UNIVERSITAS COMENIANA
ACTA MATHEMATICA UNIVERSITATIS COMENIANAE
LIV—LV—1988

ON A CERTAIN BOUNDARY-VALUE PROBLEM FOR A POLYVIBRATING EQUATION

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Abstract

The paper deals with the non-linear Goursat problem for a differential-integral polyvibrating equation of order $2p$ with boundary conditions prescribed on two curves in an arbitrary rectangle $Q \subset R^2$. The corresponding linear problem was investigated in M. N. Oğuztöreli [5] and the non-linear problem for a differential equation of the second order ($p = 1$) was closely examined by Z. Szmydt [6].
In the paper the boundary-value problem is reduced to a system of integro-differential-functional equations. Due to s. Banach's fixed point theorem, the existence of a unique local solution is proved. Then, the solution is extended to a global one.

1. Let E be a Banach space with a norm $\|\cdot\|$, $p \in N$, $Q = [-a_1, b_1] \times [-a_2, b_2]$ ($a_1, a_2, b_1, b_2 \in R_+ := [0, \infty)$) — a subset of R^2 and $b: Q^2 \times E^{p(p+2)} \to E$, $c: Q \times E^{p(p+2)} \times E \to E$, $M_j: [-a_1, b_1] \to E$ and $N_j: [-a_2, b_2] \to E$ ($j = 0, 1, \ldots, p-1$) — being given functions.

We shall consider polyvibrating differential-integral equation of order $2p$

$$L^p u(x, y) = c\left\{x, y, (D^\alpha u(x, y)), \int_0^x \int_0^y b[x, y, s, t, (D^\alpha u(s, t))] \, ds \, dt\right\} \tag{1}$$

($(x, y) \in Q$; $\alpha = (\alpha_1, \alpha_2)$; $0 \leqslant \alpha_1, \alpha_2 \leqslant p$; $|\alpha| = \alpha_1 + \alpha_2 \leqslant 2p - 1$), where $L = D_x D_y$; $L^k u = L(L^{k-1} u)$ for $1 \leqslant k \leqslant p$; $L^0 u = u$ and $(D^\alpha u)$ denote a finite sequence of all derivatives $D^\alpha u$ for $0 \leqslant \alpha_1, \alpha_2 \leqslant p$; $|\alpha| \leqslant 2p - 1$. Let us introduce two curves $y = f(x)$ and $x = h(y)$ ($(x, y) \in Q$) intersecting each other at the origin $O(0, 0)$ of the coordinate system.

The non-linear Goursat problem (G_N) relies on finding a function $u: Q \to E$ which has continuous derivatives $D^\alpha u$ ($0 \leqslant \alpha_1, \alpha_2 \leqslant p$) in Q, fulfils equation (1) at every point (x, y) of Q*) and satisfies the boundary-value conditions

$$L^j u[x, f(x)] = M_j(x); \; L^j u[h(y), y] = N_j(y) \tag{2}$$

($(x, y) \in Q$; $j = 0, 1, \ldots, p-1$).

The following lemma can be proved

Lemma 1. Let the function c be a continous one. If a function u: $Q \to E$ has continuous derivatives $D^\alpha u$ for $0 \leqslant \alpha_1, \alpha_2 \leqslant p$ and moreover satisfies the equation

$$u(x, y) = R_p^u(x, y) + \sum_{m=1}^{p} [y^{m-1}\varphi_m(x) + x^{m-1}\psi_m(y) + c_m(xy)^{m-1}] \tag{3}$$

