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ON SOME CLASES OF SETS RELATED TO GENERALIZED CONTINUITY

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Key Words

α-set, semi-open set, preopen set, semi-homeomorphism.

Abstract

Studying various types of generalized continuity notions, various classes of sets asociated to the original topology are useful. The present paper deals with the connections of such classes and mappings which are continuous with respect to them. Some completions of results of several authors are given. Also errors appearing in some papers are corrected.

Notations and notions

Throughout the present note (X, T), (Y, U) etc. will denote topological spaces. The closure of a set S will be denoted by $\operatorname{cl} S$. The interior of a set S will be denoted by $\operatorname{int} S$.

Let (X, T) be a topological space. A subset S of X is said to be

(i) an α-set if $S \subset \operatorname{int}(\operatorname{cl}(\operatorname{int} S))$ [8],

(ii) a semi-open set or semi-open if $S \subset \operatorname{cl}(\operatorname{int} S)$ [4],

(iii) a preopen set or preopen if $S \subset \operatorname{int}(\operatorname{cl} S)$ [6].

In [8] Njåstad used the term β-set for a semi-open set.

In our paper the families of all α-sets, of all semi-open sets and of all preopen sets in (X, T) are denoted by $\mathrm{AO}(X, T)$, $\mathrm{SO}(X, T)$, and $\mathrm{PO}(X, T)$ respectively.

Every open set is an α-set. Every α-set is semi-open and preopen, Conversely, if S is semi-open and preopen then S is an α-set ([9], [10]).

1 Topologies with the same systems of α-sets, semi-open sets and preopen sets

We start this section with the following simple

Proposition 1. There exist topological spaces (X, T), (X, U) such that $\mathrm{SO}(X, T) \subset \mathrm{SO}(X, U)$ while $T \not\subset U$.

Proof. It is sufficient to consider $X = \{a, b, c\}$ with $T = \{\emptyset, X, \{a\}\}$ and $U = \{\emptyset, X, \{a, b\}\}$. Then it is easy to see that a nonempty set $A \subset X$ belongs to $\mathrm{SO}(X, T)$ if and only if $a \in A$. Similarly a nonempt set $B \subset X$ belongs to $\mathrm{SO}(X, U)$ if and only if $\{a, b\} \subset B$. Thus $\mathrm{SO}(X, U) \subset \mathrm{SO}(X, T)$ while $T \not\subset U$.

Remark 1. The above proposition shows that Theorem 10 in [4] is not true. (A misunderstanding caused by unappropriate notation in the proof of the mentioned theorem was the reason of obtaining the false assertion.)

Example 1. $X = \{a, b, c\}$, $T = \{\emptyset, X, \{a\}\}$, $U = T \cup \{\{a, b\}\}$. We can see that $\mathrm{SO}(X, T) = \mathrm{SO}(X, U) = \{A \subset X;\ a \in X\} \cup \{\emptyset\}$. But evidently $T \neq U$, $T \subset U$.

Definition 1. Let T, U be two topologies on X. The topologies T, U are said to be α-equivalent if and only if $\mathrm{AO}(X, T) = \mathrm{AO}(X, U)$. The equivalence classes are called α-classes ([8]).

Remark 2. The class of all topologies on X is an union of all α-classes (which are pairwise disjoint). It was proved that if T is a topology on X then $\mathrm{AO}(X, T)$ is a topology on X too. Moreover $\mathrm{AO}(X, T)$ is a member of the α-class which is generated by T and $\mathrm{AO}(X, T)$ is the finest topology of this α-class [8].

Considering two topologies T, U on X the following equivalence holds: $\mathrm{AO}(X, T) = \mathrm{AO}(X, U) \Leftrightarrow \mathrm{SO}(X, T) = \mathrm{SO}(X, U)$ [8].

We are going to prove a similar assertion for classes of α-sets and preopen sets.

Lema 1. Let (X, T) be a topological space. Let $A \subset X$ be an α-set. Then for an arbitrary preopen set P the set $A \cap P$ is preopen.

Proof. A is open in the topological space $(X, \mathrm{AO}(X, T))$. It is well-known that intersection of an open set and a preopen one is a preopen set and by [1] $\mathrm{PO}(X, T) = \mathrm{PO}(X, \mathrm{AO}(X, T))$.

Lema 2. Let (X, T) be a topological space such that for every $a \in X$ either $\{a\}$ is closed or $\mathrm{int}(\mathrm{cl}\{a\})$ is empty. If $Z \subset X$, nonempty and such, that for an arbitrary preopen P, $Z \cap P$ is preopen, then $\mathrm{int}\, Z \neq \emptyset$.

