$$E(f|S_0) = \sum_{i=1}^{l} E(\chi_{A_i}|S_0) \cdot a_i.$$

If $\{f_m\}_{m=1}^{\infty}$ is a sequence of functions f_m: $X \to V$ then the symbol $f_m \searrow 0$ means that $f_m(x) \searrow 0$ for all $x \in X$.

Aproximations

Lemma 1: Let systems $\{l_m\}_{m=1}^{\infty}$, $\{p_{m,j}\}_{m=1}^{\infty}{}_{,}{}_{j=1}^{l_m}$ *and* $\{q_{m,j}\}_{m=1}^{\infty}{}_{,}{}_{j=1}^{l_m}$ satisfy $(3-5)$. Let X be a set, S_0 be a σ-algebra on X and $\{\varphi_{m,j}\}_{m=1}^{\infty}{}_{,}{}_{j=1}^{l_m}$ be a system of nonnegative S_0-measurable real functions defined on X such that:

$$\sum_{j=1}^{l_m} \varphi_{m,j} = 1 \tag{8}$$

$$\sum_{i=p_{m,j}}^{q_{m,j}} \varphi_{m+1,i} = \varphi_{m,j}. \tag{9}$$

For any $\varepsilon > 0$ there is a family $\{\psi_{m,i}\}_{m=1}^{\infty}{}_{,}{}_{i=1}^{l_m}$ of nonnegative simple dyadic rational valued functions such that:

$$\sum_{j=1}^{l_m} \psi_{m,j} = 1 \quad \text{for al} \quad m \tag{10}$$

$$\sum_{j=1}^{l_m} |\varphi_{m,j} - \psi_{m,j}| \leq \varepsilon(1 - 2^{-m}) \quad \text{for all} \quad m \tag{11}$$

$$\sum_{i=p_{m,j}}^{q_{m,j}} \psi_{m+1,i} = \psi_{m,j} \quad \text{for all } m \text{ and } \quad j \in \{1, \ldots, l_m\}. \tag{12}$$

Proof: The system $\{\psi_{m,i}\}$ will be constructed by induction. It is convenient to start from $m = 0$. So, put $l_0 = 1$, $\varphi_{0,1} = 1$, $q_{0,1} = l_1$, $p_{0,1} = 1$. Then (8) and (9) are satisfied for $m = 0$ also. Put $\psi_{0,1} = 1$. Then (10) and (11) are satisfied for $m = 0$. Assume that the functions $\psi_{m,j}$ are already defined for $m \leq k$ and $j \in \{1, \ldots, l_m\}$ and they have the required properties. We shall construct functions $\psi_{k+1,i}$ for $i \in \{1, \ldots, l_{k+1}\}$. Take an integer r such that

$$2 \cdot l_{k+1} \cdot 2^{-r} < \varepsilon \cdot 2^{-k}. \tag{13}$$

Take fixed $j \in \{1, \ldots, l_k\}$ and consider all $i \in \{p_{k,j}, \ldots, q_{k,j}\}$.
If $\varphi_{k,j}(x) = 0$ then for all $i \in \{p_{k,j}, \ldots, q_{k,j}\}$ $\varphi_{k+1,i}(x) = 0$ by (9).
Put $\psi_{k+1,i}(x) = 0$ when $p_{k,j} \leq i < q_{k,j}$ and $\psi_{k+1,q_{k,j}}(x) = \psi_{k,j}(x)$.
Then (12) holds for $m = k$ and

$$\sum_{i=p_{k,j}}^{q_{k,j}} |\varphi_{k+1,i}(x) - \psi_{k+1,i}(x)| = |\varphi_{k,j}(x) - \psi_{k,j}(x)|. \tag{14}$$

If $\varphi_{k,j}(x) > 0$ then put

$$\psi_{k+1,i}(x) = 2^{-r}\left[2^r \frac{\varphi_{k+1,i}(x)\, \psi_{k,j}(x)}{\varphi_{k,j}(x)}\right] \quad \text{for} \quad i < q_{k,j}$$

and

$$\psi_{k+1,q_{k,j}}(x) = \psi_{k,j}(x) - \sum_{i=p_{k,j}}^{q_{k,j}-1} \psi_{k+1,i}(x)$$

