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ON THE KURZWEIL INTEGRAL FOR FUNCTIONS WITH VALUES IN ORDERED SPACES I

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The paper contains the definition and some elementary properties of the Kurzweil integral for functions $f: \langle a, b\rangle \to X$, where X is a linear lattice. In continuation of this article some limit theorem will be proved. Throughout the paper m is a finite real-valued Borel measure.

First we recall the definition of the Riemann integral. By a decomposition of an interval $I = \langle a, b\rangle$ we mean a finite set of couples $(J_1, t_1), \ldots, (J_n, t_n)$, where $J_1, \ldots, J_n$ are non-overlapping intervals covering I and $t_i \in J_i$ $(i = 1, 2, \ldots, n)$. The corresponding integral sum is the number

$$S(f, D) = \sum_{i=1}^{n} f(t_i)\, m(J_i)$$

where $m(J_i)$ is the measure of the interval J_i. A function $f: I \to R$ is integrable in the Riemann sense if

$$\exists x \in R\ \forall \varepsilon > 0\ \exists \delta > 0\ \forall D \in A(\delta): |S(f, D) - x| < \varepsilon.$$

Here $A(\delta)$ is a set of sufficiently fine decompositions. It consists of such decomposition D that

$$J_i \subset (t_i - \delta, t_i + \delta) \qquad i = 1, 2, \ldots, n$$

Kurzweil's definition of the integral can be obtained from Riemann's, if one substitutes the number $\delta > 0$ by a positive function $\delta: I \to (0, \infty)$. So, a function $f: I \to R$ is integrable in the Kurzweil sense if

$$\exists x \in R\ \forall \varepsilon > 0\ \exists \delta \in (0, \infty)^I\ \forall D \in A(\delta): |S(f, D) - x| < \varepsilon.$$

Here $A(\delta)$ consists of all decompositions D for which $J_i \subset (t_i - \delta(t_i), t_i + \delta(t_i))$, $i = 1, \ldots, n$.

Of course, we must know that the definition is correct, i. e. that $A(\delta) \neq \emptyset$ for every δ: $I \to (0, \infty)$. Namely, if $A(\delta) = \emptyset$, then every real number satisfies the desired implication and hence it is the integral.

Lemma 1. $A(\delta) \neq \emptyset$ for every δ: $I \to (0, \infty)$.

Proof. If $I = \langle a, b \rangle$ need not be decomposed under δ, then at least one of the intervals $\left\langle a, \dfrac{a+b}{2} \right\rangle$, $\left\langle \dfrac{a+b}{2}, b \right\rangle$ has this property. So we may construct a sequence $(\langle a_n, b_n \rangle)_{n=1}^{\infty}$ of intervals such that $\langle a_{n+1}, b_{n+1} \rangle \subset \langle a_n, b_n \rangle \subset \langle a, b \rangle$ for every n and such that $\lim_{n \to \infty} (b_n - a_n) = 0$. Choose $t_0 \in \bigcap_{n=1}^{\infty} \langle a_n, b_n \rangle$. Then $\langle a_n, b_n \rangle \subset (t_0 - \delta(t_0), t_0 + \delta(t_0))$ for sufficiently large n and therefore $\langle a_n, b_n \rangle$ can be covered under δ, what is a contradiction.

In a general case when the range of f is only partially ordered we cannot use the ε technique. Namely, if $A \subset R$ and $s = \sup A$, then to every $\varepsilon > 0$ there is $a \in A$ such that $s - \varepsilon < a \leqq s$. This is not true in general linear lattices. Therefore we use the double sequence technique proposed by D. H. Fremlin [1]. The motivation is the following. If $a_{ij} \searrow 0$ $(j \to \infty, i = 1, 2, \ldots)$ and $(a_{ij})_{i,j}$ is a bounded sequence of real numbers, then to every $\varepsilon > 0$ and every i there is $\varphi(i)$ such that $a_{ij} < \varepsilon$ for every $j \geqq \varphi(i)$. Especially $a_{i\varphi(i)} < \varepsilon$ and so $\sup_i a_{i\varphi(i)} \leqq \varepsilon$. Therefore we can work with the suprema $\sup_i a_{i\varphi(i)}$ $(\varphi \in N^N)$ instead of ε positive.