Proof. Let us assume $\mathrm{int}\, Z = \emptyset$. Hence $\mathrm{cl}(X - Z) = X$. Let us choose $z \in Z$ and let us denote $P := \{z\} \cup (X - Z)$. $\mathrm{Cl}\, P = X$ holds so P is preopen. Hence $Z \cap P = \{z\}$ have to be preopen. But either $\{z\}$ is closed and $\mathrm{int}(\mathrm{cl}\{z\}) = \mathrm{int}\{z\} = \emptyset$ or $\mathrm{int}(\mathrm{cl}\{z\}) = \emptyset$. So $\{z\}$ is not preopen ($\{z\} \not\subset \mathrm{int}(\mathrm{cl}\{z\})$), and it is a contradiction.

Lema 3. Let (X, T) be a topological space such that for every $x \in X$ either $\{x\}$ is closed or $\mathrm{int}(\mathrm{cl}\{x\})$ is empty. If $A \subset X$ is such, that for every P preopen $A \cap P$ is preopen, then $A \in \mathrm{AO}(X, T)$.

Proof. Let A be nonempty. By Lema 2 $\mathrm{int}\, A \neq \emptyset$. We are going to show $A \subset \mathrm{cl}(\mathrm{int}\, A)$. Let $a \in A$. Let us assume $a \notin \mathrm{cl}(\mathrm{int}\, A)$. Let us denote $U := X - \mathrm{cl}(\mathrm{int}\, A)$. So $A \cap U$ is nonempty set. Let P be an arbitrary preopen set.

$(A \cap U) \cap P$ is preopen too, since $(A \cap U) \cap P$ equals $A \cap (U \cap P)$ and $U \cap P$ is preopen since U is open. So for an arbitrary preopen set P the set $A \cap U \cap P$ is preopen. By Lema 2 $\operatorname{int}(A \cap U) \neq \emptyset$. But it is a contradiction since $U \cap \cap \operatorname{int} A = \emptyset$.

We have proved $A \subset \operatorname{cl}(\operatorname{int} A)$ so A is semi-open. X is preopen set hence $A \cap X = A$ is preopen too. A is semi-open and preopen so by [10] A is α-set.

Remark 3. We can see that Lema 2 and Lema 3 hold for every T_1 topological space.

Lema 1, Lema 2 and Lema 3 have this

Corollary 1. Let (X, T) be T_1 topological space. Then $A \subset X$ is an α-set if and only if for every $P \in \mathrm{PO}(X, T)$ $P \cap A$ is preopen.

Example 2. Let $X = \langle 0, +\infty)$. Let $T = \{\emptyset\} \cup \{\langle 0, b); b > 0\}$. Let $P = \{0\}$ It holds: for every $A \in \mathrm{PO}(X, T)$ $A \cap P \in \mathrm{PO}(X, T)$. But $P \notin \mathrm{AO}(X, T)$.

2 Homeomorphisms

We give the following definitions 1.1, 1.2 and 1.3 from the paper [3]

Definition 1.1. A function $f: (X, T) \to (Y, U)$ is said to be irresolute if and only if for any semi-open subset S of Y $f^{-1}(S)$ is semi-open in X.

Definition 1.2. A function $f: (X, T) \to (Y, U)$ is said to be pre-semiopen if and only if, for all $A \in \mathrm{SO}(X, T), f(A) \in \mathrm{SO}(Y, U)$.

Definition 1.3. Let (X, T) and (Y, U) be topological spaces. (X, T) and (Y, U) are said to be semi-homeomorphic if and only if there exists a function $f: X \to Y$ such that f is one-to-one, onto, irresolute and pre-semiopen. Such an f is called a semi-homeomorphism.

Every Homeomorphism is a semi-homeomorphism. A semi-homeomorphism need not be a homeomorphism [3].

We are going to define a notion of an α-homeomorphism. We shall show that a function f is semi-homeomorphism if and only if it is an α-homeomorphism.

In 1980 S. N. Maheshwari and S. S. Thakur [5] introduced a notion of an α-irresolute function.

Definition 2. ([5]) A mapping $f: (X, T) \to (Y, U)$ is said to be α-irresolute if the inverse image of every α-set of Y is an α-set in X.

So a function f could be defined to be an "α-homeomorphism" in the following manner:

Definition 3. A mapping $f: (X, T) \to (Y, U)$ is said to be an α-homeomorphism if and only if f is one-to-one, onto and f and f^{-1} are α-irresolute.

Definition 4. A mapping $f: (X, T) \to (Y, U)$ is said to be a pre-homeomorphism if and only if f is one-to-one, onto and for every $A \in \mathrm{PO}(X, T)$ the set $f(A)$ belongs to $\mathrm{PO}(Y, U)$ and for every $B \in \mathrm{PO}(Y, U)$ the set $f^{-1}(B)$ belongs to $\mathrm{PO}(X, T)$.

Lema 4. Let (X, T), (Y, U) be two T_1 topological spaces. Let $f: X \to Y$ be pre-homeomorphism then f is α-homeomorphism.

