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# A GENERAL APPROACH TO INTERIOR POINT TRANSFORMATION METHODS FOR MATHEMATICAL PROGRAMMING

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This paper is devoted to a new class of parametric interior point methods which do not possess the usual barrier property but whose other properties are analogous to those of barrier methods. It turns out that this new class of methods, which we call quasibarrier methods, is complementary to the class of barrier methods in a certain sense. The quasibarrier methods may also be of interest from the computational point of view, since their speed of convergence (with respect to the parameter) is faster than that of barrier methods.

The partition of interior point methods into barrier and quasibarrier methods is a natural consequence of the general convergence theory. The general convergence theorem (convergence in the sense of function values) is proved here under very weak conditions — that the transformation function has a minimum point. (Neither continuity, nor barrier type assumptions are used.) In the second phase we use the quasibarrier assumption to prove the existence of a minimum point.

## 1 Introduction

In transformation function methods the *mathematical programming problem*

$$\operatorname{Min}\{f(x) \mid g_i(x) \geq 0 \quad (i = 1, 2, \ldots, m)\} \tag{1}$$

is transformed into a sequence of *unconstrained problems*

$$\operatorname{Min}\{T_k(x) \mid x \in R^n\} \quad k = 1, 2, 3, \ldots$$

where $T_k(x)$ are such that:

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a) $T_k(x)$ *has a minimum point* $x^k$ on some open connected subset $X_k \subset R^n$ (i.e. $x^k \in X_k$ can be found by unconstrained methods);

b) The sequence of minimum points $\{x^k\}$ *converges* to an optimal solution $\hat{x}$ of problem (1) [or at least the sequence of values $\{f(x^k)\}$ converges to the optimal value $f(\hat{x})$ of problem (1)].

These two properties of functions $T_k(x)$ justify the well known *SUMT algorithm*:*)

Step 1. Define $T_1(x)$ on an open set $X_1 \subset R^n$. Starting from any $x^0 \in X_1$ find a minimum point $x^1 \in X_1$ of $T_1(x)$.

Step 2. Define $T_2(x)$ on an open set $X_2 \subset R^n$ such that $x^1 \in X_2$. Starting from $x^1 \in X_2$ find a minimum point $x^2 \in X_2$ of $T_2(x)$.

Step 3. Define $T_3(x)$ on an open set $X_3 \subset R^n$ such that $x^2 \in X_3$. Starting from $x^2 \in X_3$ find a minimum point $x^3 \in X_3$ of $T_3(x)$. Etc.

From the point of view of how the sequence $\{T_k(x)\}$ is generated we distinguish two classes of transformation methods:

a) *Parametric methods*, if $\{T_k(x)\}$ is generated by means of one or more parameters;

b) *nonparametric methods* (e.g. method of centres).

According to the position of minimum points $\{x^k\}$ with regard to the set of feasible solutions [of problem (1)]

$$K = \{x \in R^n \mid g_i(x) \geq 0 \quad (i = 1, 2, \ldots, m)\} \tag{1a}$$

or its "interior set"

$$K^0 = \{x \in R^n \mid g_i(x) > 0 \quad (i = 1, 2, \ldots, m)\} \tag{1b}$$

we distinguish the following three classes:

a) *interior point methods* (known also as *barrier methods*) if $x^k \in K^0$ for all $k = 1, 2, 3, \ldots$

b) *exterior point methods* (known also as *penalty methods*) if $x^k \notin K$ for all $k = 1, 2, 3, \ldots$

c) other methods (e.g. Lagrange and exponential methods).

Here we shall deal only with the parametric interior point methods whose transformation functions depend linearly on the parameter. A typical interior point transformation function *with linear parameter* for the mathematical programming problem (1) is:

$$T_k(x) = T(x, r_k) = f(x) + r_k \sum_{i=1}^{m} \Gamma[g_i(x)]$$

where $\{r_k\} \downarrow 0$ and $\Gamma$: $R_{++} \to R$ ($R_{++} = \{x \in R \mid x > 0\}$)

*) Sequential Unconstrained Minimization Technique algorithm.

has some additional properties which guarantee that $T(x, r_k)$ has a minimum point $x^k = x(r_k)$ on $K^0$. In the case of convex programming problems and *barrier* transformation functions these properties are [1]:

(A) *assymptotic property*: $\lim_{\xi \to \infty} \dfrac{\Gamma(\xi)}{\xi} = 0$

(B) *barrier property*: $\lim_{\xi \downarrow 0} \Gamma(\xi) = +\infty$

(CX) *convexity property*: $\Gamma(\xi)$ is convex.

The well known examples of barrier functions are:

$\Gamma_1(\xi) = \xi^{-1}$ (hyperbolic or inverse barrier of Fiacco-McCormick [1])

$\Gamma_2(\xi) = -\ln \xi$ (logarithmic barrier of Lootsma [10]).

