UNIVERSITAS COMENIANA
ACTA MATHEMATICA UNIVERSITATIS COMENIANAE
LIV—LV—1988

ON VALUES OF THE FUNCTION $\varphi + \tau$

HELENA BEREKOVÁ — TIBOR ŠALÁT, Bratislava

1 Introduction

There are many papers devoted to the study of values of arithmetical functions $f\colon N \to N$, N being the set of all positive integers. E.g., already in [5] (p. 256) it is shown that the set of all values of the Euler's function φ has the asymptotic density 0. An analogous result concerning the values of the functions $\sigma + \varphi$, $\sigma + \tau$ is proved in [1]. In [2] it is proved that the set of all positive integers that does not belong to the sequence $\{\sigma(n) - n\}_{n=1}^{\infty}$ has a positive lower asymptotic density. It is still an open question whether there are infinitely many $n \in N$ not belonging to the sequence $\{n - \varphi(n)\}_{n=1}^{\infty}$.

In this paper we shall prove that the sets of positive values of the functions $\varphi + \tau$, $\varphi - \tau$ have the asymptotic density 0. Let us remark that this fact cannot be established by the method used in [1], since the method from [1] devoted to the investigation of values of functions $f\colon N \to N$ is based on the assumption that $f(n) \geqq n$ for all $n \in N$. But it is easy to check that if $\{p_k\}_{k=1}^{\infty}$ is the increasing sequence of all prime numbers then for each $\eta > 0$ there exists a $k_0 = k_0(\eta) \in N$ such that for each $k > k_0$ and $n = p_1 \dots p_k$ (distinct primes) we have $f(n) = \varphi(n) + \tau(n) < \eta n$.

2 The main results

The main result of this paper is the following theorem.

Theorem 1. The set of values of the function $\varphi + \tau$ has the asymptotic density 0.

For the proof of Theorem 1 we shall use two auxiliary results. The first of them is closely related to Theorem 1 from [4].

Recall the following usual denotation. If $A \subset N$, then we put

$$A(x) = \sum_{a \in A,\, a \leqq x} 1.$$

If there exists

$$\lim_{x \to \infty} \frac{A(x)}{x} = d(A),$$

then the number $d(A)$ is called the asymptotic density of the set A. The numbers

$$\liminf_{x \to \infty} \frac{A(x)}{x} = \underline{d}(A)$$

and

$$\limsup_{x \to \infty} \frac{A(x)}{x} = \bar{d}(A)$$

are called the lower and the upper asymptotic density of A, respectively.

Lemma 1. Let $x_0 \in R$ and let g, h be two continuous and positive real functions defined on the interval $(x_0, +\infty)$ with

$$\lim_{x \to \infty} g(x) = \lim_{x \to \infty} h(x) = +\infty.$$

Let N_0 be a set of positive integers satisfying the condition

$$N_0(x) = O\left(\frac{x}{g(x)}\right) \quad (x \to \infty). \tag{a}$$

Suppose, further, that

$$\frac{x}{h(x)} \to \infty \quad (x \to \infty) \tag{b}$$

and $\dfrac{x}{h(x)}$ is a non-decreasing function for $x > x_1 > x_0$.

Let

$$h(x) = o(g(x)) \quad (x \to \infty) \tag{c}$$

If f: $N \to N$ and

$$\liminf_{n \to \infty} \frac{f(n)}{\dfrac{n}{h(n)}} > 0,$$

then

$$d(f(N_0)) = 0.$$

Proof. According to the assumption of Lemma there exists a $\delta > 0$ and $n_0 \in N$ such that for each $n > n_0$ we have

$$f(n) \geqq \delta \frac{n}{h(n)}. \tag{1}$$

We can assume already that $n_0 > x_1$.

