UNIVERSITAS COMENIANA
ACTA MATHEMATICA UNIVERSITATIS COMENIANAE
LIV—LV—1988

A NOTE ON DIFFERENTIABLE FUNCTIONS

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In the problem 5955 from American Mathematical Monthly 81 [1974, 176] F. D. Hammer asks if there is such a differentiable function that maps the set of all rational numbers in itself but whose derivative maps the set of all rational numbers into the set of irrational numbers. In this paper we consider the above mentioned problem with the aim to solve the symmetrical problem.

1 Formulation of the problem

The problem 5955 reads:

Is there such a differentiable function that maps every rational number into rational but whose derivative maps any rational number into irrational one?

The official solution shows that such a function is, for example, the following function:

$$f(x) = \sum_{n=0}^{\infty} \frac{g(n!x)}{(n!)^2}, \tag{1}$$

where $g(y)$ is a periodic function with the period 1 defined on interval $\left\langle -\frac{1}{2}, \frac{1}{2} \right\rangle$ by the relation

$$g(y) = y(1 - 4y^2).$$

Then for each integer number k $g(k) = 0$ and $g'(k) = 1$.

The solution of the first part of the problem follows from that the series (1) has at most finite many nonzero rational members for every rational number x.

The proof of the second part of the problem is based upon knowledge that the series of derivations of members of series (1)

$$\sum_{n=0}^{\infty} \frac{g'(n!x)}{n!} \tag{2}$$

converges uniformly and absolutely, and that is why it converges to the derivative of function $f(x)$ (see [3], p. 183). For each rational number x series (2) differs from the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} \tag{3}$$

which sum is an irrational number e, at most in finite many rational members, and therefore $f'(x)$ equals the sum of e and the rational number.

2 Solution of a "symmetrical" problem

There is a natural requirement to change the functions f and f' in the formulation of the investigated problem from [1], and so we get problem P:

Is there such a differentiable function that maps every rational number into irrational one and whose derivative maps every rational number into rational one?

To solve this problem we will try to use the method according to which an original problem in part 1 was solved. Besides the knowledge that the sum of series $\sum_{n=0}^{\infty} \frac{1}{n!}$ is an irrational number, the core of the solution was the choice of function g. Applying a certain "methodic analysis", we will try to find out the type of reasoning of the author of this solution in choosing function g. His way of thinking might have been as follows:

It is reasonable to look for the simplest function g, which might lead to the solution of the problem among polynomic functions. The choice $g\left(-\frac{1}{2}\right) = g\left(\frac{1}{2}\right) = 0$ proves the existence of periodic extension of function g from the symmetrical interval $\left\langle -\frac{1}{2}, \frac{1}{2}\right\rangle$ onto the set of real numbers R. The choice of further properties of the function g, as for instance, $g(0) = 0$ and $g'(0) = 1$, has considerably simplified the computation and has secured main demands for functions f and f'. Polynomic function g which fulfils these requirements has three different real roots. Let us start from the polynomic function of the third degree

$$g(x) = a + bx + cx^2 + dx^3 \quad d \neq 0 \tag{4}$$

with real coefficients. Then the condition $g(0) = 0$ gives $a = 0$. Further,

$$0 = g\left(-\frac{1}{2}\right) = -\frac{1}{2}b + \frac{1}{4}c - \frac{1}{8}d$$

$$0 = g\left(\frac{1}{2}\right) = \frac{1}{2}b + \frac{1}{4}c + \frac{1}{8}d$$

and hence $c = 0$. From the condition $g'(0) = 1$ it follows that $b = 1$. Therefore $\frac{1}{2}b + \frac{1}{8}d = 0$ gives $d = -4$. From these conditions we finally get

$$g(x) = x(1 - 4x^2).$$

Solving problem P we proceed in the same way. Let us choose the conditions for the periodic function g with the period 1 in this way:

$$g\left(-\frac{1}{2}\right) = g\left(\frac{1}{2}\right) = 0 \tag{5}$$

$$g(0) = 1 \tag{6}$$

$$g'(0) = 0 \tag{7}$$

The polynomic function that fulfils these hypotheses is at least of the second degree. Then let