It turns out that the *barrier property* (B) *can be replaced* by an analogous quasibarrier property (Q) (Hamala [6]—[9])

(Q) *quasibarrier property*: $\lim_{\xi \downarrow 0} \Gamma(\xi) = \bar{\Gamma}$ (finite)

and

$$\lim_{\xi \downarrow 0} \frac{\Gamma(\xi) - \bar{\Gamma}}{\xi} = -\infty.$$

As an example of a quasibarrier function we have:

$\Gamma_3(\xi) = -\sqrt{\xi}$ (square root quasibarrier [8]).

The different behaviour of the "trajectory" $f[x(r)]$ generated by $\Gamma_1$, $\Gamma_2$, $\Gamma_3$ can be illustrated by the following trivial example:

$$\text{Min}\,\{x \mid x \geq 0\}$$

with the optimal value $f(\hat{x}) = \hat{x} = 0$. Then

$$T_1(x, r) = x + rx^{-1} \Rightarrow x(r) = \sqrt{r}$$

$$T_2(x, r) = x - r \ln x \Rightarrow x(r) = r$$

$$T_3(x, r) = x - r\sqrt{x} \Rightarrow x(r) = \frac{1}{4} r^2$$

and $f[x(r)] \to f(\hat{x})$ as $r \downarrow 0$.

To justify the replacement of the mathematical programming problem (1) by a sequence of unconstrained type problems

$$\text{Min}\,\{T(x, r_k) \mid x \in K^0\}, \quad \{r_k\} \downarrow 0 \tag{2}$$

the theory of Parametric Interior Point Methods (PIPM) must answer the following questions:

*Question 1.* Under what assumptions does the solution $x^k = x(r_k)$ of (2) exist?

*Question 2.* What are the assumptions giving convergence? (The convergence in the sense of function values $f(x^k) \to f(\hat{x})$, or in the sense of solutions $x^k \to \hat{x}$.)

*Question 3.* What is the quality of approximation for a given value $r_k > 0$? (Measured by $|f(x^k) - f(\hat{x})|$, or by $\|x^k - \hat{x}\|$.)

Note that from the practical (i.e. applied) point of view it is usually more natural to use the "function measure". E.g. the minimum $f(\hat{x}) = 0$ of the function $f$: $R_+ \to R$

$$f(x) \begin{cases} = x & \text{if} \quad 0 \leq x \leq 1 \\ = x^{-1} & \text{if} \quad 1 < x \end{cases}$$

is at $\hat{x} = 0$. But if we compare $x_1 = 0.5$ and $x_2 = 10$, we see that the "closer approximation" $x_1 = 0.5$ gives a worse result $f(x_1) = 0.5$ than the "distant point" $x_2 = 10$ for which we have $f(x_2) = 0.1$.

The main theorem of Fiacco-McCormick [1] on barrier PIPM states the *existence* and *convergence* of a minimizing sequence $\{x^k\}$ of $T(x, r_k)$, ($x^k \to \hat{x}$ as $r_k \downarrow 0$). They use various assumptions (e.g. continuity of functions, boundedness of the set of optimal solutions, barrier property, etc.). Some of these assumptions are necessary to prove the existence $x^k$ and some are needed for convergence $x^k \to \hat{x}$. But since we have found that the barrier assumption can be replaced by an equally powerful quasibarrier assumption, *we propose to separate the question of existence* of $x^k \in K^0$ *from the question of convergence* $f(x^k) \to f(\hat{x})$. In this study the theory of PIPM is treated in the following way:

In the *first phase* we assume explicitly the existence of a minimum point $x^k$ [so-called Existence property (E)] and look for the most *general conditions of convergence* (i.e. Question 2). After this we give some estimation of the magnitude $|f(x^k) - f(\hat{x})|$ (i.e. Question 3).

In the *second phase* we consider sufficient conditions for the existence property (E), (i.e. Question 1). First we define a general quasibarrier property and prove the existence theorems for the compact case, as well as for the convex case. At the end of our study we apply the general theory of PIPM to the convex programming problem and explain the relation between the barrier and quasibarrier methods by means of perturbed Kuhn-Tucker conditions.

## 2 A General Model with Linear Parameter

By a *general constrained optimization problem* (GCOP) we mean a problem of the following form:

$$\operatorname{Min}\{F(x) \mid x \in M\}, \tag{3}$$

where $M \subset R^n$ is a *closed set* ($\emptyset \neq M \neq R^n$) and $F$: $X \to R$ ($M \subset X \subset R^n$). Instead of problem (3) we shall deal with the following *weakened form* of GCOP:*)

$$\operatorname{Inf}\{F(x) \mid x \in M^0\}, \tag{$3_{\text{inf}}$}$$

where $M^0 \subset R^n$ is an *arbitrary set* such that $cl\, M^0 = M$. For our purpose ($3_{\text{inf}}$) is more convenient than (3) because it allows one to get rid of such secondary assumptions as the requirement that the feasible set $M$ be closed or that there exist an optimal solution $\hat{x} \in M$.