On the basis of the Darboux property of continuous functions we get from (b) an $x_2 > x_1$ such that each $x > x_2$, $x \in N$ has the form

$$x = \delta \frac{t(x)}{h(t(x))}, \tag{2}$$

where $t(x)$ is a suitable real number. Evidently, we have $\lim_{x \to \infty} t(x) = +\infty$.

Let $x \in N$, $x > x_2$. Let $n \in N_0$, $n > n_0$ be such that $f(n) \leqq x$. Then according to (1) and (2) we get

$$\delta \frac{t(x)}{h(t(x))} = x \geqq f(n) \geqq \delta \frac{n}{h(n)}.$$

From this we obtain

$$\frac{t(x)}{h(t(x))} \geqq \frac{n}{h(n)}.$$

Since $\frac{x}{h(x)}$ is non-decreasing (see (b)) we get from this $t(x) \geqq n$.

Put for brevity $F = f(N_0)$. We have just proved that if $x > x_2$, $x \in N$ and $n \in N_0$, $n > n_0, f(n) \leqq x$, then we have $n \leqq t(x)$. Hence each such $f(n)$ is included in the number $N_0(t(x))$. Further, the number of all $n \in N_0$, $n \leqq n_0$, is not greater than n_0. Therefore we have

$$F(x) \leqq n_0 + N_0(t(x))$$

for each $x > x_2$.

From this we get

$$\frac{F(x)}{x} \leqq \frac{n_0}{x} + \frac{N_0(t(x))}{x} =$$

$$= \frac{n_0}{x} + \frac{N_0(t(x))}{t(x)} \cdot \frac{t(x)}{x} \quad (x > x_2). \tag{3}$$

On the basis of (a) there is a $K > 0$ such that for $x > x_3 > x_2$ we have

$$\frac{N_0(t(x))}{t(x)} \leqq K \frac{t(x)}{t(x)g(t(x))} = K \frac{1}{g(t(x))}.$$

Hence

$$\frac{N_0(t(x))}{x} \leqq K \frac{1}{g(t(x))} \cdot \frac{t(x)}{x} \quad (x > x_3). \tag{4}$$

But then from (2) and (4) we get (for $x > x_3$)

$$\frac{N_0(t(x))}{x} \leqq \frac{K}{\delta} \frac{h(t(x))}{g(t(x))}.$$

If $x \to \infty$, then $t(x) \to \infty$ and then according to (c) the right-hand side has the limit 0. So by (3) we get

$$d(F) = d(f(N_0)) = 0.$$

The proof is finished.

Lemma 2. Denote by M_k ($k \geqq 0$) the set of all $n \in N$'s of the form

$$n = q_1 \dots q_k \cdot a^2, \tag{5}$$

where $a \in N$ and $q_1, \dots, q_k$ are distinct prime numbers. Then we have

$$M_k(x) = O\left(\frac{x(\log\log x)^k}{\log x}\right) \quad (x \to \infty). \tag{6}$$

Proof. In what follows we shall use the following wellknown result:

Let $k \geqq 1$ and $\pi_k(x)$ denote the number of all n's with $n \leqq x$ which have the form $n = q_1 \cdot q_2 \dots q_k$ (k is fixed, $q_1, q_2, \dots, q_k$ are distinct primes). Then we have

$$\pi_k(x) \sim \frac{x(\log\log x)^{k-1}}{\log x} \quad (x \to \infty) \tag{7}$$

(cf. [3], pp. 368—370).

Evidently, (6) holds for $k = 0$. In this case (see (5)) the set M_0 equals the set $\{1^2, 2^2, \dots, n^2, \dots\}$ and therefore

$$M_0(x) = O(\sqrt{x}) = O\left(\frac{x}{\log x}\right).$$

Hence in what follows we can assume that the number k is a fixed positive integer. Let

$$n = q_1 \dots q_k a^2 \leqq x, \quad n \in M_k. \tag{8}$$

If a is fixed, then the number of n's, $n \in M_k$ satisfying (8), equals $\pi_k\left(\frac{x}{a^2}\right)$.