$((x, y) \in Q)$, where

$$R_p^u(x, y) = \int_0^x \int_0^y \frac{[(x-s)(y-t)]^{p-1}}{[(p-1)!]^2} c\{s, t, (D^\alpha u(s, t)),$$

$$\int_0^s \int_0^t b[s, t, \xi, \eta, (D^\alpha u(\xi, \eta))] \,d\xi \,d\eta\} \,ds \,dt \tag{4}$$

and $\varphi_m: [-a_1, b_1] \to E$, $\psi_m: [-a_2, b_2] \to E$ $(m = 1, 2, \ldots, p)$ are functions of class C^p and $c_m \in E$ $(m = 1, 2, \ldots, p)$ — some constants, then u is a solution of equation (1) in Q. Inversely, if u is a solution of equation (1) in Q, then there exist constants $c_m \in E$ and functions φ_m: $[-a_1, b_1] \to E$, ψ_m: $[-a_2, b_2] \to E$ $(m = 1, 2, \ldots, p)$ of class C^p and such that $\varphi_m^{(1)}(0) = \varphi_m^{(1)}(0) = 0$ $(0 \leqslant 1 \leqslant \leqslant m - 1 \leqslant p - 1)$ and u is a solution of equation (3) in Q.

We assume the following

A. The curves f: $[-a_1, b_1] \to [-a_2, b_2]$ and h: $[-a_2, b_2] \to [-a_1, b_1]$ are of class C^p; $f(0) = h(0) = 0$; $0 \leqslant f'(0)h'(0) =: g_0 < 1$ and have no common points in $Q - \{0\}$. Moreover, $xf(x) \geqslant 0$ and $yh(y) \geqslant 0$ for $(x, y) \in Q$.

B. The functions M_j: $[-a_1, b_1] \to E$ and N_j: $[-a_2, b_2] \to E$ $(j = 0, 1, \ldots, p-1)$ are of class C^{p-j} respectively and satisfy the condition

$$M_j(0) = N_j(0) \quad (j = 0, 1, \ldots, p-1) \tag{5}$$

C. The functions b: $Q^2 \times E^{p(p+2)} \to E$ and c: $Q \times E^{p(p+2)} \times E \to E$ are continuous and fulfil the Lipschitz condition with respect to $p(p+2)$ and $p(p+2)+1$ the last variables respectively.

Define a sequence of functions $\{z^n\}_{n \in N}$, where

$$z^1(x) = h \circ f(x); \; z^{n+1}(x) = z \circ z^n(x) \quad \text{for } n \in N \tag{6}$$

and sign $\circ$ denotes the composition of functions. Assumption **A** and Dini's theorem yield $z^n \to 0$ for $n \to \infty$ uniformly with respect to x (cp. [2], Lemma 3 and [4], Th. 0.4).

Assuming that u is a solution of (G_N) in Q and imposing on u the boundary

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*) Every such function will be called a solution of equation (1) in Q.

conditions (2) we notice the functions u, φ_m, ψ_m $(m = 1, 2, \ldots, p)$ satisfy equation (3) with $c_m = M_{m-1}(0)/[(m-1)!]^2$ and the following system of integro-differential-functional equations

$$\begin{aligned} \varphi_{j+1}^{(j)}(x) + \psi_{j+1}^{(j)} \circ f(x) &= V_j^u(x); \\ \varphi_{j+1}^{(j)} \circ h(y) + \psi_{j+1}^{(j)}(y) &= W_j^u(y) \end{aligned} \tag{7}$$

$((x, y) \in Q; j = 0, 1, \ldots, p-1)$ with initial conditions

$$\varphi_{j+1}^{(1)}(0) = \psi_{j+1}^{(1)}(0) = 0 \tag{8}$$

$(0 \leqslant 1 \leqslant p-1)$, where*)

$$V_j^u(x) = \frac{1}{j!}\left\{M_j(x) - \sum_{m=j+1}^{p} \frac{M_{m-1}(0)}{[(m-j-1)!]^2}[xf(x)]^{m-j-1} - R_{p-j}^u[x, f(x)]\right\} -$$

$$- \sum_{m=j+2}^{p} \binom{m-1}{j} \{[f(x)]^{m-j-1} \varphi_m^{(j)}(x) + x^{m-j-1} \psi_m^{(j)} \circ f(x)\}; \tag{9}$$

$$W_j^u(y) = \frac{1}{j!}\left\{N_j(y) - \sum_{m=j+1}^{p} \frac{M_{m-1}(0)}{[(m-j-1)!]^2}[yh(y)]^{m-j-1} - R_{p-j}^u[h(y), y]\right\} -$$

$$- \sum_{m=j+2}^{p} \binom{m-1}{j} \{y^{m-j-1} \varphi_m^{(j)} \circ h(y) + [h(y)]^{m-j-1} \psi_m^{(j)}(y)\} \tag{10}$$