(the symbol $[a]$ denotes the integer part of a).
Then we have:

$$\sum_{i=p_{k,j}}^{q_{k,j}} \psi_{k+1,i}(x) = \psi_{k,j}(x) \tag{15}$$

$$0 \leq \psi_{k+1,i}(x) \leq \frac{\varphi_{k+1,i}(x)\, \psi_{k,j}(x)}{\varphi_{k,j}(x)} < \psi_{k+1,i}(x) + 2^{-r} \tag{16}$$

for $i < q_{k,j}$.
Relation (9) implies equality

$$\sum_{i=p_{k,j}}^{q_{k,j}} \frac{\varphi_{k+1,i}(x)\, \psi_{k,j}(x)}{\varphi_{k,j}(x)} = \psi_{k,j}(x). \tag{17}$$

Comparing relations (15), (16) and (17), we have

$$\psi_{k+1,q_{k,j}}(x) - \frac{\varphi_{k+1,q_{k,j}}(x)\, \psi_{k,j}(x)}{\varphi_{k,j}(x)} = \tag{18}$$

$$= \sum_{i=p_{k,j}}^{q_{k,j}-1} \left(\frac{\varphi_{k+1,i}(x)\, \psi_{k,j}(x)}{\varphi_{k,j}(x)} - \psi_{k+1,i}(x)\right)$$

Relations (16) and (18) give

$$\text{(19)} \qquad \sum_{i=p_{k,j}}^{q_{k,j}} \left| \frac{\varphi_{k+1,i}(x)\,\psi_{k,j}(x)}{\varphi_{k,j}(x)} - \psi_{k+1,i}(x) \right| =$$

$$= 2 \sum_{i=p_{k,j}}^{q_{k,j}-1} \left| \frac{\varphi_{k+1,i}(x)\,\psi_{k,j}(x)}{\varphi_{k,j}(x)} - \psi_{k+1,i}(x) \right| < 2(q_{k,j} - p_{k,j})\,2^{-r}.$$

Using (9) and (19), we obtain the following estimate:

$$\sum_{i=p_{k,j}}^{q_{k,j}} |\varphi_{k+1,i}(x) - \psi_{k+1,i}(x)| \le \sum_{i=p_{k,j}}^{q_{k,j}} \left| \varphi_{k+1,i}(x) - \frac{\varphi_{k+1,i}(x)\,\psi_{k,j}(x)}{\varphi_{k,j}(x)} \right| +$$

$$+ \sum_{i=p_{k,j}}^{q_{k,j}} \left| \frac{\varphi_{k+1,i}(x)\,\psi_{k,j}(x)}{\varphi_{k,j}(x)} - \psi_{k+1,i}(x) \right| =$$

$$= \left| 1 - \frac{\psi_{k,j}(x)}{\varphi_{k,j}(x)} \right| \sum_{i=p_{k,j}}^{q_{k,j}} \varphi_{k+1,i}(x) + \sum_{i=p_{k,j}}^{q_{k,j}} \left| \frac{\varphi_{k+1,i}(x)\,\psi_{k,j}(x)}{\varphi_{k,j}(x)} - \psi_{k+1,i}(x) \right| \le$$

$$\le |\varphi_{k,j}(x) - \psi_{k,j}(x)| + 2(q_{k,j} - p_{k,j})\,2^{-r}.$$

It means that

$$\sum_{i=p_{k,j}}^{q_{k,j}} |\varphi_{k+1,i}(x) - \varphi_{k+1,i}(x)| \le |\varphi_{k,j}(x) - \psi_{k,j}(x)| + 2(q_{k,j} - p_{k,j})\,2^{-r}$$

whenever $\varphi_{k,j}(x) > 0$.