Consider that from (8) we get for a the estimation $a \leqq \sqrt{\frac{x}{2}}$. Thus we have

$$M_k(x) = \sum_{a \leqq \sqrt{\frac{x}{2}}} \pi_k\left(\frac{x}{a^2}\right). \tag{9}$$

In view of (7) there is a $K > 0$ such that for each $x > e$

$$\pi_k(x) = K\frac{x(\log\log x)^{k-1}}{\log x} \tag{10}$$

holds.

If

$$\frac{x}{a^2} > e^2, \tag{11}$$

then according to (10) we get

$$\pi_k\left(\frac{x}{a^2}\right) \leqq K\frac{x\left(\log\log\frac{x}{a^2}\right)^{k-1}}{a^2\log\frac{x}{a^2}}.$$

The condition (11) is equivalent to the condition $a < \sqrt{\frac{x}{e^2}}$. Consider that

$$M_k(x) = \sum_{1 \leqq a < \sqrt{\frac{x}{e^2}}} \pi_k\left(\frac{x}{a^2}\right) + \sum_{\sqrt{\frac{x}{e^2}} \leqq a \leqq \sqrt{\frac{x}{2}}} \pi_k\left(\frac{x}{a^2}\right).$$

We shall investigate the second summand on the right-hand side. The condition (for a)

$$\sqrt{\frac{x}{e^2}} \leqq a \leqq \sqrt{\frac{x}{2}}$$

is equivalent to the condition

$$2 \leqq \frac{x}{a^2} \leqq e^2 (< 9)$$

and therefore

$$\pi_k\left(\frac{x}{a^2}\right) \leqq \pi_k(9) \leqq 9.$$

Consider that the number of positive integers lying in the interval

$$\left\langle\sqrt{\frac{x}{e^2}}, \sqrt{\frac{x}{2}}\right\rangle$$

does not exceed the number

$$\sqrt{\frac{x}{2}} - \sqrt{\frac{x}{e^2}} + 1.$$

Therefore

$$\sum_{\sqrt{\frac{x}{e^2}} \leqq a \leqq \sqrt{\frac{x}{2}}} \pi_k\left(\frac{x}{a^2}\right) \leqq 9\left(\sqrt{\frac{x}{2}} - \sqrt{\frac{x}{e^2}} + 1\right) = O(\sqrt{x}).$$

Hence

$$M_k(x) \leqq \sum_{1 \leqq a < \sqrt{\frac{x}{e^2}}} \pi_k\left(\frac{x}{a^2}\right) + O(\sqrt{x}). \tag{12}$$

The first summand on the right-hand side of (12) can be expressed in the form

$$\sum_{1 \leqq a < \sqrt{\frac{x}{e^2}}} \pi_k\left(\frac{x}{a^2}\right) = S_1(x) + S_2(x),$$

where

$$S_1(x) = \sum_{1 \leqq a \leqq \log x}, \quad S_2(x) = \sum_{\log x < a < \sqrt{\frac{x}{e^2}}}.$$

Considering (10) we get the following estimations

$$S_1(x) \leqq K \sum_{1 \leqq a \leqq \log x} \frac{x\left(\log\log\frac{x}{a^2}\right)^{k-1}}{a^2 \log\frac{x}{a^2}},$$

$$S_2(x) \leqq K \sum_{\log x < a < \sqrt{\frac{x}{e^2}}} \frac{x\left(\log\log\frac{x}{a^2}\right)^{k-1}}{a^2 \log\frac{x}{a^2}}.$$