**Remark.** Obviously, if $F(x)$ is continuous on $M$ and attains its minimum there (in $\hat{x} \in M$), then

$$\hat{F} = \inf_{x \in M^0} F(x) = \min_{x \in M} F(x) = F(\hat{x}),$$

i.e. the *optimal value* $\hat{F}$ of ($3_{\text{inf}}$) equals the optimal value $F(\hat{x})$ of (3).

Related to the problem ($3_{\text{inf}}$) we shall consider the following *parametrized problem*

$$\operatorname{Min}\{T(x, r) = F(x) + rG(x) \mid x \in M^0\}, \tag{4}$$

where $r > 0$ is a positive scalar parameter and $G$: $M^0 \to R$ is an *arbitrary function*. Hence $T$: $M^0 \times R_{++} \to R$ and, for any fixed $x^0 \in M^0$, $T(x^0, r)$ is linear (affine) in $r > 0$. In what follows, we shall assume that for any $r > 0$ the parametrized problem (4) has an optimal solution $x(r) \in M^0$. This basic assumption on the function $T(x, r)$ we call *"Existence property"* (E). Thus

$$\text{(E)} \qquad \forall r > 0 \ \exists x(r) \in M^0: T[x(r), r] = \min_{x \in M^0} T(x, r)$$

**Remark.** This (E) property is the only assumption we shall use to prove the convergence $F[x(r)] \to \hat{F}$ (as $r \downarrow 0$).

We shall treat the above-mentioned main questions of the theory of PIPM in the following sequence and with the following results:

Q2. We shall show that for the convergence $F[x(r)] \to \hat{F}$ (as $r \downarrow 0$) we need only the "Existence property" (E) (Theorem 4). For the convergence $x(r) \to \hat{x}$ it is sufficient to assume that $F(x)$ is continuous (Corollary 5).

Q3. We shall show that for the so-called quasibarrier type we have $F[x(r)] - \hat{F} = r\,.\,\omega(r)$ where $\omega(r) \to 0$ (as $r \downarrow 0$) (Theorem 6).

Q1. Together with the barrier property we shall define the quasibarrier property which, combined with a convexity or compactness assumption, implies the existence property (E) (Lemma 7 and Theorem 9).

*) Problem ($3_{\text{inf}}$) means "to find the infimum of $F(x)$ on $M^0$".

## 3 Monotonicity and Convergence in the General Model

Let $r_1 > r_2 > \ldots > r_k > r_{k+1} > \ldots > 0$
be a strictly decreasing sequence of positive numbers, such that $\lim r_k = 0$, and

$$T(x, r_k) = F(x) + r_k G(x)$$

be the function defined on an *arbitrary set* $M^0 \neq \emptyset$ satisfying the existence property

(E) $$\forall r_k > 0 \, \exists x^k = x(r_k) \colon T(x^k, r_k) = \min_{x \in M^0} T(x, r_k).$$

Denote:

$$x^k = x(r_k) \text{ — the minimum point of } T(x, r_k)$$

$$F_k = F(x^k); \quad G_k = G(x^k); \quad T_k = T(x^k, r_k).$$

**Lemma 1.** Let $0 < r_2 < r_1$. Then

a) $F_2 \leq F_1$
b) $G_2 \geq G_1$
c) $G_1 > 0 \Rightarrow T_2 < T_1$
d) $G_2 < 0 \Rightarrow T_2 > T_1$

**Proof.** For the proof of parts a), b), c) see Theorem 25 in Fiacco-McCormick [1].

d) If $G_2 < 0$, then obviously

$$T_2 = F_2 + r_2 G_2 > F_2 + r_1 G_2 \geq T_1$$

Q.E.D.

**Remark.** Fiacco-McCormick [1] considered only the first three relations of Lemma 1, which are involved in the theory of barrier functions. The last relation is fundamental in the case of quasibarrier functions.

**Corollary 2.** Let $\{r_k\} \downarrow 0$ and $\hat{F} = \inf_{x \in M^0} F(x)$.

a) If $G_1 > 0$, then

$$\hat{F} \leq F_{k+1} \leq F_k < T_k < T_1.$$

b) If $G(x) \leq 0$ for all $x \in M^0$, then

$$T_k \leq T_{k+1} \leq \hat{F} \leq F_{k+1} \leq F_k.$$

**Proof.**

a) Follows immediately from Lemma 1a, c.

b) It is necessary to show only that $T_k \leq \hat{F}$ (the other inequalities follow from Lemma 1a, d).
Since $G(x) \leq 0$ for all $x \in M^0$, we have

$$T_k \leq T(x, r_k) = F(x) + r_k G(x) \leq F(x), \text{ for all } x \in M^0.$$

Thus

$$T_k \leq \inf_{x \in M^0} F(x) = \hat{F}.$$

Q.E.D.

**Remark.** There is an essential difference between the relation of Corollary 2a (typical for barrier methods) and the relation of Corollary 2b (typical for quasibarrier methods). In the second case, successive steps of the SUMT algorithm provide us with *monotone sequences of lower and upper bounds* of the optimal value $\hat{F}$. On the other hand, the lower bound for barrier methods is obtainable only in the convex case, by the weak duality theorem.