Looking at (14), we see that the last inequality holds for all x. Using (3), (4), (5), the inductive assumption and (13) we obtain:

$$\sum_{i=1}^{l_{k+1}} |\varphi_{k+1,i}(x) - \psi_{k+1,i}(x)| = \sum_{j=1}^{l_k} \sum_{i=p_{k,j}}^{q_{k,j}} |\varphi_{k+1,i}(x) - \psi_{k+1,i}(x)| \le$$

$$\le \sum_{j=1}^{l_k} |\varphi_{k,j}(x) - \psi_{k,j}(x)| + \sum_{j=1}^{l_k} 2(q_{k,j} - p_{k,j})\,2^{-r} \le$$

$$\le \varepsilon(1 - 2^{-k}) + 2l_{k+1}2^{-r} \le \varepsilon(1 - 2^{-k}) + \varepsilon 2^{-k-1} = \varepsilon(1 - 2^{-k-1}).$$

It means that (11) holds for $m = k + 1$.

The other properties of $\psi_{k+1,i}$ are obvious.

The proof is complete.

Now, suppose we have a probability measure space (X, S, P) and fixed systems of integers $\{l_m\}_{m=1}^{\infty}$, $\{p_{m,j}\}_{m=1}^{\infty}{}_{j=1}^{l_m}$ and $\{q_{m,j}\}_{m=1}^{\infty}{}_{j=1}^{l_m}$ and a system $\{A_{m,j}\}_{m=1}^{\infty}{}_{j=1}^{l_m}$ of measurable sets which satisfy relations (2)—(6).

Definition 1: A sequence $\{A_{m,j_m}\}_{m=1}^{\infty}$ is said to be acceptable, if $j_m \in \{1, \ldots, l_m\}$ and $j_{m+1} \in \{p_{m,j_m}, \ldots, q_{m,j_m}\}$ for all m. An acceptable sequence $\{A_{m,j_m}\}_{m=1}^{\infty}$ is said to be disappearing if $\bigcap_{m=1}^{\infty} A_{m,j_m} = \emptyset$.

A set $A \in S$ is said to be catching if for any disappearing sequence $\{A_{m,j_m}\}_{m=1}^{\infty}$ there exists m_0 such that $A_{m_0,j_{m_0}} \subset A$.

Lemma 2:

(i) If A and B are catching sets, then $A \cap B$ is catching

(ii) If (X, S, P) is inner regular, then for any $\varepsilon > 0$ there is a catching set A such that $P(A) < \varepsilon$.

Proof: Part (i) is obvious. We shall prove (ii).

Let $\varepsilon > 0$ be fixed. We shall construct a system $\{K_{m,j}\}_{m=1}^{\infty},{}_{j=1}^{l_m}$ such that:

$$\forall m\ \forall j \in \{1, \ldots, l_m\}: K_{m,j} \subset A_{m,j},\ K_{m,j} \in \mathcal{K} \quad \text{or} \quad K_{m,j} = \emptyset \tag{20}$$

$$\forall m\ \forall j \in \{1, \ldots, l_m\}\ \forall i \in \{p_{m,j}, \ldots, q_{m,j}\}: K_{m+1,i} \subset K_{m,j} \tag{21}$$

$$\forall m: \sum_{j=1}^{l_m} P(K_{m,j}) > 1 - \varepsilon(1 - 2^{-m}). \tag{22}$$

The system $\{K_{m,j}\}_{m=1}^{\infty},{}_{j=1}^{l_m}$ will be constructed by induction.

Let $m = 1$. For all $j \in \{1, \ldots, l_1\}$ take $K_{1,j} \in \mathcal{K}$ or $K_{1,j} = \emptyset$ such that:

$$K_{1,j} \subset A_{1,j} \quad \text{and} \quad P(A_{1,j} - K_{1,j}) < \frac{\varepsilon}{2l_1}.$$

Then $\{K_{1,j}\}_{j=1}^{l_1}$ has the required properties. Suppose that for all $m' \leq m$ the systems $\{K_{m',j}\}_{j=1}^{l_{m'}}$ are already constructed. We are going to construct the system $\{K_{m+1,i}\}_{i=1}^{l_{m+1}}$. Let $j \in \{1, \ldots, l_m\}$ be fixed. Consider all $i \in \{p_{m,j}, \ldots, q_{m,j}\}$.