Put for $t > e$

$$v(t) = \frac{\sqrt{t}\,(\log\log t)^{k-1}}{\log t}$$

(k is a fixed positive integer). An easy calculation shows that

$$v'(t) = \frac{1}{\log^2 t}\left\{\frac{1}{2}\frac{1}{\sqrt{t}}(\log\log t)^{k-1} + \sqrt{t}(k-1)\,.\right.$$

$$\left.\cdot\frac{(\log\log t)^{k-2}}{t\log t}\right\}\log t - \frac{1}{\log^2 t}\left\{\frac{\sqrt{t}(\log\log t)^{k-1}}{t}\right\} =$$

$$= \frac{1}{\log^2 t}\left\{\frac{1}{\sqrt{t}}(\log\log t)^{k-1}\left(\frac{\log t}{2} - 1\right) + \right.$$

$$\left. + \frac{k-1}{\sqrt{t}}\frac{(\log\log t)^{k-2}}{\log t}\right\}.$$

From this we see that $v'(t) > 0$ if $t > e^2 > e$. Therefore the function v is increasing in the interval $(e^2, +\infty)$.

Using the function v we get

$$S_1(x) \leqq K \sum_{1 \leqq a \leqq \log x} \sqrt{\frac{x}{a^2}}\, v\left(\frac{x}{a^2}\right). \tag{13}$$

Since the function v is increasing and positive in $(e^2, +\infty)$ we get $v\left(\frac{x}{a^2}\right) \leqq v(x)$ for each $a \geqq 1$. Then (13) yields

$$S_1(x) \leqq Kv(x)\sqrt{x} \sum_{1 \leqq a \leqq \log x} \frac{1}{a}.$$

But for $y > 0$

$$\sum_{a \leqq y} \frac{1}{a} = O(\log y)$$

(cf. [3], p. 266). Hence

$$S_1(x) \leqq K\sqrt{x}\,\frac{\sqrt{x}(\log\log x)^{k-1}}{\log x}\,O(\log\log x) =$$

$$= O\left(\frac{x(\log\log x)^k}{\log x}\right).$$

So we get

$$S_1(x) = O\left(\frac{x(\log\log x)^k}{\log x}\right) \quad (x \to \infty). \tag{14}$$

We shall estimate the sum $S_2(x)$.

The condition

$$\log x < a < \sqrt{\frac{x}{e^2}}$$

is equivalent to the condition

$$e^2 < \frac{x}{a^2} < \frac{x}{\log^2 x}. \tag{15}$$

Put $t = \frac{x}{a^2}$. Since the function v is increasing in $(e^2, +\infty)$, for each a satisfying (15) we get

$$v\left(\frac{x}{a^2}\right) \leqq v\left(\frac{x}{\log^2 x}\right)$$

But then a simple estimation gives

$$S_2(x) \leqq K \sum_{\log x < a < \sqrt{\frac{x}{e^2}}} \frac{\sqrt{x}}{a} v\left(\frac{x}{a^2}\right) \leqq$$

$$\leqq K\sqrt{x}\, v\left(\frac{x}{\log^2 x}\right) \sum_{a \leqq \sqrt{x}} \frac{1}{a} =$$

$$= K\sqrt{x}\, v\left(\frac{x}{\log^2 x}\right) O(\log x) =$$

$$= K\sqrt{x} \frac{\sqrt{x}\left(\log\log \frac{x}{\log^2 x}\right)^{k-1}}{\log x \,.\, \log \frac{x}{\log^2 x}} O(\log x).$$

Since $x > e^2$, we have $\log x > 2$, $\frac{x}{\log^2 x} < x$, $\log\log \frac{x}{\log^2 x} = O(\log\log x)$.

Further,

$$\log \frac{x}{\log^2 x} \sim \log x \quad (x \to \infty).$$

Therefore we get

$$S_2(x) = O\left(\frac{x(\log\log x)^{k-1}}{\log x}\right) \quad (x \to \infty). \tag{16}$$

By (12), (14), (16) we get at once

$$M_k(x) = O\left(\frac{x(\log\log x)^k}{\log x}\right) \quad (x \to \infty).$$

The proof is finished.

We can now prove Theorem 1.

