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ON LOGARITHMS OF RATIONAL NUMBERS

MILAN PAŠTÉKA, TIBOR ŠALÁT, Bratislava

It is well known that the decadic logarithm of a rational number $r > 0$ is rational if and only if $r = 10^n$, n being an integer (cf. [1], p. 24; [2], p. 210). It can be also proved that $\log_b r$ (b, r are positive rational numbers, $b \neq 1$) is rational if and only if $b^n = r^m$ for some integers m, n (cf. [1], p. 25). In this paper we restrict ourselves to the study of logarithms of rational numbers at integer bases $b > 1$. It is a natural request from the point of view of teaching mathematics to determine the set of all positive integers $b > 1$ that have a property similar to the mentioned property of the number 10, i.e. $\log_b r$ is rational for rational $r > 0$ if and only if $r = b^n$, n being an integer. The aim of this note is to give a characterization of numbers $b > 1$ with the mentioned property and to show that "almost all" positive integers $b > 1$ have this property.

In connection with the formulated aim of this paper we introduce the following definition. In what follows the symbols Q^+, Q and Z denote the set of all positive rational numbers, the set of all rational numbers and the set of all integers, respectively.

Definition 1. An integer $b > 1$ is said to have the property (L) provided that the number $\log_b r$ for $r \in Q^+$ is rational if and only if $r = b^n$, $n \in Z$.

The following theorem gives a characterization of numbers having the property (L).

Theorem 1. An integer $b > 1$, $b = p_1^{\alpha_1} . p_2^{\alpha_2} \dots p_m^{\alpha_m}$ (the standard form of b) has the property (L) if and only if

$$(\alpha_1, \alpha_2, \dots, \alpha_m) = 1. \tag{1}$$

Proof. 1. Suppose that (1) holds. Let $r \in Q^+$, $r = \frac{p}{q}$, $p, q \in N$, $(p, q) = 1$. Let $\log_b r \in Q$. We can already assume that $r > 1$. Hence

$$\log_b r = \frac{c}{d},\ c, d \in N,\ (c, d) = 1.$$

From this we get

$$b^c q^d = p^d. \tag{2}$$

and so $q|p$. Since $(p, q) = 1$, we have $q = 1$. But then (2) has the form

$$b^c = p^d \tag{3}$$

For the standard form of p we get on account of (3),

$$p = p_1^{\beta_1}, p_2^{\beta_2}, \ldots, p_m^{\beta_m}. \tag{3'}$$

It follows from (3), (3′) that

$$d\beta_j = c\alpha_j \quad (j = 1, 2, \ldots, m).$$

Since $(c, d) = 1$, we get $d|\alpha_j$ $(j = 1, 2, \ldots, m)$ and then according to (1) we obtain $d = 1$. But then (3) yields $p = b^c$, $c \in Z$.

2. Suppose that (1) does not hold. Then $(\alpha_1, \alpha_2 \ldots \alpha_m) = v > 1$. Put $\alpha_j = v\alpha_j'$ $(j = 1, 2, \ldots, m)$. Then $(\alpha_1', \alpha_2', \ldots, \alpha_m') = 1$. Put $z = b^{\frac{1}{v}}$. Then $z \in Q^+$ and $\log_b z = \frac{1}{v} \in Q$. But z has not the form b^n, $n \in Z$. The proof is finished.

Let $H \subset N = \{1, 2, \ldots, n, \ldots\}$. Put

$$H(x) = \sum_{a \in H,\, a \leq x} 1$$

and

$$d(H) = \lim_{x \to \infty} \frac{H(x)}{x}$$

if the limit on the right-hand side exists. The number $d(H)$ is said to be the asymptotic density of the set H. Clearly, we have $d(H) \in \langle 0, 1 \rangle$ (cf. [3], p. 100).

Let (P) be a property of positive integers. Denote by M(P) the set of all $n \in N$ having the property (P). We say that almost all positive integers have the property (P) provided that the set M(P) has the asymptotic density equal to 1.

Theorem 2. Almost all positive integers have the property (L).

We shall give two different proofs of Theorem 2.

Proof I. Let M(L) have the defined meaning, i.e. M(L) is the set of all $n \in N$ having the property (L).

We set

$$A = N \setminus (\{1\} \cup \mathrm{M(L)}).$$

If suffices to prove that $d(A) = 0$.

