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Label: Article

Jahr: 1989

PURL: https://resolver.sub.uni-goettingen.de/purl?312901348_54-55|log7

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ACTA MATHEMATICA UNIVERSITATIS COMENIANAE
LIV—LV—1988

A REMARK ON A FLOW WITH A HOMOCLINIC TRAJECTORY

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1 Introduction

We show the existence of nontrivial positively invariant sets for certain flows with homoclinic trajectories.

Our proof is based on the Ważewski theorem (on its version stated in [1]) and the main tool is the fundamental group. An analogous result for a flow with a knotted trajectory can be found in [1].

2 Ważewski principle

Let T be a topological space and let R denote the set of all reals. If $f\colon T \times R \to T$ is a flow on T then we denote $f(\gamma, t) = \gamma . t$ for all $\gamma \in T$ and $t \in R$. The set

$$\gamma . R = \{\gamma . t;\ t \in R\}$$

is called the trajectory of the flow f determined by the point γ.

Let W be a subset of T. We denote

$$W_0 = \{\gamma \in W;\ \exists\, t > 0,\ \ g . t \notin W\}$$

and

$$W^- = \{\gamma \in W;\ \forall\, t > 0,\ \gamma . [0, t) \not\subset W\}.$$

The set W^- is called the exit set of W.

If W^- is closed relative to W_0 and for every $\gamma \in W$ it holds

$$\gamma . [0, t] \subset \mathrm{cl}(W) \Rightarrow \gamma . [0, t] \subset W,$$

where $\mathrm{cl}(W)$ is the closure of W, then the set W is called a Ważewski set.

In [1] one can find the following

Theorem (Ważewski): If W is a Ważewski set, then W^- is a strong deformation retract of W_0 and W_0 is open relative to W.

If W^- is not a strong deformation retract of W, then $W - W_0$ is not empty, i.e. there exists a trajectory which stays in W for all positive time.

3 Topological prelimilaries

If we want to prove the existence of a trajectory which stays in W for all positive time, it is sufficient to show that W and W^- are not of the same homotopy type.

Let X be a linearly connected topological space. We denote the fundamental group of X by $\pi(X)$. It is an invariant of the homotopy type.

If the fundamental groups of W and W^- are not isomorphic, then W and W^- have different homotopy types and we can conclude that there exists some trajectory which stays in W for all positive time. This is the main idea of our paper.

Let G_1, G_2 be arbitrary groups. We denote the free product of the groups G_1 and G_2 by $G_1 \times G_2$.

The computation of the fundamental group of a more complicated topological space can be reduced to computations of fundamental groups of simpler spaces by

Theorem (Seifert-van Kampen, [2]): Let X be a linearly connected topological space. Let $X = U_1 \cup U_2$, where U_1 and U_2 are open sets, $U_1 \cap U_2$ is linearly connected. The inclusion maps i: $U_1 \cap U_2 \to U_1$ and j: $U_1 \cap U_2 \to U_2$ induce homomorphisms of groups i_*: $\pi(U_1 \cap U_2) \to \pi(U_1)$ and j_*: $\pi(U_1 \cap U_2) \to \pi(U_2)$. Let N be the smallest normal subgroup of $\pi(U_1) \times \pi(U_2)$ which contains the set $\{i_*(\gamma) . (j_*(\gamma))^{-1};\ \gamma \in \pi(U_1 \cap U_2)\}$. Then $\pi(X) \cong \pi(U_1) \times \pi(U_2)/_N$.

4 Flows in a cylinder

Suppose that f is a flow on R^3 such that its trajectories run downward outside a solid cylinder V of a finite height. Further let there exist points γ_0, γ_1, γ_2, γ_3 from the interior of V such that

$\gamma_0 . R = \{\gamma_0\}$

$\omega(\gamma_1) = \{\gamma_0\}$ and $\gamma_1 . t_1 \notin V$ for some $t_1 \leqslant 0$

$\alpha(\gamma_2) = \{\gamma_0\}$ and $\gamma_2 . t_2 \notin V$ for some $t_2 \geqslant 0$

$\omega(\gamma_3) = \alpha(\gamma_3) = \{\gamma_0\}$ and

the trajectory $\gamma_3 . R$ is contained in the interior of V.