Note that in the first case $\hat{F}$ can be finite as well as $-\infty$, but in the second case $\hat{F}$ is finite. (Thus, the existence property (E) together with the assumption that $G(x) \leq 0$ for all $x \in M^0$, imply that $\hat{F}$ is finite.)

**Corollary 3.** Let $\{r_k\} \downarrow 0$ and $\hat{F} = \inf_{x \in M^0} F(x)$. Then

a) $\lim F_k \geq \hat{F}$

b) $\lim G_k \triangleq \hat{G} > -\infty$

c) $G_1 > 0 \Rightarrow \lim T_k \geq \hat{F}$

d) For all $x \in M^0$ $G(x) \leq 0 \Rightarrow \lim T_k \leq \hat{F}$.

**Proof.** Follows immediately from Lemma 1 and Corollary 2.

Note that the relation b) defines a quantity $\hat{G}$ wich is either finite or $+\infty$.

**Theorem 4** (Convergence). Let $\{r_k\} \downarrow 0$. Then

a) $\lim T_k = \hat{F}$

b) $\lim F_k = \hat{F}$.

Moreover, if $\hat{F} = \inf_{x \in M^0} F(x)$ is finite, then also

c) $\lim r_k G_k = 0$.

**Proof.**

**I.** Let $\hat{F} = -\infty$.

Ia)

$$\forall N > 0 \quad \exists x_N \in M^0 \colon F(x_N) < -N \tag{5}$$

Now

$$T_k \leq T(x_N, r_k) = F(x_N) + r_k G(x_N) < -N + r_k G(x_N)$$

and since $\{r_k\} \downarrow 0$

$$\forall N > 0 \quad \limsup T_k \leq -N,$$

which implies

$$\lim T_k = -\infty \, (= \hat{F}).$$

Ib) The monotonicity of $\{G_k\}$ (Lemma 1b) implies

$$\limsup(-r_k G_k) < +\infty$$

Now, applying this and the proved part Ia) to the relation

$$F_k = T_k - r_k G_k$$

we get

$$\limsup F_k = -\infty \ (= \hat{F}).$$

**II.** Let $\hat{F}$ be finite.

IIc)

$$\forall \varepsilon > 0 \quad \exists x_\varepsilon \in M^0\colon F(x_\varepsilon) < \hat{F} + \varepsilon \tag{6}$$

Now

$$\begin{aligned} \hat{F} + r_k G_k \le F_k + r_k G_k = T_k \le T(x_\varepsilon, r_k) = \\ = F(x_\varepsilon) + r_k G(x_\varepsilon) < \hat{F} + \varepsilon + r_k G(x_\varepsilon) \end{aligned} \tag{7}$$

or

$$\forall \varepsilon > 0\colon r_k G_k \le \varepsilon + r_k G(x_\varepsilon),$$

which implies

$$\limsup r_k G_k \le 0.$$

But by Lemma 1b

$$\liminf r_k G_k \ge 0,$$

therefore

$$\lim r_k G_k = 0.$$

IIa) By (7) and IIc)

$$\forall \varepsilon > 0\colon \hat{F} \le \liminf T_k \le \limsup T_k \le \hat{F} + \varepsilon,$$

which implies

$$\hat{F} = \lim T_k.$$

IIb) Applying IIa) and IIc) to the relation $T_k - r_k G_k = F_k$ we get

$$\hat{F} = \lim F_k.$$

Q.E.D.

**Remark.** The last relation in Theorem 4 does not hold in general, if $\hat{F} = -\infty$. For example, if we take $M^0 = R_{++}$, $F(x) = \ln x$, $G(x) = x^{-1}$,

$$T(x, r) = \ln x + rx^{-1}, \text{ then } \hat{F} = -\infty,\ x^k = r_k \text{ and } r_k G_k = r_k r_k^{-1} = 1.$$

**Corollary 5.** Let $F(x)$ be continuous on $M = \operatorname{cl} M^0$, the set $\hat{M}$ of all optimal solutions $\hat{x}$ of (3) be nonempty and let $\{r_k\} \downarrow 0$. Then every limit point of $\{x^k\}$ belongs to $\hat{M}$.

**Proof.** Let $\bar{x} \in M$ be an accumulation point of $\{x^k\}$. (For the sake of simplicity of notation suppose that $x^k \to \bar{x}$.) Then from continuity of $F(x)$ it follows that

$$\lim F(x^k) = F(\bar{x}).$$

But according to Theorem 4b

$$\lim F(x^k) = \hat{F}$$

and so $\bar{x} \in \hat{M}$. Q.E.D.

**Remark.** We have proved the convergence Theorem 4 under the assumption of the validity of the existence property (E). It can be shown that the convergence takes place even under a weakened assumption:

$(E_{\inf})$ $\qquad \forall r > 0\colon t(r) = \inf_{x \in M^0} T(x.r) \quad$ is finite.