It can be easily deduced from Theorem 1 that

$$A = \bigcup_{k=2}^{\infty} A_k, \tag{4}$$

where

$$A_k = \{2^k, 3^k, \ldots, n^k, \ldots\} \qquad (k = 2, 3, \ldots).$$

Let $x > 2$. Then $A_k(x)$ is evidently not greater than the number $\left[x^{\frac{1}{k}}\right]$. Hence $A_k(x) = 0$ for $k > \dfrac{\log x}{\log 2}$. Thereofore according to (4) we get

$$A(x) \leqq \sum_{2 \leqq k \leqq \frac{\log x}{\log 2}} \left[x^{\frac{1}{k}}\right].$$

A simple estimation gives

$$A(x) \leqq x^{\frac{1}{2}} \cdot \frac{\log x}{\log 2}$$

hence $A(x) = 0\ (\sqrt{x} \,.\, \log x)\ (x \to \infty)$. From this we get at once

$$\frac{A(x)}{x} = 0\left(\frac{\log x}{\sqrt{x}}\right) = \mathrm{o}\,(x) \quad (x \to \infty),$$

thus $d(A) = 0$. This ends the proof.

Proof II. This proof is based on the follwing result:

Let $H \subset N$ and $\sum_{h \in H} h^{-1} < +\infty$. Then $d(H) = 0$ (cf. [1], p. 257, Exercise 2; [3], p. 100).

Let the set A have the same meaning as in Proof I. On account of the mentioned result it suffices to prove that $\sum_{a \in A} a^{-1} < +\infty$. Since the set A consists of all positive integers n^k, where $n \geqq 2$, $k \geqq 2$, it suffices to show that

$$\sum_{k=2}^{\infty} \left(\sum_{n=2}^{\infty} n^{-k}\right) < +\infty$$

A simple estimation using the improper integral yields

$$\sum_{n=2}^{\infty} n^{-k} < 2^{-k} + \int_2^{\infty} \frac{\mathrm{d}t}{t^k} = 2^{-k} + \frac{2^{-k+1}}{k-1}.$$

Thus

$$\sum_{k=2}^{\infty} \left(\sum_{n=2}^{\infty} n^{-k}\right) < \sum_{k=2}^{\infty} 2^{-k} + 2 \sum_{k=2}^{\infty} 2^{-k} = 3.$$

The proof is finished.

REFERENCES

1. Niven, I.: Irrational Numbers. The Carus Mathematical Monographs. John Wiley, New Jersey, 1956.
2. Šalát, T.: Reálne čísla. Alfa, Bratislava, 1982.
3. Šalát, T.: Nekonečné rady. Academia, Praha, 1974.

Author's addresses:

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Tibor Šalát
Katedra algebry a teórie čísel MFF UK
Mlynská dolina
842 15 Bratislava

Milan Paštéka
Katedra matematiky a deskriptívnej geometrie
Stavebnej fakulty SVŠT
Radlinského 11
813 68 Bratislava

SÚHRN

O LOGARITMOCH RACIONÁLNYCH ČÍSEL

Milan Paštéka — Tibor Šalát, Bratislava

Hovoríme, že prirodzené číslo $b > 1$ má vlastnosť (L), ak pre ľubovoľné racionálne číslo $r > 0$ je číslo $\log_b r$ racionálne vtedy a len vtedy, keď $r = b^n$, n je celé číslo.

V článku je odvodená nutná a postačujúca podmienka k tomu, aby číslo $b > 1$ malo vlastnosť (L) a je ukázané, že skoro všetky prirodzené čísla v zmysle asymptotickej hustoty majú vlastnosť (L).

РЕЗЮМЕ

О ЛОГАРИФМАХ РАЦИОНАЛЬНЫХ ЧИСЕЛ

Милан Паштека — Тибор Шалат, Братислава

Говорим, что натуральное число $b > 1$ имеет свойство (L), если для любого рационального числа $r > 0$ число $\log_b r$ рациональное тогда и только тогда когда $r = b^n$, где n — целое число. В работе показано необходимое и достаточное условие для того, чтобы натуральное число $b > 1$ имело свойство (L) и доказано, что почти все натуральные числа (в смысле ассимптотической плотности) имеют свойство (L).