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ON THE SEQUENCES OF BOUNDED VARIATION

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Introduction

The sequences of bounded variation have a significant place in formulation of different criteria for convergence of functional series.

Let (X, d) be a metric space and $x = \{x_n\}_{n=1}^{\infty}$ be a sequence of its elements. The number

$$V(x) = \sum_{k=1}^{\infty} d(x_{k+1}, x_k) \in \langle 0, +\infty \rangle$$

is said to be a variation of the sequence x. If $V(x) < +\infty$, then x is said to be a sequence of bounded variation.

The present paper deals with the structure of the sequences of bounded variation in the space of all convergent sequences. Moreover, in this paper we shall study the properties of functions which preserve bounded variation of sequences.

1 Sequences of Bounded Variation in the Space of Convergent Sequences

In this part of the paper we shall investigate the position of the set W of all sequences of real numbers of bounded variation in the space C of all convergent sequences.

The space C is a Banach normed linear space with the norm

$$\|x\| = \sup_{n=1,2,\dots} |x_n| \qquad (x = \{x_n\}_{n=1}^{\infty} \in C)$$

Proposition 1.1. If the sequence $x = \{x_n\}_{n=1}^{\infty}$ of elements of a metric space X has bounded variation then it is fundamental.

Proof. Let $\varepsilon > 0$. Since $V(x) < +\infty$, there exists an n_0 such that

$$\sum_{k > n_0} d(x_{k+1}, x_k) < \varepsilon.$$

Let $p \in N$ (N is the set of positive integers). Using the triangular inequality we obtain

$$d(x_{n+1}, x_{n+p}) \leq d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + \ldots + d(x_{n+p-1}, x_{n+p}) < \varepsilon.$$

The proof is finished.

Corollary 1.1. If $x = \{x_n\}_{n=1}^{\infty}$ has bounded variation and (X, d) is a complete metric space, then x is a convergent sequence. Hence we have $W \subset C$.

The position of the set W in the set C is described by the following theorem.

Theorem 1.1. The set W is an F_σ set, dense in C, first Baire category in C.

Proof. At first we shall prove, that W is dense in C. Let $c = \{c_n\}_{n=1}^{\infty} \in C$, and $\varepsilon > 0$. We shall prove that there is such a sequence $a = \{a_n\}_{n=1}^{\infty} \in W$ that $\|a - c\| < \varepsilon$. Since $c \in C$, there is $t \in R$ such that $c_n \to t$. Then there is n_0 such that for $n > n_0$ we have $|c_n - t| < \varepsilon/2$. Define $a = \{a_n\}_{n=1}^{\infty}$ in the following way:

$$a_n = c_n \quad \text{for} \quad n \leq n_0$$

$$a_n = t \quad \text{for} \quad n > n_0$$

It is obvious that $a \in W$ and for each $n \in N$ we have $|c_n - a_n| \leq \varepsilon/2$ and so $\|c - a\| < \varepsilon$.

Let $x = \{x_n\}_{n=1}^{\infty} \in C$. We put

$$V_n(x) = |x_2 - x_1| + \ldots + |x_n - x_{n-1}| \qquad (n = 1, 2, \ldots)$$

For $n \in N$ and $K \in N$ we define

$$B(n, K) = \{x \colon W_n(x) \leq K\}$$

Then it is obvious that

$$W = \bigcup_{K=1}^{\infty} \bigcap_{n=1}^{\infty} B(n, K). \tag{1}$$

It is easy to prove that the set $B(n, K)$ is closed in C for each n and K. therefore according to (1) the set W is an F_σ set in C.

We shall prove that W is the set of the first Baire category in C. It is sufficient to prove that the set

$$H = \bigcap_{n=1}^{\infty} B(n, K), \qquad K \in N$$

is nowhere dense in C. Since this set is closed in C, it is sufficient to prove that the set $C - H$ is dense in C.

We prove that for each $x \in C$ and $\delta > 0$ there is a sequence $y = \{y_n\}_{n=1}^{\infty}$ such that $\|y - x\| < \delta$ and $W_n(y) > K$ for all sufficiently large n. Since $x \in C$, there is an $a \in R$ such that $x_j \to a$. Choose $N_0 > 1$ such that $|x_j - a| < \delta/2$ for $j \geq N_0$.