Then $\qquad \forall r > 0 \, \exists \, y(r) \in M^0\colon t(r) \leq T[y(r), r] < t(r) + r$

and it can be proved that

$$\lim_{r \downarrow 0} t(r) = \lim_{r \downarrow 0} T[y(r), r] = \hat{F} \tag{8}$$

$$\lim_{r \downarrow 0} F[y(r)] = \hat{F}. \tag{9}$$

Moreover, if $\hat{F}$ is finite, then

$$\lim_{r \downarrow 0} r \,.\, G[y(r)] = 0. \tag{10}$$

**Remark.** If for any $r > 0$ we solve the parametrized problem (4) with better than $\delta$-accuracy, i.e. if for any $r > 0$ we find $z(r) \in M^0$ such that

$$t(r) \leq T[z(r), r] < t(r) + \delta$$

and if $\hat{F}$ is finite, then

$$\hat{F} \leq \liminf_{r \downarrow 0} T[z(r), r] \leq \limsup_{r \downarrow 0} T[z(r), r] \leq \hat{F} + \delta$$

$$\hat{F} \leq \liminf_{r \downarrow 0} F[z(r)] \leq \limsup_{r \downarrow 0} F[z(r)] \leq \hat{F} + \delta$$

$$0 \leq \liminf_{r \downarrow 0} r \,.\, G[z(r)] \leq \limsup_{r \downarrow 0} r \,.\, G[z(r)] \leq \delta.$$

Moreover, if $G(x) \leq 0$ for all $x \in M^0$, then

$$T[z(r), r] - \delta \leq \hat{F} \leq F[z(r)]$$

and

$$\lim_{r \downarrow 0} r \,.\, G[z(r)] = 0.$$

Let us return once more to Corollary 3b. We see that there can be two qualitatively different cases: $\hat{G} = +\infty$ and $\hat{G}$ finite. Accordingly, we introduce two notions:

**Definition.** We shall say that the transformation function $T(x, r) = F(x) + r \,.\, G(x)$ is of

a) *B-type*, if $\quad \lim G_k = +\infty$;

b) *Q-type*, if $\quad \lim G_k = \hat{G}$ (finite).

**Theorem 6.** Let $\{r_k\} \downarrow 0$ and $\hat{F}$, $\hat{G}$ be finite. Then

$$0 \leq F_k - \hat{F} \leq r_k \omega_k,$$

where

$$\lim_{k \to \infty} \omega_k = 0,$$

i.e., the rate of convergence of a *Q-type* method is *superlinear* with respect to the parameter.

**Proof.** Let $0 < r_j < r_k$ $(j > k)$. Then

$$T_k = F_k + r_k G_k \leq F_j + r_k G_j.$$

Now, applying Theorem 4b and Corollary 3b $(j \to +\infty)$ we get

$$F_k + r_k G_k \leq \hat{F} + r_k \hat{G}$$

or

$$0 \leq F_k - \hat{F} \leq r_k(\hat{G} - G_k) \triangleq r_k \omega_k,$$

where

$$\lim \omega_k = \lim (\hat{G} - G_k) = 0.$$

Q.E.D.

**Remark.** There does not exist any analogous theorem for *B-type* methods. For example in the case of the logarithmic or hyperbolic barrier function we have only (see also the trivial example in the Introduction):

$$F_k - \hat{F} \approx c_1 r_k \quad \text{(linear rate of convergence [10], [11])}$$

or

$$F_k - \hat{F} \approx c_2 \sqrt{r_k} \quad \text{(worse than linear rate [10], [11])}.$$

## 4 Sufficient Conditions for the Existence Property

Consider a function $H$: $M^0 \to R$, where $\emptyset \neq M^0 \subset R^n$ is an *open set*, and its "Existence property"

$$(\mathrm{E_H}) \qquad \exists\, x^0 \in M^0\colon H(x^0) = \min_{x \in M^0} H(x).$$

In the case of a continuous function $H(x)$ and a bounded set $M^0$ the obvious condition for this $(\mathrm{E_H})$ property to hold is that for any point $y \in M^0$ which is "sufficiently close" to the boundary of $M^0$ we have

$$H(y) > \inf_{x \in M^0} H(x).$$

For example, the class of the well-known barrier functions has this property. We shall define another class of functions (so-called quasibarrier functions) which also satisfies the above relation.

**Definition.** Let $\emptyset \neq M^0 \subset R^n$ be an open set and $M = \mathrm{cl}\, M^0$. A sequence $\{y^k\} \subset M^0$ is said to be a *boundary sequence* if $y^k \to \bar{y} \in \partial M = M - M^0$. A function $H$: $M^0 \to R$ is said to be

a) a *barrier function*, if *for any* boundary sequence

$$\lim_{k \to \infty} H(y^k) = +\infty,$$

b) a *quasibarrier* function, if for each boundary point $\bar{y} \in \partial M$ *there exists at least one* sequence $y^k \to \bar{y}$ such that

$$\lim H(y^k) = \bar{H}_{\{y^k\}} \text{ (finite)} \tag{11}$$

and

$$\lim_{k \to \infty} \frac{H(y^k) - \bar{H}_{\{y^k\}}}{\|y^k - \bar{y}\|} = -\infty. \tag{12}$$

**Lemma 7.** (The existence property for a continuous function and a bounded set.)

Let $\emptyset \neq M^0 \subset R^n$ be open and *bounded.*

a) If $H(x)$ is continuous on $M^0$ and barrier, then it has property $(\mathrm{E_H})$.

b) If $H(x)$ is continuous on $M = \mathrm{cl}\, M^0$ and quasibarrier on $M^0$, then it has property $(\mathrm{E_H})$.