If $P(A_{m+1,i} \cap K_{m,j}) = 0$ put $K_{m+1,i} = \emptyset$.

If $P(A_{m+1,i}) \cap K_{m,j}) > 0$ take $K_{m+1,i} \in \mathcal{K}$ such that:

$$K_{m+1,i} \subset (A_{m+1,i} \cap K_{m,j})$$

and

$$P((A_{m+1,i} \cap K_{m,j}) - K_{m+1,i}) < \frac{\varepsilon \cdot 2^{-m}}{2 \cdot l_{m+1}}.$$

The system $\{K_{m+1,i}\}_{i=1}^{l_{m+1}}$ has the required properties and the system $\{K_{m,j}\}_{m=1}^{\infty},{}_{j=1}^{l_m}$ is constructed. Put $A = X - \bigcap_{m=1}^{\infty} \bigcup_{j=1}^{l_m} K_{m,j}$. Obviously $P(A) < \varepsilon$.

We are going to prove that A is a catching set. Let $\{j_m\}_{m=1}^{\infty}$ be a sequence such that:

$$j_m \in \{1, \ldots, l_m\}, \quad A_{m+1.j_{m+1}} \subset A_{m.j_m}$$

for all m and

$$\bigcap_{m=1}^{\infty} A_{m.j_m} = \emptyset.$$

Then for some m_0 we must have $K_{m_0.j_{m_0}} = \emptyset$. In the opposite case we should have a sequence $\{K_{m.j_m}\}_{m=1}^{\infty}$ such that:

$$K_{m.j_m} \neq \emptyset, \quad K_{m.j_m} \in \mathscr{K}, \quad K_{m.j_m} \subset A_{m.j_m}, \quad K_{m+1.j_{m+1}} \subset K_{m.j_m}$$

for all m and

$$\emptyset \neq \bigcap_{m=1}^{\infty} K_{m.j_m} \subset \bigcap_{m=1}^{\infty} A_{m.j_m},$$

which is a contradiction.

Since

$$K_{m_0.j_{m_0}} = \emptyset, \quad \text{we have} \quad A_{m_0.j_{m_0}} \cap K_{m_0.j_{m_0}} = \emptyset.$$

If $j \in \{1, \ldots, l_{m_0}\}$ and $j \neq j_{m_0}$ then $A_{m_0.j_{m_0}} \cap K_{m_0.j} = \emptyset$ because $K_{m_0.j} \subset A_{m_0.j}$ and $A_{m_0.j_{m_0}} \cap A_{m_0.j} = \emptyset$ by (2). Therefore

$$A_{m_0.j_{m_0}} \subset X - \bigcup_{j=1}^{l_{m_0}} K_{m_0.j} \subset X - \bigcap_{m=1}^{\infty} \bigcup_{j=1}^{l_m} K_{m.j} = A.$$

The proof of (ii) is complete.

The monotone limit convergence theorem

Theorem: Let (X, S, P) be an inner regular probability measure space, S_0 be a σ-subalgebra of S and V be a σ-complete vector lattice which satisfies DTC. For any sequence $\{f_m\}_{m=1}^{\infty}$ of simple V-valued functions defined on X we have:

$$f_m \searrow 0 \Rightarrow E(f_m \mid S_0) \searrow 0 \quad \text{a.e.}$$

Proof: Let $\{f_m\}_{m=1}^{\infty}$, $\{l_m\}_{m=1}^{\infty}$, $\{p_{m.j}\}_{m=1,j=1}^{\infty\ \ l_m}$, $\{q_{m.j}\}_{m=1,j=1}^{\infty\ \ l_m}$, $\{A_{m.j}\}_{m=1,j=1}^{\infty\ \ l_m}$ and $\{a_{m.j}\}_{m=1,j=1}^{\infty\ \ l_m}$ satisfy (1)—(6) and

(23) $$f_m \searrow 0.$$

Put $\varphi_{m.j} = E(\chi_{A_{m.j}} \mid S_0)$.