Proof of Theorem 1. Let $\varepsilon > 0$. Choose an $m \in N$ such that

$$2^{-m} < \varepsilon. \tag{17}$$

Denote by $N_1(N_2)$ the set of all $n \in N'$ of the form $n = q_1 . q_2 \dots q_k . a^2$, where $k > m$ $(k \leqq m)$, $q_1, \dots, q_k$ are distinct prime numbers and $a \in N$. Then, evidently, $N = N_1 \cup N_2$, $N_1 \cap N_2 = \emptyset$.

Put for brevity $f = \varphi + \tau$. If $n \in N_1$, then in the standard form of n there are at least m odd primes. From the well-known equalities

$$\varphi(n) = n \prod_{p \mid n}\left(1 - \frac{1}{p}\right), \quad \tau(n) = \prod_{p} (\alpha(p) + 1)$$

$\left(n = \prod_{p \mid n} p^{\alpha(p)}\right)$ we get then $2^m \mid \varphi(n)$, $2^m \mid \tau(n)$, hence $2^m \mid f(n)$. But then the inclusion

$$f(N_1) \subset \{1 . 2^m, 2 . 2^m, \dots, j . 2^m, \dots\}$$

holds. From this we have (see (17))

$$\bar{d}(f(N_1)) \leqq 2^{-m} < \varepsilon. \tag{18}$$

Further, according to the definition of the set N_2 we see that

$$N_2 = M_0 \cup M_1 \cup \dots \cup M_m.$$

By Lemma 2 we have

$$N_2(x) = O\left(\frac{x(\log\log x)^m}{\log x}\right) \quad (x \to \infty). \tag{19}$$

Further,

$$\liminf_{n \to \infty} \frac{f(n)}{\left(\dfrac{n}{\log\log n}\right)} \geqq \liminf_{n \to \infty} \frac{\varphi(n)}{\left(\dfrac{n}{\log\log n}\right)} = e^{-\gamma} > 0 \tag{20}$$

(cf. [3], p. 267), where $\gamma > 0$ is the Euler's constant.

Using the denotation from Lemma 1 we choose $N_0 = N_2$, $x_0 = e^2$, $g(x) = \dfrac{\log x}{(\log\log x)^m}$, $h(x) = \log\log x$.

We shall verify that the assumptions of Lemma 1 are satisfied. Evidently we have

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} h(x) = +\infty,$$

$$g(x) > 0,\ h(x) > 0 \quad \text{for } x \in (e^2,\ +\infty).$$

Further,

$$N_0(x) = N_2(x) = O\left(\frac{x}{g(x)}\right) = O\left(\frac{x(\log\log x)^m}{\log x}\right)$$

(see (19)).

Hence the condition (a) in Lemma 1 is satisfied. The (b) holds, too, since the function

$$\frac{x}{h(x)} = \frac{x}{\log\log x}$$

is continuous and increasing in $(e^2,\ +\infty)$ and its limit if $x \to \infty$ equals $+\infty$.

Also (c) is satisfied because of $h(x) = \log\log x = o\left(\frac{\log x}{(\log\log x)^m}\right) = o(g(x))$ $(x \to \infty)$.

The condition

$$\liminf_{n\to\infty} \frac{f(n)}{\left(\frac{n}{h(n)}\right)} > 0$$

follows from (20).

Therefore according to Lemma 1 we have

$$d(f(N_2)) = 0. \tag{21}$$

Since $f(N) = f(N_1) \cup f(N_2)$, it follows from (18) and (21) that $\bar{d}(f(N)) < \varepsilon$. But ε is an arbitrary positive real number. Therefore $d(f(N)) = 0$, i.e. $d((\varphi + \tau)(N)) = 0$. This ends the proof.

For each $n > 30$ we have $\varphi(n) - \tau(n) > 0$ (cf. [6], p. 143, Exercise 7). In connection with Theorem 1 the following question arises: How large is the asymptotic density of the set of values of the sequence $\{\varphi(n) - \tau(n)\}_{n=31}^{\infty}$?