2. Let us introduce a Banach space $C_*^p(Q)$ of functions $u \colon Q \to E$ possesing in Q continuous derivatives $D^\alpha u$ $(0 \leqslant \alpha_1, \alpha_2 \leqslant p;\ |\alpha| \leqslant 2p-1)$ with the norm

$$\|u\|_* := \max_{\substack{0 \leqslant \alpha_1, \alpha_2 \leqslant p \\ |\alpha| \leqslant 2p-1}} \{\max_Q \|D^\alpha u(x, y)\|\} \tag{11}$$

It is easy to observed that the inclusion $C_*^p(Q) \subset C^p(Q)$ holds.

The following lemma deals with a solution of the initial problem (7), (8).

Lemma 2. Let $u \in C_*^p(Q)$ be fixed and Assumptions **A** – **C** be satisfied. Then the system $\varphi_1^u, \varphi_2^u, \ldots, \varphi_p^u; \psi_1^u, \psi_2^u, \ldots, \psi_p^u$ of functions given recursively**) by the formulae

$$\varphi_m^u(x) = (1 - \delta_{1m}) \int_0^x \frac{(x-s)^{m-2}}{(m-2)!} S_{m-1}^u(s)\,ds + \delta_{1m} S_0^u(x); \tag{12}$$

*) $\sum_{k=l}^{m} a_k = 0$ if $l > m$; symbol $R_m^u (m \in N)$ denotes the expression given by formula (4) where p is replaced by m.

**) Due to formulae (9), (10), (12) — (14) the functions φ_m^u and ψ_m^u (for fixed $m < p$) depend on φ_j^u, ψ_j^u with $m < j \leqslant p$, but the functions φ_p^u and ψ_p^u can be obtained directly.

$$\psi_m^u(y) = (1 - \delta_{1m}) \int_0^y \frac{(y-t)^{m-2}}{(m-2)!} \tilde{S}_{m-1}^u(t)\,\mathrm{d}t + \delta_{1m} \tilde{S}_0^u(y) \tag{13}$$

($(x, y) \in Q$; $m = 1, 2, \ldots, p$; δ_{1m} — Kronecker's symbol; $0^\bullet(\pm\infty) := 0$), where

$$\begin{gathered} S_j^u(x) = \sum_{n=0}^{\infty} [V_j^u \circ z^n(x) - W_j^u \circ f \circ z^n(x)]; \\ \tilde{S}_j^u(y) = W_j^u(y) - S_j^u \circ h(y) \end{gathered} \tag{14}$$

($j = 0, 1, \ldots, p-1$) is the only solution of the initial problem (7), (8). Moreover, the functions φ_m^u and ψ_m^u ($m = 1, 2, \ldots, p$) are of class C^p.

Proof. If $u \in C_*^p(Q)$ is fixed, then there exists $\varrho \in R_+$ such that $\|u\|_* \leqslant \varrho$. For $j = p - 1$ cp. [2], pp. 113—114 (cp. also [4], pp. 101—103). When $p = 1$ the proof is complete, otherwise ($p > 1$); the method of mathematical induction implies the assertion.

Remark 1 (cp. [5]). When linear Goursat problem is considered (viz., function c does not depend on u and its derivatives), then u has explicit form of (3), where $\varphi_1, \varphi_2, \ldots, \varphi_p$; $\psi_1, \psi_2, \ldots, \psi_p$ is a solution of differential-functional system (7) with initial conditions (8).