**Proof.**

a) See Fiacco — McCormick [1] (Corollary 8, p. 46).

b) Since $H(x)$ is continuous on the compact set $M = \mathrm{cl}\, M^0$, there exists $y^0 \in M$ such that

$$H(y^0) = \min_{x \in M} H(x). \tag{13}$$

Suppose that $y^0 \in M - M^0$. Then by definition there exists a boundary sequence $y^k \to y^0$ such that

$$\lim_{k \to \infty} \frac{H(y^k) - H(y^0)}{\|y^k - y^0\|} = -\infty.$$

But this implies that for sufficiently large index $k$, $H(y^k) < H(y^0)$, contradicting (13).

Q.E.D.

**Remark.** In Lemma 7 it is sufficient (in both cases) to require only the lower semicontinuity of $H(x)$ on $M^0$.

**Remark.** By excluding the possible uncertainty of expressions of the type $(-\infty) + (+\infty)$, we can immediately apply the notion of a barrier and quasibarrier function to the transformation function $T(x, r) = F(x) + r \,.\, G(x)$. The details are given in the following lemma.

**Lemma 8.** The transformation function $T(x, r) = F(x) + r \,.\, G(x)$, defined on an open set $M^0$ is:

a) *barrier* if $G(x)$ is a barrier function and for any boundary sequence $\{y^k\}$

$$\liminf_{k \to \infty} F(y^k) > -\infty, \tag{14}$$

b) *quasibarrier* if $G(x)$ is a quasibarrier function and for any boundary sequence $y^k \to \bar{y}$

$$\lim F(y^k) = F_{\{y^k\}} \quad \text{(finite)} \tag{15}$$

and

$$\limsup_{k \to \infty} \frac{F(y^k) - F_{\{y^k\}}}{\|y^k - \bar{y}\|} < +\infty. \tag{16}$$

**Remark.** The conditions (14), (15) are obviously fulfilled if $F(x)$ is continuous on $M = \operatorname{cl} M^0$. The condition (16) is fulfilled if $F(x)$ satisfies a Lipschitz condition on $M = \operatorname{cl} M^0$.

**Theorem 9.** (The existence property for a convex transformation function.) Let:

a) $\emptyset \neq M^0 \subset R^n$ be an *open convex* set, $M = \operatorname{cl} M^0$;

b) the set $\hat{M}$ of all optimal solutions of the problem (3) is *nonempty* and *bounded*;

c) if $M$ is *unbounded*, then for any ray $x + \mu s \subset M$ ($\mu \in R_+$, $0 \neq s \in R^n$)

$$\lim_{\mu \to \infty} \frac{1}{\mu} G(x + \mu s) = 0; \tag{17}$$

d) $F(x)$ is *convex* and *continuous* on $M$;

and $e_B$) $G(x)$ is *convex* and *barrier* on $M^0$,

or $e_Q$) $G(x)$ is *convex*, *continuous* on $M$ and *quasibarrier* on $M^0$.

Then for any $r > 0$ the transformation function $T(x, r) = F(x) + r \,.\, G(x)$ has the (E) property.

**Proof.** I. For the case of barrier assumption ($e_B$) see Fiacco-McCormick [1] (Theorem 25, p. 97).

II. Consider the quasibarrier assumption ($e_Q$). First we shall show that the set

$$S_r = \{x \in M \mid T(x, r) \leq T(\hat{x}, r)\}, \quad \hat{x} \in \hat{M}$$

is bounded. (Obviously, $S_r$ is convex and closed.) Suppose that $S_r$ is unbounded.

Then it contains a ray

$$w = w(\mu) = \hat{x} + \mu(z - \hat{x}), \quad \mu > 1,$$

where $z \in S_r$ can be chosen in such a way (by assuption b) that

$$\varepsilon = F(z) - F(\hat{x}) > 0.$$

Then by convexity of $F(x)$:

$$F(w) \geq F(\hat{x}) + \mu . \varepsilon$$

and

$$T(w, r) = F(w) + r . G(w) \geq F(\hat{x}) + \mu\varepsilon + r . G[\hat{x} + \mu(z - \hat{x})] =$$

$$= F(\hat{x}) + \mu\left\{\varepsilon + r\frac{1}{\mu}G[\hat{x} + \mu(z - \hat{x})]\right\}.$$

But by assumption c) the last expression tends to $+\infty$ as $\mu \to \infty$. This contradiction proves the boundedness of $S_r$.