Let y be a sequence defined in the following way:

$$x_j = x_j \quad \text{for} \quad j < N_0$$

Choose a positive integer L such that $L\delta > K$ and put

$$y_j = a + \delta/2 \quad \text{for} \quad N_0 \le j \le N_0 + L + 1, \quad j \text{ is odd,}$$
$$y_j = a - \delta/2 \quad \text{for} \quad N_0 \le j \le N_0 + L + 1, \quad j \text{ is even,}$$

$$y_j = a \quad \text{for} \quad j > N_0 + L + 1.$$

Obviously $y \in C$, and

$$W_{N_0 + L + 1}(y) \ge \sum_{j = N_0 + 1}^{N_0 + L + 1} |a + \delta/2 - (a - \delta/2)| = \delta(L + 1) > K$$

and $\|x - y\| = \sup_{j = 1, 2, \dots} |x_j - y_j| < \delta.$

The proof is finished.

2 On Mappings that Preserve Boundedness of Variation of Sequences

Definition 2.1. Let (X, d) be a metric space and $f\colon X \to X$. The function f is said to preserve boundedness of variation of sequences if for each $x = \{x_n\}_{n=1}^{\infty}$ $x_n \in X$, $V(x) < +\infty$ we have $V(f(x)) < +\infty$, where $f(x) = \{f(x_1), f(x_2), \dots\}$.

The following theorem characterizes the functions $f\colon X \to X$ that preserve the bounded variation of sequences. We recall the notion of the local Lipschitz condition (see [2], [1], [4])

Definition 2.2. Let (X, d) and (Y, d') be two metric spaces, let $f\colon X \to Y$. The function f is said to be locally Lipschitz at $p \in X$, if there is an $M > 0$ and $\delta > 0$, such that for each $y \in K(p, \delta) = \{y \in K\colon d(p, y) < \delta\}$ we have

$$d'(f(y), f(p)) \le Md(y, p)$$

The function f is said to be strongly locally Lipschitz at $p \in X$, if there is an $M > 0$ and $\delta > 0$, such that for every $x, y \in K(p, \delta)$ we have $d'(f(x), f(y)) \le Md(x, y)$.

A function $f\colon X \to Y$ is said to be strongly locally Lipschitz on X if it is strongly locally Lipschitz at every point $p \in X$.

Theorem 2.1. Let (X, d) be a complete metric space. A function $f\colon X \to X$ preserves boundedness of variation of sequences if and only if it is strongly locally Lipschitz on X.

Proof. 1. Let the function f be strongly locally Lipschitz on X. We shall prove that it preserves boundedness of variation of sequences. Let $x = \{x_n\}_{n=1}^{\infty}$ be a sequence of elements of X. Let

(2) $$V(x) = \sum_{k=1}^{\infty} d(x_{k+1}, x_k) < +\infty.$$

According to corollary of Proposition 1.1 we have

$$\lim_{k \to \infty} x_k = x_0 \in X$$

Since the function f is strongly locally Lipschitz at x_0, there is an $M > 0$ and $\delta > 0$, such that

(3) $$y_1, y_2 \in K(x_0, \delta) \Rightarrow d(f(y_1), f(y_2)) \leq Md(y_1, y_2)$$

It follows from $x_k \to x_0$ that there is an n_0 such that for each $k > n_0$ we have $x_k \in K(x_0, \delta)$. Then according to (2) and (3) for $n > n_0$ we have

$$\sum_{n > n_0} d(f(x_{n+1}), f(x_n)) \leq M \sum_{n > n_0} d(x_{n+1}, x_n) < +\infty$$

and therefore

$$\sum_{n=1}^{\infty} d(f(x_{n+1}), f(x_n)) < +\infty.$$

Thus we have proved that $V(f(x)) < +\infty$.

2. Let a function $f: X \to X$ be not strongly locally Lipschitz on X. Then there is a point $q \in X$, such that f is not strongly locally Lipschitz at q. Hence to every $A > 0$ and $\delta > 0$ there are such points $y_1, y_2 \in K(q, \delta)$ that

$$d(f(y_1), f(y_2)) > Ad(y_1, y_2).$$

Put $A = 2^n$, $\delta = 2^{-n-1}$ $\quad (n = 1, 2, \dots)$

Hence to each $n = 1, 2, \dots$ there exist points $y_n, z_n \in K(q, 2^{-n-1})$ such that

(4) $$d(f(y_n), f(z_n)) > 2^n d(y_n, z_n)$$

and so $y_n \neq z_n$.