Then $E(f_m \mid S_0) = \sum_{j=1}^{l_m} a_{m.j} \cdot \varphi_{m.j}$ and $\varphi_{m.j}$ satisfy (8) and (9) almost everywhere.

We shall assume that (8) and (9) are satisfied everywhere. It is easy to see that $E(f_{m+1} | S_0) \leq E(f_m | S_0)$ for all m.

If we show that $\bigwedge_{m=1}^{\infty} E(f_m | S_0) = 0$, the proof will be finished.

For every natural s take a catching set A_s such that

$$P(A_s) < \frac{1}{s}. \tag{24}$$

We may assume that

$$A_{s+1} \subset A_s \quad \text{for all } s. \tag{25}$$

Now, let s be fixed. Put $\varphi_s = E(\chi_{A_s} | S_0)$ and

$$b_{m,j} = \begin{cases} a_{m,j} & \text{when} \quad A_{m,j} \not\subset A_s \\ 0 & \text{when} \quad A_{m,j} \subset A_s \end{cases}. \tag{26}$$

Then

$$b_{m,j_m} \searrow 0 \text{ for every acceptable sequence } \{A_{m,j_m}\}_{m=1}^{\infty}, \tag{27}$$

because $b_{m,j_m} = f_m(x)$ for some $x \in \bigcap_{m=1}^{\infty} A_{m,j_m}$, when $\{A_{m,j_m}\}_{m=1}^{\infty}$ is not disappearing, and $b_{m,j_m} = 0$ for sufficiently large m, when $\{A_{m,j_m}\}_{m=1}^{\infty}$ is disappearing. Put

$$a = \bigvee_{j=1}^{l_1} a_{1,j}. \tag{28}$$

Then:

$$E(f_m | S_0) = \sum_{j=1}^{l_m} a_{m,j}\varphi_{m,j} \leq a\varphi_s + \sum_{j=1}^{l_m} b_{m,j}\varphi_{m,j}. \tag{29}$$

Take $\varepsilon = \frac{1}{s}$ and the functions $\psi_{m,j}$ from Lemma 1.

Using (29), (11) and (28), we obtain:

$$E(f_m | S_0) \leq a \cdot \varphi_s + \sum_{j=1}^{l_m} b_{m,j}\psi_{m,j} + \sum_{j=1}^{l_m} b_{m,j}(\varphi_{m,j} - \psi_{m,j}) \leq$$

$$\leq a\varphi_s + \sum_{j=1}^{l_m} b_{m,j}\psi_{m,j} + \sum_{j=1}^{l_m} b_{m,j}|\varphi_{m,j} - \psi_{m,j}| \leq$$

$$\leq a\varphi_s + \sum_{j=1}^{l_m} b_{m,j}\psi_{m,j} + a\sum_{j=1}^{l_m} |\varphi_{m,j} - \psi_{m,j}| \leq$$

$$\leq a\varphi_s + \sum_{j=1}^{l_m} b_{m,j}\psi_{m,j} + a\frac{1}{s} =$$

$$= a\left(\frac{1}{s} + \varphi_s\right) + \sum_{j=1}^{l_m} b_{m,j}\psi_{m,j}, \quad \text{i.e.}$$

(30) $$E(f_m \mid S_0) \leq a\left(\frac{1}{s} + \varphi_s\right) + \sum_{j=1}^{l_m} b_{m,j}\,\psi_{m,j}.$$

Now, we shall show that $\bigwedge_{m=1}^{\infty}\left(\sum_{j=1}^{l_m} b_{m,j}\psi_{m,j}(x)\right) = 0$ for all $x \in X$. Let $x \in X$ be fixed. We shall construct a dyadic tree $\{c(n,k)\}_{n=0,\,k=1}^{\infty,\;2^n}$, which is decreasing to 0 and

$$\sum_{j=1}^{l_m} b_{m,j}\psi_{m,j}(x) = 2^{-n_m}\sum_{k=1}^{2^{n_m}} c(n_m, k)$$

for a suitable increasing sequence $\{n_m\}_{m=1}^{\infty}$ of natural numbers. Since all values $\psi_{m,j}(x)$ are dyadic rational, there are integers $r_{m,j} \geq 0$ and $n_m > 0$ such that