$$g(x) = a + bx + cx^2,$$

where a, b, c ($c \neq 0$) are real numbers. Then from conditions (5), (6), (7) it easily follows that periodic function g with period 1 on interval $\left\langle -\frac{1}{2}, \frac{1}{2}\right\rangle$ is inevitably defined by relation

$$g(x) = 1 - 4x^2.$$

A function defined like that does not have any derivative in points $\pm\frac{(2k+1)}{2}$, ($k \in Z$, Z -- the set of integers), therefore we cannot use the theorem on the derivation of functional series for series (1) and so it is not sufficient to take a quadratic function.

Let us assume, therefore, a polynomic function of the third degres with real coefficients of the shape (4). Then from conditions (5) and (6) we have

$$1 - \frac{1}{2}b + \frac{1}{4}c - \frac{1}{8}d = 0$$

$$1 + \frac{1}{2}b + \frac{1}{4}c + \frac{1}{8}d = 0$$

and so $c = -4$. Further, from the condition (7) we get $b = 0$. Hence, it follows that $d = 0$, which is a contradiction with the fact that $g(x)$ is of the third degres.

Let us try to broaden these considerations into polynomic functions

$$g(x) = a + bx + cx^2 + dx^3 + ex^4 \quad e \neq 0$$

of the fourth degree with real coefficients. From the condition (6) it follows that $a = 1$. Then from the condition (5) we get equations

$$1 - \frac{1}{2}b + \frac{1}{4}c - \frac{1}{8}d + \frac{1}{16}e = 0$$

$$1 + \frac{1}{2}b + \frac{1}{4}c + \frac{1}{8}d + \frac{1}{16}e = 0$$

and hence $e = -16 - 4c$. Further, condition (7) gives $b = 0$.

Thus the function $g(x)$ gets the shape

$$g(x) = 1 + cx^2 + dx^3 - (16 + 4c)x^4$$

for the computation of the missing coefficients we will choose further conditions

$$g'\left(-\frac{1}{2}\right) = g'\left(\frac{1}{2}\right) = 0 \tag{8}$$

that will secure us the existence of the derivative of periodic extension in every real number. From this condition we have equations

$$-2c\frac{1}{2} + 3d\frac{1}{4} + 4(16 + 4c)\frac{1}{8} = 0$$

$$2c\frac{1}{2} + 3d\frac{1}{4} - 4(16 + 4c)\frac{1}{8} = 0$$

and from then we get $d = 0$. Then, further, $c = -8$ and finally $e = -16 - 4c = 16$.

So we get the polynomic function

$$g(x) = 1 - 8x^2 + 16x^4 \tag{9}$$

that evidently satisfies the conditions (5), (6), (7), (8). If we now consider function $g(x)$ as periodic extension of the function (9) from interval $\left\langle -\frac{1}{2}, \frac{1}{2}\right\rangle$ onto the whole set of real numbers R, then it holds

$$g(m) = 1 \quad \text{and} \quad g'(m) = 0 \quad \text{for } m \in Z. \tag{10}$$

Then, similarly as above, when we put

$$f(x) = \sum_{n=1}^{\infty} \frac{g(n!x)}{(n!)^2}, \tag{11}$$

then

$$f'(x) = \sum_{n=1}^{\infty} \frac{g'(n!x)}{n!} \tag{12}$$

because the series on the right hand side of these equalities uniformly converge.

It follows from (10) that for every rational number x the series (12) has at most a finite number of non-zero rational members, and therefore $f'(x)$ is a rational number.

From the first equality (10) it further follows that for any rational number x, the series (11) differs from series

$$\sum_{n=1}^{\infty} \frac{1}{(n!)^2} \tag{13}$$

at most in finite many rational members. If we choose $g_k = (k + 1)^2$, $k = 1, 2, \ldots$, so on the basis of Theorem 1.2 from [2] (see p. 270) number (13) is irrational. Therefore $f(x)$ is also an irrational number for every rational number x.

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Received: 24. 4. 1987