Now the continuous function $T(x, r)$ on a compact set $S_r$ attains its minimum at some point $x(r) \in S_r \subset M$. But, obviously,

$$T[x(r), r] = \min_{x \in S_r} T(x, r) = \inf_{x \in M} T(x, r).$$

So $T(x, r)$ attains its minimum on $M$. Now, by the same argument as in the proof of Lemma 7b), it can be shown that $x(r)$ cannot be a boundary point of $M$, i.e. $x(r) \in M^0$.

Q.E.D.

## 5 Quasibarrier Functions for Convex Programming

Here we apply the results obtained for the general model (3), (4) to the convex programming problem (1) with convex functions $f$, $-g_i$ defined on $R^n$, the nonempty open set $K^0$ (1b) and the convex transformation function

$$T(x, r) = f(x) + r \cdot \sum_{i=1}^{m} \Gamma[g_i(x)] = f(x) + r . G(x), \tag{18}$$

where $\Gamma$: $R_{++} \to R$. Now for the function

$$G(x) = \sum_{i=1}^{m} \Gamma[g_i(x)] \tag{19}$$

— *to be convex*, it is sufficient to require that

(CX) $\Gamma(\xi)$ be convex for $\xi > 0$, and

(M) $\Gamma(\xi)$ be (monotonically) nonincreasing;

— *to be quasibarrier*, it is sufficient (as is proved below) to require that

$$\text{(Q)} \qquad \begin{cases} \lim\limits_{\xi \downarrow 0} \Gamma(\xi) = \bar{\Gamma} \quad \text{(finite)} \\ \lim\limits_{\xi \downarrow 0} \dfrac{\Gamma(\xi) - \bar{\Gamma}}{\xi} = -\infty; \end{cases}$$

— *to be "asymptotic"* in the sense of (17), it is sufficent (as is proved below) to require that

$$\text{(A)} \qquad \lim_{\xi \to \infty} \frac{\Gamma(\xi)}{\xi} = 0.$$

These four assumptions (CX), (M), (Q), (A) are not independent. It is elementary to prove the following.

**Lemma 10.** Assumptions (CX), (A) imply (M).

**Remark.** Since convexity of $\Gamma(\xi)$ implies continuity for $\xi > 0$, we can continuously extend $\Gamma(\xi)$ to $\xi = 0$ by

$$\Gamma(0) = \lim_{\xi \downarrow 0} \Gamma(\xi) = \bar{\Gamma}.$$

Hence $\Gamma(\xi)$ can be considered as convex continuous on $R_+$ and property (Q) can be defined as

$$\lim_{\xi \downarrow 0} \frac{\Gamma(\xi) - \Gamma(0)}{\xi} = -\infty.$$

We are now in a position to prove a convergence theorem for convex programming and quasibarrier transformation functions which is a complete analogue to the basic convergence Theorem 25 ([1] p. 97) of Fiacco-McCormick concerning barrier transformation functions.

**Theorem 11.** Consider the convex programming problem (1) with the nonempty set $K^0$ (1b) and a nonempty bounded set $\hat{K}$ of all optimal solutions $\hat{x}$. Let $\Gamma$: $R_{++} \to R$ have the properties (A), (CX), (Q). Then the *quasibarrier* transformation function (18) has the following properties:

(i) For any $r_k > 0$ there exists $x^k = x(r_k) \in K^0$ which minimizes $T(x, r_k)$ on $K^0$;
For $\{r_k\} \downarrow 0$

$$\text{(ii)} \qquad \lim_{k \to \infty} T(x^k, r_k) = \lim_{k \to \infty} f(x^k) = f(\hat{x});$$

(iii) Every limit point of $\{x^k\}$ belongs to $\hat{K}$;

(iv) The sequence $\{T(x^k, r_k)\}$ is monotonically strictly increasing if $\Gamma(\xi) < 0$ for $\xi > 0$;

(v) The sequence $\{f(x^k)\}$ is monotonically decreasing;

(vl) The sequence $\{G(x^k)\}$ defined by (19) is monotonically increasing and converges to a finite value.

**Proof.** (i) To prove the existence property (E) we shall show that the assumptions of Theorem 9 are fulfilled. The assumptions a), b), d) are obviously fulfilled. We need only to show that the function $G(x)$ (19) has the asymptotic property (17) and is quasibarrier.

**I.** First we shall show that $G(x)$ is quasibarrier on $K^0$. Let $\bar{y} \in K - K^0$ be a boundary point of $K^0$. We shall construct a sequence $\{y^k\} \subset K^0$, $y^k \to \bar{y}$ which satisfies the relations (11), (12) of the definition of a quasibarrier function. Let $x^0 \in K^0$, $\bar{s} = x^0 - \bar{y}$, $\{\lambda_k\} \subset R_{++}$, $\lambda_k \downarrow 0$, $y^k = \bar{y} + \lambda_k \bar{s}$, $\bar{I} = \{i | g_i(\bar{y}) = 0\}$. Obviously,

$$\lim_{k \to \infty} G(y^k) = \sum_{i=1}^{m} \lim_{k \to \infty} \Gamma[g_i(y^k)] = \sum_{i=1}^{m} \Gamma[g_i(\bar{y})] = \bar{G}$$