Let us remark that

(5) $$d(y_n, z_n) < 2^{-n}.$$

Let M_n be the smallest positive integer for which the inequality $2^{-n} \leq M_n d(y_n, z_n)$ holds. By (5) we have that $M_n \geq 2$ and by the definition of M_n we have

$$(M_n - 1)d(y_n, z_n) < 2^{-n}.$$

Hence according to (5)

$$M_n d(y_n, z_n) < 2^{-n} + 2^{-n} = 2^{-n+1}$$

and so

(6) $$2^{-n} \leq M_n d(y_n, z_n) < 2^{-n+1} \qquad (n = 1, 2, \ldots).$$

We shall construct a sequence $t = \{t_1, t_2, \ldots\}$ of points of the space X in the following way. In the first $M_1 + 1$ places the members are the following elements:

$$y_1, z_1, y_1, z_1, \ldots, y_1, z_1, (y_1)$$

next $M_2 + 1$ members are the following elements:

$$y_2, z_2, y_2, z_2, \ldots, y_2, z_2, (y_2).$$

We can continue this construction by induction. It is obvious (see (6)) that

$$\sum_{n=1}^{\infty} d(t_{n+1}, t_n) \leq \sum_{i=1}^{\infty} M_i d(y_i, z_i) + \sum_{i=1}^{\infty} d(y_i, y_{i+1}) + \sum_{i=1}^{\infty} d(z_i, y_{i+1}) <$$

$$< \sum_{i=1}^{\infty} 2^{-i+1} + \sum_{i=1}^{\infty} 2^{-i} + \sum_{i=1}^{\infty} 2^{-i} < +\infty.$$

On the other hand by the definition of the sequence t and (4) we have

$$\sum_{n=1}^{\infty} d(f(t_{n+1}), f(t_n)) \geq \sum_{n=1}^{\infty} M_n d(f(y_n), f(z_n)) \geq$$

$$\geq \sum_{n=1}^{\infty} M_n 2^n d(y_n, z_n) \geqq \sum_{n=1}^{\infty} 1 = +\infty.$$

The proof of the theorem is finished.

3 The Class of Functions Preserving Boundedness of Variation of Sequences in the Space of Continuous Functions

In what follows $P(R)$ denotes the set of all real functions of a real variable which preserve boundedness of variation of sequences. By $C(R, R)$ denote the linear space of all continuous functions $f\colon R \to R$.

Theorem 3.1. $P(R)$ is a linear subspace of the space $C(R, R)$

Proof: We shall prove that if $f, g \in P(R)$ and $a, b \in R$ then $(af + bg) \in P(R)$. Let $c = \{c_n\}_{n=1}^{\infty}$ be a sequence of real numbers of bounded variation. Then

$$V((af + bg)(c)) = \sum_{k=1}^{\infty} |(af + bg)(c_{k+1}) - (af + bg)(c_k)| =$$

$$= \sum_{k=1}^{\infty} |af(c_{k+1}) - af(c_k) + bg(c_{k+1}) - bg(c_k)| \leq$$

$$\leq \sum_{k=1}^{\infty} (|a|\,|f(c_{k+1}) - f(c_k)| + |b|\,|g(c_{k+1}) - g(c_k)|) =$$

$$= |a|\, V(f(c)) + |b|\, V(g(c)) < +\infty .$$

Hence $af + bg \in P(R)$ and $P(R)$ is a linear space.

Definition 3.1. Let (X, d) and (Y, d') be two metric spaces and $f: X \to Y$. Let $g, h: X \to R^+$. The function f is said to be strongly locally Lipschitz with respect to g and h, if for every $p \in X$ the following holds: if $y, z \in K(p, h(p))$, then

$$d'(f(y), f(z)) \leq g(p)\, d(y, z).$$

The class of functions is said to be a class of equally strongly locally Lipschitz functions if there are functions g and h such that each $f \in M$ is strongly locally Lipschitz with respect to g and h.

Note 3.1. In the similar way we can define equally locally Lipschitz class of functions and equally Lipschitz class of functions.

Note 3.2. If M is a class of functions each of which is strongly locally Lipschitz, then the class M need not be a class of equally strongly locally Lipschitz functions.

Theorem 3.2. Let $M = \{f_1, f_2, \ldots\}, f_i: X \to Y, i = 1, 2, \ldots$ be a class of strongly locally Lipschitz functions with respect to g and h. Let $\{f_n\}_{n=1}^{\infty}$ be uniformly convergent to a limit function f. Then f is a strongly locally Lipschitz function with respect to g and h.