Now some **definitions.** A non-empty set X is a linear lattice, if X is a real linear space, X is a lattice with the lattice operations $x \vee y$, $x \wedge y$ and the following implications hold: $x \leqq y \Rightarrow x + z \leqq y + z$; $x \leqq y$, $\alpha > 0$, $\alpha \in R \Rightarrow \alpha x \leqq \alpha y$. A linear lattice X is called boundedly σ-complete, if every bounded sequence $(a_i)_i$ of elements of X has the supremum $\bigvee_{i=1}^{\infty} a_i$.

In the real valued case, if $0 \leqq x < \varepsilon$ for every $\varepsilon > 0$, then $x = 0$. In the double sequence technique we need something similar. This means is given by the notion of a weakly σ-distributive lattice (see [4]).

Definition 2. A boundedly σ-complete linear lattice X is said to be weakly σ-distributive, if for every bounded double sequence $(a_{ij})_{i,j}$ such that $a_{ij} \searrow 0$ $(j \to \infty, i = 1, 2, \ldots)$ it is

$$\bigwedge_{\varphi \in N^N} \bigvee_i a_{i\varphi(i)} = 0.$$

In the paper we shall assume that X is a given weakly σ-distributive linear lattice.

Definition 3. A function f: $I \to X$ is called integrable (in the Kurzweil sense), if there exist $x \in X$ and a bounded double sequence $(a_{ij})_{i,j}$ such that $a_{ij} \searrow 0$ $(j \to \infty, i = 1, 2, \ldots)$ and for every φ: $N \to N$ there exists δ: $I \to (0, \infty)$ such that for every $D \in A(\delta)$

$$|x - S(f, D)| < \bigvee_i a_{i\varphi(i)}.$$

The element x is defined uniquely. Indeed, let $y \in X$ be another element satisfying the conditions of the preceding definition. Then

$$|y - S(f, D)| < \bigvee_i b_{i\psi(i)}$$

for every $D \in A(\delta)$, where $\delta: I \to (0, \infty)$ is a function corresponding to both functions $\varphi: N \to N$ and $\psi: N \to N$.

Then

$$|x - y| \leqq |x - S(f, D)| + |y - S(f, D)| < \bigvee_i a_{i\varphi(i)} + \bigvee_i b_{i\psi(i)}.$$

Therefore

$$|x - y| \leqq \bigwedge_{\varphi \in N^N} \bigvee_i a_{i\varphi(i)} + \bigvee_i b_{i\psi(i)} = \bigvee_i b_{i\psi(i)}$$

and

$$|x - y| \leqq \bigwedge_{\psi \in N^N} \bigvee_i b_{i\psi(i)} = 0.$$

Therefore we may put

$$x = \int f \, dm.$$

Theorem 4. If f, g: $I \to X$ are integrable and $\alpha, \beta \in R$, then $\alpha f + \beta g$ is integrable and

$$\int (\alpha f + \beta g) \, dm = \alpha \int f \, dm + \beta \int g \, dm.$$

Proof. We prove the theorem in two steps.

1. If f, g are integrable, then $f + g$ is integrable and

$$\int (f + g) \, dm = \int f \, dm + \int g \, dm.$$

Indeed, there are $b_{ij} \searrow 0$, $c_{ij} \searrow 0$ ($j \to \infty$, $i = 1, 2, \ldots$) such that for every $\varphi: N \to N$ there is $\delta: I \to (0, \infty)$ such that for every $D \in A(\delta)$

$$\left|\int f \, dm - S(f, D)\right| < \bigvee_i b_{i\varphi(i)},$$

$$\left|\int g \, dm - S(g, D)\right| < \bigvee_i c_{i\varphi(i)}.$$

Evidently $S(f + g, D) = S(f, D) + S(g, D)$. Therefore

$$\left|\int f \, dm + \int g \, dm - S(f + g, D)\right| < \bigvee_i b_{i\varphi(i)} + \bigvee_i c_{i\varphi(i)} \leqq \bigvee_i a_{i\varphi(i)},$$

where $a_{ij} = b_{ij} + c_{ij}$.