This question can be solved by a small modification of the method used in the proof of Theorem 1. A little greater change we must do only in the proof of the inequality

$$\liminf_{n\to\infty} \frac{\psi(n)}{\left(\frac{n}{\log\log n}\right)} > 0,$$

where $\psi(n) = \varphi(n) - \tau(n)$ (see (20)).

But here in suffices to remark that

$$\liminf_{n \to \infty} \frac{\psi(n)}{\left(\frac{n}{\log \log n}\right)} = \liminf_{n \to \infty} \frac{\varphi(n)}{\left(\frac{n}{\log \log n}\right)} - \lim_{n \to \infty} \frac{\tau(n)}{\left(\frac{n}{\log \log n}\right)} \tag{22}$$

and observe that $\tau(n) = O(n^{\varepsilon})$ for each $\varepsilon > 0$ (cf. [4], pp. 260, 267). Then we have

$$\lim_{n \to \infty} \frac{\tau(n)}{\left(\frac{n}{\log \log n}\right)} = 0$$

and (20) gives

$$\liminf_{n \to \infty} \frac{\psi(n)}{\left(\frac{n}{\log \log n}\right)} = e^{-\gamma} > 0.$$

The remaining modifications of the mentioned method are only slight and therefore they can be left to the reader. So we get the proof of the following result.

Theorem 2. The set of terms of the sequence $\{\varphi(n) - \tau(n)\}_{n=31}^{\infty}$ has the asymptotic density 0.

REFERENCES

1. Bero, P.: On sets of values of some arithmetical functions, Acta Fac. Rer. Nat. Univ. Com. XXXIII (1977), 179—182.
2. Erdös, P.: Über die Zahlen der Form $\sigma(n) - n$ und $n - \varphi(n)$, Elem. Math. 28 (1973), 83—86.
3. Hardy, G. H.—Wright, E. M.: An Introduction to the Theory of Numbers. Oxford, At the Clavendon Press, 1954.
4. Kanold, H. J.: Über zahlentheoretische Funktionen. J. rein angew. Math. 195 (1955), 180—191.
5. Niven, I.—Zuckerman, H. S.: An Introduction to the Theory of Numbers. John Wiley and Sons INC., New York—London—Sydney, 1967.
6. Sierpiński, W.: Teoria liczb. Warszawa—Wroclaw, 1950.

Authors' addresses:

Received: 2. 12. 1987

Helena Bereková
Katedra geometrie MFF UK
Mlynská dolina
842 15 Bratislava

Tibor Šalát
Katedra algebry a teórie čísel MFF UK
Mlynská dolina
842 15 Bratislava

SÚHRN

O HODNOTÁCH FUNKCIE $\varphi + \tau$

Helena Bereková — Tibor Šalát, Bratislava

Nech $\tau(n)$ označuje počet prirodzených deliteľov prirodzeného čísla n a φ označuje Eulerovu funkciu. V práci je dokázané, že množina členov každej z postupností $\{\varphi(n) + \tau(n)\}_{n=1}^{\infty}$, $\{\varphi(n) - \tau(n)\}_{n=31}^{\infty}$ má nulovú asymptotickú hodnotu.

РЕЗЮМЕ

О ЗНАЧЕНИЯХ ФУНКЦИИ $\varphi + \tau$

Гелена Берекова — Тибор Шалат, Братислава

Пусть $\tau(n)$ обозначает число натуральных делителей числа n и φ обозначает функцию Эйлера. В работе показано, что множество членов каждой из последовательностей $\{\varphi(n) + \tau(n)\}_{n=1}^{\infty}$, $\{\varphi(n) - \tau(n)\}_{n=31}^{\infty}$ имеет нулевую асимптотическую плотность.

DIDAKTIKA MATEMATIKY