Proof. Let $p \in X$. According to the assumption of the theorem for each $n \in N$ and $x, y \in K(p, h(p))$ we have

$$d'(f_n(x), f_n(y)) \leq g(p)\, d(x, y). \tag{7}$$

We show, that

$$d'(f(x), f(y)) \leq g(p)\, d(x, y). \tag{8}$$

Let $\varepsilon > 0$. Then there is an n_0 such that for $n > n_0$ we have

$$\sup_{x \in X} d(f(x), f_n(x)) < \varepsilon.$$

Then from (7) we obtain

$$d'(f(x), f(y)) \leq d'(f_n(x), f_n(y)) + d'(f_n(x), f(x)) +$$

$$+ d'(f(y), f_n(y)) \leq g(p)\, d(x, y) + 2\varepsilon.$$

The latest inequality is true for every $\varepsilon > 0$ and $x, y \in K(p, h(p))$. Hence (8) is true.

The proof of the theorem is finished.

Corollary 3.1. If the space X is complete, then for each g, h: $X \to R^+$ the class of all functions $\{f_i | f_i: X \to Y, i \in I\}$ that are strongly locally Lipschitz with respect to g and h (we shall denote it $L(X, Y, g, h)$ is colosed.

Lemma 3.1. Let X be a compact space and let

$$M = \{f_i; f_i: X \to Y, i \in I\}$$

be the set of strongly locally Lipschitz functions with respect to g and h. Then there exists an $L > 0$ such that all functions $f_i \in M$ are Lipschitz with the constant L.

Proof. Let $J = \{K(p, h(p)) : p \in X\}$ be a covering of the space X. Since the space X is compact, there is a finite subcovering of X $O = \{K(p, h(p)) : p \in P\}$, where P is a finite set. Denote

$$L = \max_{p \in P} g(p)$$

Using the triangle inequality and mathematical induction it can be easily proved that L satisfies the theorem.

Corollary 3.2. If the function f is strongly locally Lipschitz on a compact space X, then f is Lipschitz on X.

Corollary 3.3. Let X be compact. Then each strongly locally Lipschitz function belongs to a certain class $L(X, Y, n, 1)$.

By $P(X)$ we shall denote the set of all functions f_i: $X \to X$, $i \in I$, that preserve the boundedness of variation of sequences.

From corollary 3.1 and 3.3 and theorem 2.1 we obtain the following theorem.

Theorem 3.3. Let X be a compact metric space. Then $P(X)$ is an F_σ-set.

Proof. The assertion follows at once from the obvious equality

$$P(X) = \bigcup_{n=1}^{\infty} L(X, X, n, 1)$$

In the following the class $C(R, R)$ is considered as a space endowed with the topology of uniform convergence.

Theorem 3.4. The class $P(R)$ is a dense set in $C(R, R)$.

Proof. Let $f \in C(R, R)$. It suffices to show that there exists a function $g \in P(R)$ such that

$$(9) \qquad |f(x) - g(x)| < \varepsilon \quad \text{for each} \quad x \in R.$$

We partion R into a countable union of closed intervals $\langle n, n+1 \rangle$ (n is an integer). According to the well-known Weierstrass Theorem we can construct for each $n = 0, \pm 1, \pm 2, \ldots$ a polynomial P_n such that for each $x \in \langle n, n+1 \rangle$ we have $|f(x) - P_n(x)| < \varepsilon$. This construction can be realized in such a way, that $P_n(n+1) = P_{n+1}(n+1)$ $(n = 0, \pm 1, \pm 2, \ldots)$. Then it suffices to put $g(x) =$

$= P_n(x)$ for $x \in \langle n, n+1 \rangle$ ($n = 0, \pm 1, \pm 2, \ldots$) and, since each polynomial is a Lipschitz function, (9) follows.

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SÚHRN

O POSTUPNOSTIACH S KONEČNOU VARIÁCIOU

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V práci je preskúmaná štruktúra postupností s konečnou variáciou v priestore všetkých konvergentných postupností. Okrem toho sa v práci študujú vlastnosti funkcií, ktoré zachovávajú konečnú variáciu postupnosti.

РЕЗЮМЕ

О ПОСЛЕДОВАТЕЛЬНОСТЯХ С КОНЕЧНОЙ ВАРИАЦИЕЙ

ПАВОЛ РАЛБОВСКИ, Братислава

В статье рассматривается структура последовательностей с конечной вариацией в пространстве всех сходящихся последовательностей. Кроме того в статье рассматриваются свойства фунций сохраняющих конечную вариацию последовательностей.