3. For every $u \in C_*^p(Q)$ define operation T by the right-hand side of equation (3), where $\varphi_m = \varphi_m^u$ and $\psi_m = \psi_m^u$ ($m = 1, 2, \ldots, p$) are given by (12) and (13) respectively. Due to the formula

$$\begin{aligned} D^\alpha Tu(x, y) = D^\alpha R_p^u(x, y) &+ \sum_{m=\alpha_2+1}^{p} \frac{(m-1)!}{(m-\alpha_2-1)!} y^{m-\alpha_2-1} (\varphi_m^u)^{(\alpha_1)}(x) + \\ &+ \sum_{m=\alpha_1+1}^{p} \frac{(m-1)!}{(m-\alpha_1-1)!} x^{m-\alpha_1-1} (\psi_m^u)^{(\alpha_2)}(y) + \\ &+ \sum_{m=\max(\alpha_1,\alpha_2)+1}^{p} \frac{M_{m-1}(0)}{(m-\alpha_1-1)!(m-\alpha_2-1)!} x^{m-\alpha_1-1} y^{m-\alpha_2-1} \end{aligned} \tag{15}$$

($(x, y) \in Q$; $0 \leqslant \alpha_1, \alpha_2 \leqslant p$; $|\alpha| \leq 2p - 1$), Lemma 2 and Assumption **C**, the relation $Tu \in C_*^p(Q)$ holds. We want to prove that under additional assumption, operation T is a contraction. To do this we need

Lemma 3. If $u, v \in C_*^p(Q)$ and Assumptions **A**—**C** are satisfied, then the inequalities

$$\|(\varphi_m^u)^{(l)}(x) - (\varphi_m^v)^{(l)}(x)\| \leqslant \text{const}\, \|u - v\|_* |x|^{2p-m-l+1}; \tag{16}$$

$$\|(\psi_m^u)^{(l)}(y) - (\psi_m^v)^{(l)}(y)\| \leqslant \text{const}\, \|u - v\|_* |y|^{2p-m-l+1} \tag{17}$$

($(x, y) \in Q$; $m = 1, 2, \ldots, p$; $l = 0, 1, \ldots, p$) hold.

The proof can be obtained by the direct calculation and application of mathematical induction.

One can notice that Lemma 3, formula (15) and Assumption **C** lead to the estimations

$$\|D^{\alpha}Tu(x, y) - D^{\alpha}Tv(x, y)\| \leqslant \text{const}\, \|u - v\|_{*} \Delta^{2p - |\alpha|} \tag{18}$$

$(u, v \in C_{*}^{p}(Q);\ (x, y) \in Q;\ 0 \leqslant \alpha_1,\ \alpha_2 \leqslant p;\ |\alpha| \leqslant 2p - 1)$, where $\Delta: = \max(a_1, a_2, b_1, b_2)$. Hence

$$\|Tu - Tv\|_{*} \leqslant \text{const}\, \|u - v\|_{*} \max_{\alpha} \Delta^{2p - |\alpha|} \tag{19}$$

$(u, v \in C_{*}^{p}(Q))$ and the operation T is a contraction if the number Δ is sufficiently small to fulfil relations

$$\text{const}\, \Delta^{2p - |\alpha|} < 1 \tag{20}$$

$(0 \leqslant \alpha_1,\ \alpha_2 \leqslant p;\ |\alpha| \leq 2p - 1)$. In virtue of Banach's fixed point theorem and Lemma 1 we can formulate

Theorem 1. Under Assumptions **A**—**C** and if number $\Delta = \max(a_1, a_2, b_1, b_2)$ is sufficiently small to fulfil relations (20), then there exists exactly one solution of the non-linear Goursat problem (G_N) in Q.

Remark 2. If $f(x) = 0$ and $h(y) = 0$ $((x, y) \in Q)$ and denoting $U = L^p u$, then the problem can be reduced to a non-linear Volterra equation

$$U(x, y) = c\left\{x, y, (U_{\alpha}(x, y)), \int_0^x \int_0^y b[x, y, s, t, (U_{\alpha}(s, t))]\, \mathrm{d}s\, \mathrm{d}t\right\} \tag{21}$$