(31) $$\psi_{m,j}(x) = r_{m,j}2^{-n_m} \quad \text{for all } m \text{ and } \quad j \in \{1, \ldots, l_m\}.$$

We may assume that

(32) $$n_{m+1} > n_m \quad \text{for all } m.$$

Then (31), (10) and (12) imply

(33) $$\sum_{j=1}^{l_m} r_{m,j} = 2^{n_m}$$

(34) $$\sum_{i=p_{m,j}}^{q_{m,j}} r_{m+1,i} = r_{m,j}2^{n_{m+1}-n_m}$$

If $0 \leq n < n_1$, put

(35) $$c(n,k) = a \quad \text{for all} \quad k \in \{1, \ldots, 2^n\},$$

where a is defined by (28).

If $n_m \leq n < n_{m+1}$ and $k \in \{1, \ldots, 2^n\}$, then put

(36) $$c(n, k) = b_{m,j_{n,k}},$$

where $j_{n,k}$ is a natural number such that $j_{n,k} \in \{1, \ldots, l_m\}$ and

$$\left(\sum_{i=1}^{j_{n,k}-1} r_{m,i}\right)2^{n-n_m} < k \leq \left(\sum_{i=1}^{j_{n,k}} r_{m,i}\right)2^{n-n_m}. \tag{37}$$

$\Bigg($Note that the sequence $\left\{\sum_{i=1}^{j} r_{m,i} \cdot 2^{n-n_m}\right\}_{j=0}^{l_m}$ is nondecreasing and

$$\sum_{i=1}^{l_m} r_{m,i} \cdot 2^{n-n_m} = 2^{n_m} \cdot 2^{n-n_m} = 2^n \quad \text{by (33).}$$

It means that $j_{n,k}$ is uniquely determined by n and $k.\Bigg)$

We shall show that the dyadic tree $\{c(n, k)\}$ is decreasing to 0.

Let $\{k_n\}_{n=0}^{\infty}$ be a sequence such that

$$k_0 = 1 \quad \text{and} \quad k_{n+1} \in \{2k_n - 1, 2k_n\}. \tag{38}$$

Put

$$j_m = j_{n_m, k_{n_m}}. \tag{39}$$

Then we have:

$$j_{n,k_n} = j_m \quad \text{whenever} \quad n_m \leq n < n_{m+1} \tag{40}$$

and

$$j_{m+1} \in \{p_{m,j_m}, \ldots, q_{m,j_m}\}. \tag{41}$$

We shall prove (40). Relations (39) and (37) imply

$$\sum_{i=1}^{j_m-1} r_{m,i} < k_{n_m} \leq \sum_{i=1}^{j_m} r_{m,i},$$

which may be rewritten as

$$\sum_{i=1}^{j_m-1} r_{m,i} \leq (k_{n_m} - 1) < k_{n_m} \leq \sum_{i=1}^{j_m} r_{m,i}. \tag{42}$$

Relation (38) implies inequality

$$2^{n-n_m}(k_{n_m} - 1) < k_n \leq 2^{n-n_m} \cdot k_{n_m} \quad \text{whenever} \quad n \geq n_m. \tag{43}$$

Relations (42) and (43) give

$$2^{n-n_m} \sum_{i=1}^{j_m-1} r_{m,i} < k_n \leq 2^{n-n_m} \sum_{i=1}^{j_m} r_{m,i}.$$