2. If f is integrable and $c \in R$ then cf is integrable and

$$\int cf \, \mathrm{d}m = c \int f \, \mathrm{d}m.$$

Namely, if $(a_{ij})_{i,j}$ is bounded and $a_{ij} \searrow 0$ ($j \to \infty$, $i = 1, 2, \ldots$) then $(|c| \, a_{ij})_{i,j}$ has these properties, too. further

$$|c \int f \, \mathrm{d}m - S(cf, D)| = |c| \, |\int f \, \mathrm{d}m - S(f, D)| \leqq \bigvee_i |c| a_{i\varphi(i)}$$

for sufficiently fine D.

Theorem 5. If f is integrable and nonnegative, then

$$\int f \, \mathrm{d}m \geqq 0.$$

Proof. There is a bounded $(a_{ij})_{i,j}$ such that $a_{ij} \searrow 0$ ($j \to \infty$ $i = 1, 2, \ldots$) and

$$\int f \, \mathrm{d}m \geqq S(f, D) - \bigvee_i a_{i\varphi(i)}$$

for any φ: $N \to N$ and any sufficiently fine D. Evidently $S(f, D) \geqq 0$. Therefore

$$-\int f \, \mathrm{d}m \leqq \bigvee_i a_{i\varphi(i)}$$

for every φ: $N \to N$, hence by the weak σ-distributivity of X we obtain $-\int f \, \mathrm{d}m \leqq 0$.

Lemma 6. Let X be boundedly complete. Then a function f: $I \to X$ is integrable if and only if the following condition is satisfied:

There exists a bounded sequence $(a_{ij})_{i,j}$ of elements of X such that $a_{ij} \searrow 0$ ($j \to \infty$, $i = 1, 2, \ldots$) and for every φ: $N \to N$ there is δ: $I \to (0, \infty)$ with

$$|S(f, D_1) - S(f, D_2)| < \bigvee_i a_{i\varphi(i)}$$

for every D_1, $D_2 \in A(\delta)$.

Proof. Evidently the condition is necessary. Now we prove that it is sufficient. To every φ: $N \to N$ there is $\delta(\varphi)$ with the property stated allow. Put

$$T = \{\delta \colon I \to (0, \infty); \ \exists \varphi \colon N \to N, \ \delta = \delta(\varphi)\}.$$

Then for $\delta \in T$ the set

$$\{S(f, D); \ D \in A(\delta)\}$$

is bounded. Since X is boundedly complete, there exist

$$a_\delta = \bigwedge_{D \in A(\delta)} S(f, D), \quad b_\delta = \bigvee_{D \in A(\delta)} S(f, D).$$

If δ_1, $\delta_2 \in T$, then $\delta = \min(\delta_1, \delta_2) \leqq \delta_1, \delta_2$, so $A(\delta) \subset A(\delta_1) \cap A(\delta_2)$, hence

$\{S(f, D);\ D \in A(\delta)\}$ is bounded, too, and

$$a_{\delta_1} = \bigwedge_{D \in A(\delta_1)} S(f, D) \leqq \bigwedge_{D \in A(\delta)} S(f, D) \leqq \bigvee_{D \in A(\delta)} S(f, D) \leqq \bigvee_{D \in A(\delta_2)} S(f, D) = b_{\delta_2}.$$

Therefore

$$\bigvee_{\delta \in T} a_\delta \leqq \bigwedge_{\delta \in T} b_\delta,$$

hence there exists $x \in X$ such that

$$a_\delta \leqq x \leqq b_\delta$$

for all $\delta \in T$. Now let $\varphi\colon N \to N$. Then there is $\delta(\varphi)\colon I \to (0, \infty)$ such that

$$S(f, D_1) \leqq S(f, D_2) + \bigvee_i a_{i\varphi(i)}$$

for all $D_1, D_2 \in A(\delta(\varphi))$. Fix D_2. Then

$$b_{\delta(\varphi)} \leqq S(f, D_2) + \bigvee_i a_{i\varphi(i)}.$$

Since the inequality holds for every $D_2 \in A(\delta(\varphi))$, we have

$$b_{\delta(\varphi)} \leqq a_{\delta(\varphi)} + \bigvee_i a_{i\varphi(i)}.$$

By the weak σ-distributivity of X we obtain $\bigwedge_\varphi \bigvee_i a_{i\varphi(i)} = 0$ and so

$$\bigwedge_\varphi b_{\delta(\varphi)} - \bigvee_\varphi a_{\delta(\varphi)} = \bigwedge_\varphi (b_{\delta(\varphi)} - a_{\delta(\varphi)}) \leqq 0$$

hence

$$x = \bigwedge_\varphi b_{\delta(\varphi)} = \bigvee_\varphi a_{\delta(\varphi)}.$$