$((x, y) \in Q;\ 0 \leqslant \alpha_1,\ \alpha_2 \leqslant p;\ |\alpha| \leqslant 2p - 1)$, where

$$U_{\alpha}(x, y) = w_{\alpha}(x, y) + D^{\alpha}\left\{\int_0^x \int_0^y \frac{[(x - s)(y - t)]^{p-1}}{[(p - 1)!]^2} U(s, t)\, \mathrm{d}s\, \mathrm{d}t\right\}, \tag{22}$$

functions w_{α} depend on M_j and N_j $(j = 0, 1, \ldots, p - 1)$ and the meaning of the symbol (U_{α}) is this same as in equation (1). Applying the renorming technique introduced by A. Bielecki (cp. [1]) and then Banach's fixed point theorem, the existence of a unique solution of a continuous solution of (21) might be proved. Hence, there exists exactly one solution of (G_N) in Q (cp. Remark 1). Moreover (cp. [3]), if $a_1 = a_2 = b_1 = b_2 = \infty$, then the problem is equivalent to the solvability of the equation

$$u(x, y) = G(x, y) + R_p^u(x, y) \tag{23}$$

$((x, y) \in R^2)$, where G depends on functions M_j and N_j $(j = 0, 1, \ldots, p - 1)$. Applying Tychonoff's fixed point theorem G. Hecquet proves the existence of a solution of equation (23) in R^2.

4. Finally, we assume the monotonicity of the curves f and h.

According to Theorem 1 there exists rectangle $Q_1 = [-a_1^1, b_1^1] \times [-a_2^1, b_2^1]$; $a_1^1, a_2^1, b_1^1, b_2^1 \in R_+$, such that (G_N) has a unique solution u_1 in Q_1. Assuming the curve $y = f(x)$ intersects the segment $x = b_1^1$; $y \in [-a_2^1, b_2^1]$ we consider the Goursat problem for equation (1) with boundary conditions (2) prescribed on curve $y = f(x)$; $(x, y) \in Q$ and segment $x = b_1^1$; $y \in [-a_2^1, b_2^1]$. There exists rectangle Q_2 — a neighbourhood of the intersection point (x_0, y_0) of the above-mentioned curve and segment such that the problem has exactly one solution u_2 in Q_2. In virtue of int $Q_1 \cap Q_2 \neq \emptyset$ and uniqueness of the problem, $u_1 = u_2$ in $Q_1 \cap Q_2$ and the function $u = u_1$ in Q_1 and $u = u_2$ in Q_2 is a solution in $Q_1 \cup Q_2$. Concerning equation (1) with boundary conditions (2) prescribed on two segments $y = y_0$; $(x, y) \in Q_1 \cup Q_2$ and $x = 0$; $y \in [-a_2^1, b_2^1]$, the problem has a unique solution in the rectangle Q_3 obtained as a Cartesian product of the afore-said segments. In this way (cp. also [6]), step by step, local solution of (G_N) can be extended to a global one. The above considerations lead to

Theorem 2. If Assumptions A—C are fulfilled and moreover the curves $y = f(x)$ and $x = h(y)$ $((x, y) \in Q)$ are non-decreasing, then the problem (G_N) has exactly one global solution in Q.

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Received: 19. 12. 1985

SÚHRN

O ISTOM OKRAJOVOM PROBLÉME PRE JEDNU POLYVIBRAČNÚ ROVNICU

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Práca pojednáva o nelineárnej Gourzatovej úlohe pre polyvibračnú rovnicu $2p$-teho rádu s okrajovými podmienkami na dvoch krivkách z daného obdĺžnika v R^2. Transformáciou tejto úlohy na integro-diferenciálnu rovnicu a použitím Banachovho princípu kontrakcie sa dokazuje existencia a jednoznačnosť lokálneho riešenia, z ktorého sa postupným rozširovaním získa globálne riešenie úlohy.

РЕЗЮМЕ

ОБ ОДНОЙ КРАЕВОЙ ЗАДАЧЕ ДЛЯ ПОЛИВИБРАЦИОННОГО УРАВНЕНИЯ

Марек В. Михалски, Варшава

Работа занимается нелинейной задачей Гурса для поливибрационного уравнения $2p$-того порядка. Краевые условия задачи на двух кривых прямоугольника из R^2. После трансформации этой задачи на интегрально-дифференциальное уравнение при помощи контракции Банаха доказывается существование и единственность локального решения. Из этого решения получается расширением глобальное решение.