Proof. We shall prove that if $A \in \mathrm{AO}(X, T)$ then $f(A) \in \mathrm{AO}(Y, U)$. Since Y is T_1 it is sufficient to prove (see Lema 3) that intersection of $f(A)$ with an arbitrary preopen set is a preopen set. Let $P \subset Y$ be preopen. $f(A) \cap P = f(A) \cap f(B)$ where $B = f^{-1}(P)$ is preopen. $f(A) \cap f(B) = f(A \cap B)$ which is an image of a preopen set (Corollary 1); therefore it is a preopen set.

Lema 5. Let $f: (X, T) \to (Y, U)$ be a semi-homeomorphism. Then f is a pre-homeomorphism.

Proof. $f: (X, \mathrm{AO}(X, T)) \to (Y, \mathrm{AO}(Y, U))$ is a homeomorphism (by [3]). Moreover $\mathrm{AO}(X, T)$ and T ($\mathrm{AO}(Y, U)$ and U respectively) have the same systems of preopen sets ([1] Theorem 2.9). Now it is sufficient to prove that a homeomorphism is a pre-homeomorphism.

Let $g: (X_1, T_1) \to (Y_1, U_1)$ be a homeomorphism. Let $P \subset X_1$, $P \subset \operatorname{int}\operatorname{cl} P$ then $g(P) \subset g(\operatorname{int}\operatorname{cl} P)$, g is a homeomorphism so $g(\operatorname{int}\operatorname{cl} P) = \operatorname{int}\operatorname{cl} g(P)$ hence $g(P) \subset \operatorname{int}\operatorname{cl} g(P)$ and $g(P)$ is preopen.

Remark 4. It is easy to show that every α-homeomorphism is a semi-homeomorphism. In fact if $f: (X, T) \to (Y, U)$ is an α-homeomorphism then $f: (X, \mathrm{AO}(X, T)) \to (Y, \mathrm{AO}(Y, U))$ is a homeomorphism. So $f: (X, \mathrm{AO}(X, T)) \to$ $\to (Y, \mathrm{AO}(Y, U))$ is a semi-homeomorphism. But $\mathrm{SO}(X, T)$ equals $\mathrm{SO}(X, \mathrm{AO}(X, T))$ and $\mathrm{SO}(Y, U)$ equals $\mathrm{SO}(Y, \mathrm{AO}(Y, U))$. Hence $f: (X, T) \to (Y, U)$ is a semi-homeomorphism too.

Lema 4, Lema 5 and Remark 4 give us the following

Theorem 1. Let (X, T), (Y, U) be two T_1 topological spaces. Then $f: (X, T) \to (Y, U)$ is a pre-homeomorphism $\Leftrightarrow f$ is a semi-homeomorphism $\Leftrightarrow f$ is an α-homeomorphism.

Corollary 2. Let (X, T), (X, U) be two T_1 topological spaces. Then $\mathrm{AO}(X, T) = \mathrm{AO}(X, U) \Leftrightarrow \mathrm{PO}(X, T) = \mathrm{PO}(X, U) \Leftrightarrow \mathrm{SO}(X, T) = \mathrm{SO}(X, U)$.

Example 3. Let $X = \{1, 2, 3\}$. Let $T = 2^X$. Let $U = \{\emptyset, \{1\}, \{2, 3\}\}$. Then $\mathrm{PO}(X, T) = \mathrm{PO}(X, U) = 2^X$. But evidently $\mathrm{AO}(X, T) \neq \mathrm{AO}(X, U)$.

It is an open problem to find two T_0 topologies T, U on a set X such, that $\mathrm{PO}(X, T) = \mathrm{PO}(X, U)$ but $\mathrm{AO}(X, T) \neq \mathrm{AO}(X, U)$.

3 Remark on semi-topological groups

Bohn and Lee introduced the notion of a semi-topological group in [2]. Unfortunately a slight error in their considerations in Theorem 5 in [2] caused that this theorem and also some further assertions in their article are false. We give here some counterexamples to the mentioned statements.

Definition 5. A set M_x is a semi-neighbourhood of a point $x \in X$ if there exists $A \in \mathrm{SO}(X, T)$ such that $x \in A \subset M_x$.

Definition 6. For subsets A, B of a group $(G, +)$ symbols $-A$ and $A + B$ denote the following sets: $-A = \{a;\ -a \in A\}$; $A + B = \{a + b;\ a \in A \text{ and } b \in B\}$.

Definition 7. A triple $(G, +, T)$ is termed a semi-topological group if $(G, +)$ is a group, $X = (G, T)$ is a topological space and for each neighbourhood $N_{(x-y)}$ of $x - y$ there exist semi-neighbourhoods M_x and M_y (of x and y) such that $M_x + (-M_y) \subset N_{(x-y)}$.