Then for every $D \in A(\delta(\varphi))$

$$S(f, D) - x \leqq b_{\delta(\varphi)} - a_{\delta(\varphi)} \leqq \bigvee_i a_{i\varphi(i)}$$

and similarly

$$x - S(f, D) \leqq b_{\delta(\varphi)} - a_{\delta(\varphi)} \leqq \bigvee_i a_{i\varphi(i)}$$

so that

$$|S(f, D) - x| \leqq \bigvee_i a_{i\varphi(i)}$$

and the proof is complete.

Theorem 7. Let X be boundedly complete, E, F, G be compact intervals, F,

G be non-overlapping and $E = F \cup G$. If $f \colon E \to X$ is integrable, then the restrictions $f|F$ and $f|G$ are integrable too and

$$\int_E f \, \mathrm{d}m = \int_F f \, \mathrm{d}m + \int_G f \, \mathrm{d}m.$$

Proof. By Lemma 6 there is a_{ij} such that for every $\varphi \colon N \to N$ there is $\delta \colon E \to (0, \infty)$ that

$$|S(f, D_1) - S(f, D_2)| < \bigvee_i a_{i\varphi(i)}$$

for every $D_1, D_2 \in A(\delta)$. Put $\delta_0 = \delta|F$ and let $D, D' \in A(\delta_0)$.

By Lemma 1 there exists $D_0 \in A(\delta|G)$. Put $D_1 = D \cup D_0$, $D_2 = D' \cup D_0$. Then $D_1, D_2 \in A(\delta)$, so that

$$|S(f, D_1) - S(f, D_2)| < \bigvee_i a_{i\varphi(i)}.$$

But

$$S(f, D_1) = S(f, D) + S(f, D_0)$$

$$S(f, D_2) = S(f, D') + S(f, D_0),$$

so that

$$|S(f, D) - S(f, D')| < \bigvee_i a_{i\varphi(i)}$$

for all $D, D' \in A(\delta_0)$. Hence f is integrable on F by Lemma 6.

Since f is integrable on E, we have

$$\left| S(f, D) - \int_E f \, \mathrm{d}m \right| < \bigvee_i a_{i\varphi(i)}$$

for all $D \in A(\delta)$. Choose $\delta_1 < \delta|F$ such that

$$\left| S(f, D_1) - \int_F f \, \mathrm{d}m \right| < \bigvee_i b_{i\varphi(i)}$$

for every $D_1 \in A(\delta_1)$. Similarly there exists $\delta_2 < \delta|G$ such that

$$\left| S(f, D_2) - \int_G f \, \mathrm{d}m \right| < \bigvee_i c_{i\varphi(i)}$$

for every $D_2 \in A(\delta_2)$. Evidently $D_1 \cup D_2 \in A(\delta)$, so that

$$\left| S(f, D_1 \cup D_2) - \int_E f \, \mathrm{d}m \right| < \bigvee_i a_{i\varphi(i)}$$

Since $S(f, D_1 \cup D_2) = S(f, D_1) + S(f, D_2)$, we obtain

$$\left|\int_E f\,\mathrm{d}m - \int_F f\,\mathrm{d}m - \int_G f\,\mathrm{d}m\right| \leqq \left|\int_E f\,\mathrm{d}m - S(f, D_1 \cup D_2)\right| +$$

$$+ \left|s(f, D_1) - \int_F f\,\mathrm{d}m\right| + \left|s(f, D_2) - \int_G f\,\mathrm{d}m\right| <$$

$$< \bigvee_i a_{i\varphi(i)} + \bigvee_i b_{i\varphi(i)} + \bigvee_i c_{i\varphi(i)}$$