Looking at (37) we seet that $j_{n,k_n} = j_m$ which proves (40).
We shall show (41). Using (34) we may rewrite (42) in the following form:

$$2^{n_m - n_{m+1}} \sum_{i=1}^{j_m - 1} \sum_{j=p_{m,i}}^{q_{m,i}} r_{m+1,j} \leq (k_{n_m} - 1) < k_{n_m} \leq 2^{n_m - n_{m+1}} \sum_{i=1}^{j_m} \sum_{j=p_{m,i}}^{q_{m,i}} r_{m+1,j},$$

or equivalently:

$$\sum_{j=1}^{q_{m,j_m}-1} r_{m+1,j} \leq 2^{n_{m+1}-n_m}(k_{n_m} - 1) < 2^{n_{m+1}-n_m} \cdot k_{n_m} \leq \sum_{j=1}^{q_{m,j_m}} r_{m+1,j}.$$

Comparing the last inequality with (43) we obtain

$$\sum_{j=1}^{q_{m,j_m}-1} r_{m+1,j} < k_{n_{m+1}} \leq \sum_{j=1}^{q_{m,j_m}} r_{m+1,j}.$$

Looking at (37) we see that $j_{n_{m+1},k_{n_{m+1}}}$ must belong to the set $\{1 + q_{m,j_m - 1}, \ldots, q_{m,j_m}\}$, i.e. to the set $\{p_{m,j_m}, \ldots, q_{m,j_m}\}$, which proves (41).
From (40), (39) and (37) we obtain

$$c(n, k_n) = b_{m,j_m} \quad \text{whenever} \quad n_m \leq n < n_{m+1}.$$

The sequence $\{A_{m,j_m}\}_{m=1}^{\infty}$ is an acceptable sequence by (41). Therefore $c(n, k_n) \searrow 0$ by (27).

We have just proved that the dyadic tree $\{c(n,k)\}_{n=0,k=1}^{\infty \; 2^n}$ is decreasing to zero. Since V satisfies DTC,

$$\left(2^{-n} \cdot \sum_{k=1}^{2^n} c(n,k)\right) \searrow 0,$$

which means

$$2^{-n_m}\left(\sum_{k=1}^{2^{n_m}} c(n_m, k)\right) \searrow 0.$$

Looking at (36) we see that

$$\left(\sum_{j=1}^{l_m} r_{m,j} \cdot 2^{-n_m} \cdot b_{m,j}\right) \searrow 0.$$

It means

$$\bigwedge_{m=1}^{\infty} \left(\sum_{j=1}^{l_m} b_{m,j} \psi_{m,j}(x)\right) = 0$$

for all $x \in X$.

Returning to (30) we obtain:

$$\bigwedge_{m=1}^{\infty} E(f_m \mid S_0) \leq a\left(\frac{1}{s} + \varphi_s\right)$$

for any natural s.

Relation (25) gives $\varphi_{s+1} \leq \varphi_s$ for all s and

$$\int_X \varphi_s \, \mathrm{d}P = \int_X \chi_{A_s} \, \mathrm{d}P \leq \frac{1}{s} \to 0.$$

Therefore $\varphi_s \searrow 0$ a.e. and $\bigwedge_{m=1}^{\infty} E(f_m \mid S_0) = 0$ a.e.

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SÚHRN

O PODMIENENEJ STREDNEJ HODNOTE NÁHODNÝCH PREMENNÝCH S HODNOTAMI VO VEKTOROVOM ZVÄZE

PETER MALIČKÝ, Liptovský Mikuláš

V článku je dokázaná veta o monotónnej konvergencii podmienenej strednej hodnoty pre náhodné premenné definované na vnútorne regulárnych pravdepodobnostných priestoroch s hodnotami v σ-úplnom vektorovom zväze, ktorý spĺňa podmienku DTC (podmienku o dyadických stromoch).

РЕЗЮМЕ

ОБ УСЛОВНОМ СРЕДНЕМ ВЕКТОРНО-ЗНАЧНЫХ СЛУЧАЙНЫХ ВЕЛИЧИН

ПЕТЕР МАЛИЧКИ, Липтовски Микулаш

В работе рассмотрены простые случайные величины, определённые на внутренне регулярных вероятностных пространствах принимающие значения в σ-полной векторной решётке, которая удовлетворяет одному условию о двоичных деревьях. Для таких случайных величин доказана теорема о монотонной сходимости условного математичского ожидания.