In the following examples a semi-topological group is given, which does not fulfil some assertions of [2].

Example 4. Let U denote a usual topology on R (real line). Let $T = \{V \subset R;\ V = W - \{1\} \text{ and } W \in U\} \cup \{R\}$. Let us denote $(R, +, T) = S$. We can see that (R, T) is a topological space. For proving that S is a semi-topological group it is sufficient to prove the following statement: If $c \in R$, $c = a - b$ then for any open O (such that $c \in O$) there exist semi-neighbourhoods V_a, V_b of a, b such, that $V_a + (-V_b) \subset O$.

First we prove that every open interval (a, b) belongs to $\mathrm{SO}(R, T)$. There are two cases.

(a) if $1 \notin (a, b)$ then $(a, b) \in T$,

(b) if $1 \in (a, b)$ then $(a, b) - \{1\}$ belongs to T and $(a, b) \in \mathrm{SO}(R, T)$.

In fact $(a, b) = ((a, b) - \{1\}) \cup \{1\}$. So putting $L = (a, b) - \{1\}$ we have that $L \in T$ and $L \subset (a, b) \subset \mathrm{cl}\, L$.

So let $c = a - b$.

(i) Let $c = 1$. R is the only neighbourhood of the point 1 so $O = R$ and the proof is trivial.

(ii) Let $c \neq 1$. Let V be a neighbourhood of the point c. From the definition of the topology T it follows that there exists $d > 0$ such that an open interval $(c - 2d, c + 2d)$ is a subset of V. Let us choose $V_a = (a - d, a + d)$, $V_b = (b - d, b + d)$. V_a and V_b are semi-neighbourhoods of a, b respectively and $V_a + (-V_b) = (a - b - 2d,\ a - b + 2d) = (c - 2d,\ c + 2d) \subset V$ holds.

S is proved to be a semi-topological group.

Example 5. Let us consider $S = (R, +, T)$ defined above. Let $A = \{1\} \cup \cup (0; 0{,}5)$. $A \in \mathrm{SO}(R, T)$ but $-A = \{-1\} \cup (-0{,}5; 0)$ is not semi-open. So in a semi-topological group it holds: if A is semi-open then $-A$ need not be semi-open. Thus the assertion of Theorem 5 in [2] is false.

Example 6. Note that the translation $A + a$ of a semi-open set in a semi-topological group need not be semi-open what is a contradiction with Theorem 8 in [2]. Take $A = \{1\} \cup (0; 0{,}5)$. $A \in \mathrm{SO}(R, T)$. Let $a = 10$. Then $A + a = \{11\} \cup (10; 10{,}5)$. But this set is not semi-open in $(R, +, T)$.

Example 7. There is a false assertion in the proof of the Theorem 5 in [2]. This assertion says: if $(G, +, T)$ is a semi-topological group and $V \in T$ then $-(\mathrm{cl}\, V) = \mathrm{cl}(-V)$. Let us take $V = (0; 0{,}5)$. $V \in T$. $\mathrm{cl}\, V = \langle 0; 0{,}5 \rangle \cup \{1\}$. $-(\mathrm{cl}\, V) = \{-1\} \cup \langle -0{,}5; 0 \rangle$ but $\mathrm{cl}(-V) = \langle -0{,}5; 0 \rangle \cup \{1\}$ so $-\mathrm{cl}\, V \neq \mathrm{cl}(-V)$.

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SÚHRN

O NIEKTORÝCH SYSTÉMOCH MNOŽÍN SÚVISIACICH SO ZOVŠEOBECNENOU SPOJITOSŤOU

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Pri štúdiu rozličných druhov zovšeobecnenej spojitosti je užitočné skúmať rôzne systémy množín, súvisiace istým spôsobom s pôvodnou topológiou. Článok sa zaoberá vzťahmi medzi systémami semiotvorených množín, preotvorených množín a α-množín. Skúmajú sa rôzne typy homeomorfizmov spojité vzhľadom na tieto typy množín. Článok dopĺňa výsledky niekoľkých autorov a opravuje chyby, vyskytujúce sa v niektorých prácach.

РЕЗЮМЕ

О НЕКОТОРЫХ КЛАССАХ МНОЖЕСТВ СВЯЗАННЫХ С ОБОБЩЁННОЙ НЕПРЕРЫВНОСТЬЮ

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Изучая разные виды обобщённой непрерывности полезно рассматривать разные классы множеств связанные определённым образом с начальной топологией. В работе

рассматриваются взаимоотношения между системами полуоткрытых множеств, предоткрытых множеств и α-множеств. Изучаются разные виды гомеоморфизмов непрерывных в отношении к этим типом множеств. Статья дополняет результаты некоторых авторов и исправляет ошибки, находящиеся в некоторых работах.