Theorem 8. If $f\colon I \to X$ is a simple measurable function, $f = \sum_{i=1}^{n} x_i \chi_{E_i}$, where $x_i \in X$, E_i are disjoint Borel subsets of I and m is a finite Borel measure defined on the σ-algebra of Borel subsets of I, then f is integrable and

$$\int f\,\mathrm{d}m = \sum_{i=1}^{n} x_i m(E_i).$$

Proof. Evidently, by Theorem 4 it suffices to prove that $x\chi_E$ is integrable (for $x \in X$ and E Borel) and $\int x\chi_E\,\mathrm{d}m = m(E)\,x$. Assume first that E is compact and $x \geqq 0$. To every positive real number ε there is an open set $U \supset E$ such that $m(U \setminus E) < \varepsilon$. Since E is compact there is $\delta\colon I \to (0, \infty)$ such that

$$(t - \delta(t), t + \delta(t)) \subset U \quad \text{for} \quad t \in E$$

$$(t - \delta(t), t + \delta(t)) \cap E = \emptyset \quad \text{for} \quad t \notin E.$$

Let $D \in A(\delta)$, $D = \{(E_1, t_1), \ldots, (E_n, t_n)\}$. Then $E \subset \cup\{E_i;\, t_i \in E\}$, hence

$$m(E) \leqq \sum_{t_i \in E} m(E_i) = m\left(\bigcup_{t_i \in E} E_i\right) \leqq m(U) \leqq m(E) + \varepsilon.$$

Further

$$\sum_{t_i \in E} xm(E_i) = \sum_{t_i \in E} x\chi_E(t_i)\,m(E_i) + \sum_{t_i \notin E} x\chi_E(t_i)\,m(E_i) =$$

$$= \sum_{i=1}^{n} x\chi_E(t_i)\,m(E_i) = S(x\chi_E, D).$$

So

$$xm(E) \leqq S(x\chi_E, D) \leqq xm(E) + \varepsilon x,$$

$$|S(x\chi_E, D) - xm(E)| \leqq \varepsilon x$$

hence it suffices to put $a_{ij} = x\frac{1}{j}$ $(i = 1, 2, \ldots)$. If E is an arbitrary Borel set, then there are compact F and open U such that $F \subset E \subset U$ $m(U \setminus F) < \varepsilon$. By the preceding

$$\int x\chi_F \, \mathrm{d}m = m(F)x.$$

Evidently $\chi_U = 1 - \chi_{I \setminus U}$, hence also

$$\int x\chi_U \, \mathrm{d}m = \int x \, \mathrm{d}m - \int x\chi_{I \setminus U} \, \mathrm{d}m = xm(I) - xm(I \setminus U) = m(U)x.$$

Further

$$S(x\chi_F, D) \leqq S(x\chi_E, D) \leqq S(x\chi_U, D),$$

so

$$|S(x\chi_E, D) - m(E)x| < 2\varepsilon x$$

for sufficiently fine D. Hence we have proved the equality $\int x\chi_E \, \mathrm{d}m = xm(E)$ for nonnegative x. In the general case one can use the Riesz decomposition $x = x^+ - x^-$ and the linearity of the integral.

Now we may summarize. We started with a finite Borel measure m on a compact interval and functions with values in a weakly σ-distributive linear lattice X. Recall that the conditions imposed on X are very general, because the weak σ-distributivity of X is a necessary and sufficient condition for the extendability of X-valued measures and X-valued Daniell integrals [4]. Under the weak conditions we defined the Kurzweil integral and proved that it is a linear positive operator and it coincides with the Lebesgue integral on simple measurable functions. Only the additivity of the integral (Theorem 7) was proved with an additional condition that X is boundedly complete (the weak σ-distributivity asks only bounded σ-completness).

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SÚHRN

O KURZWEILOVOM INTEGRÁLI PRE FUNKCIE S HODNOTAMI V USPORIADANÝCH PRIESTOROCH I

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V článku je uvedená definícia a niektoré elementárne vlastnosti Kurzweilovho integrálu pre funkcie s hodnotami v lineárnom zväze.

РЕЗЮМЕ

ОБ ИНТЕГРАЛЕ КУРЗВЕИЛА ДЛЯ ФУНКЦИЙ СО ЗНАЧЕНИЯМИ В УПОРЯДОЧЕННЫХ ПРОСТРАНСТВАХ I

БЕЛОСЛАВ РИЕЧАН, Липтовски Микулаш

В статье приведено определение и некоторые основные свойства интеграла Курзвеила для функций со значениями в линейной решетке.