Finally, let the set $W = V - \bigcup_{i=0}^{3} \gamma_i^t$, where

$$\gamma_i^t = \{\gamma \in V;\ \exists\, t \in R,\ \gamma_i . t = \gamma\} \quad \text{for } i = 0, 1, 2, 3.$$

Moreover, suppose that the set W has the same homotopy type as a solid cylinder of a finite height without its axis and without a circular line which lies in the interior of the cylinder and has exactly one common point with that axis. See *Figure 1*.

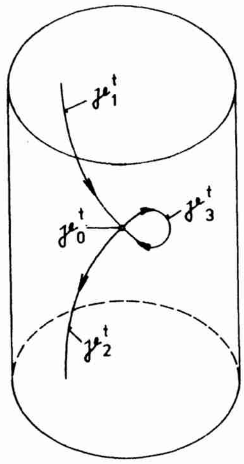

Figure 1

It is easy to see that such a flow exists.

Using the above notation we state the following theorem:

Theorem 1: There is a $\gamma^+ \in W$ ($\gamma^- \in W$) such that the positive (resp. negative) part of the trajectory running through γ^+ (resp. γ^-) is contained in W.

Proof: It is obvious that W is a Ważewsi set. The set W^- consists of all points of the bottom of V without one of its internal points. W^- is therefore homotopically equivalent to the circle S^1, hence $\pi(W^-)$ is isomorphic to the group of integers denoted by Z.

Now we shall compute the fundamental group of W. Let us choose the sets U_1, U_2 as follows:

U_1 is the set W without some surface σ with the boundary consisting of the trajectories $\gamma_3 . R$ and $\gamma_0 . R$,

U_2 is the open solid torus contained in the interior of V.

The position of the sets U_1 and U_2 is depicted in *Figure 2*.

The set $U_1 \cap U_2$ is homeomorphic to an open cylinder, hence it is contractible to a point and $\pi(U_1 \cap U_2)$ is trivial.

According to the theorem of Seifert — van Kampen we have $\pi(W) \cong Z \times Z$.

Since the group $Z \times Z$ is not commutative an Z is commutative, the groups $\pi(W)$ and $\pi(W^-)$ are not isomorphic.

The previous arguments show that there exists a trajectory $\gamma^+ . R$ which stays in W for all positive time.

Reversing the flow we get also the existence of a trajectory $\gamma^- . R$ which stays in W for all negative time.

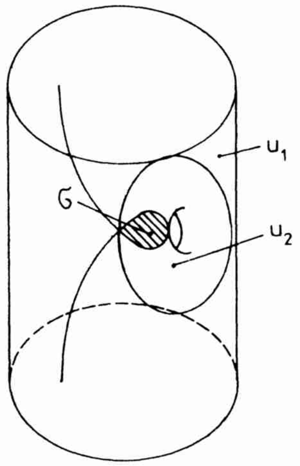

Figure 2

Remark: If we assume that γ_0 is a hyperbolic rest point of the flow f and there is no trajectory which is contained in V for all time except of $\gamma_0 . R$, $\gamma_3 . R$, then $\gamma_0 . R \cup \gamma_3 . R$ is either the ω — limit set of $\gamma^+ . R$ or the α — limit set of $\gamma^- . R$.

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Received: 15. 11. 1985

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SÚHRN

POZNÁMKA K TOKOM S HOMOKLINICKOU TRAJEKTÓRIOU

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Pre istý typ tokov s homoklinickou trajektóriou je dokázaná existencia netriviálnej pozitívne invariantnej množiny.

Dôkaz je založený na Ważewského vete.

РЕЗЮМЕ

ЗАМЕЧАНИЕ О ПОТОКАХ С ГОМОКЛИНИЧЕСКОЙ ТРАЙЕКТОРИЕЙ

Марек Фила — Франтишек Марко, Братислава

Для некоторых потоков с гомоклинической трайекторией показано существование нетривиального позитивно инвариантного множества.

Доказательство основано на теореме Важевского.