where $\bar{G}$ is finite. Now

$$\frac{G(y^k) - \bar{G}}{\|y^k - \bar{y}\|} = \frac{G(y^k) - \bar{G}}{\lambda_k \|\bar{s}\|} = \frac{1}{\lambda_k \|\bar{s}\|} \sum_{i=1}^{m} \{\Gamma[g_i(y^k)] - \Gamma[g_i(\bar{y})]\} =$$

$$= \frac{1}{\|\bar{s}\|} \sum_{i \notin \bar{I}} \frac{\Gamma[g_i(\bar{y} + \lambda_k \bar{s})] - \Gamma[g_i(\bar{y})]}{\lambda_k} +$$

$$+ \frac{1}{\|\bar{s}\|} \sum_{i \in \bar{I}} \left\{ \frac{\Gamma[g_i(\bar{y} + \lambda_k \bar{s})] - \bar{\Gamma}}{g_i(\bar{y} + \lambda_k \bar{s})} \right\} \cdot \frac{g_i(\bar{y} + \lambda_k \bar{s})}{\lambda_k}.$$

The first sum ($i \notin \bar{I}$) is finite as $\lambda_k \downarrow 0$ (directional derivative of a convex function). By the (Q) assumption, the first terms in the second sum ($i \in \bar{I}$) tend to $-\infty$ as $\lambda_k \downarrow 0$. The terms

$$\frac{g_i(\bar{y} + \lambda_k \bar{s})}{\lambda_k}, \quad i \in \bar{I}$$

are positive and (by concavity of $g_i$) monotonically increasing as $\lambda_k \downarrow 0$. Hence

$$\lim_{k \to \infty} \frac{G(y^k) - \bar{G}}{\|y^k - \bar{y}\|} = -\infty.$$

**II.** Now we shall show that for any ray $x + \mu s \in K$ (1a) the function $G(x)$ (19) has the asymptotic property (17). Consider the ray $w = w(\mu) = \bar{x} + \mu \bar{s}$, ($\mu > 1$) in $K$, where $\bar{x} \in K$, $\bar{s} = \bar{z} - \bar{x}$, $\bar{z} \in K$, $\bar{z} \neq \bar{x}$.

By concavity of $g_i(x)$ we have

$$0 \leq g_i(w) \leq g_i(\bar{x}) + \mu . \delta_i, \quad (\mu > 1) \tag{20}$$

where $\delta_i = g_i(\bar{z}) - g_i(\bar{x}) \geq 0$. (The fact that $\delta_i \geq 0$ follows from (20); otherwise the right-hand term of (20) would become negative as $\mu$ goes to infinity.) Now by Lemma 10

$$G(\bar{x} + \mu\bar{s}) = G(w) = \sum_{i=1}^{m} \Gamma[g_i(w)] \geq \sum_{i=1}^{m} \Gamma[g_i(\bar{x}) + \mu \,.\, \delta_i],$$

which, by assumption (A), implies

$$\lim_{\mu \to \infty} \frac{1}{\mu} G(\bar{x} + \mu \,.\, \bar{s}) \geq 0. \tag{21}$$

From (20) we see that the concave function

$$\gamma_i(\mu) = g_i(w) = g_i(\bar{x} + \mu\bar{s})$$

is nondecreasing. (Otherwise, there are $1 < \mu_1 < \mu_2$ such that $\gamma(\mu_1) > \gamma(\mu_2)$, and by concavity of $\gamma(\mu)$ we get

$$\gamma(\mu) \leq \gamma(\mu_2) + \frac{\mu - \mu_2}{\mu_2 - \mu_1} [\gamma(\mu_2) - \gamma(\mu_1)] \xrightarrow[\mu \to \infty]{} -\infty$$

contradicting the first inequality of (20)). Hence by Lemma 10 the convex function

$$\varphi_i(\mu) = \Gamma[\gamma_i(\mu)] = \Gamma[g_i(\bar{x} + \mu\bar{s})]$$

is nonincreasing, which implies

$$\lim_{\mu \to \infty} \frac{1}{\mu} G(\bar{x} + \mu\bar{s}) \leq 0. \tag{22}$$

The relation (22) together with (21) imply (17). Now the statement (i) of Theorem 11 follows from Theorem 9.

(ii) Follows from the convergence Theorem 4.
(iii) Follows from Corollary 5.
(iv) Follows from Lemma 1d.
(v) Follows from Lemma 1a.
(vi) Follows from Lemma 1b and the continuity of $G(x)$ on $K$ (1a).

Q.E.D.

**Theorem 12** (Dual convergence). If in addition to the assumptions of Theorem 11 the functions $f$, $g_i$, $\Gamma$ are assumed to be differentiable and we put

$$u_i^k = -r_k \Gamma'[g_i(x^k)], \quad (i = 1, 2, \ldots, m) \tag{23}$$

then the point $(x^k, u^k)$ is a feasible solution to the dual problem

$$\operatorname{Max} \{L(x, u) \mid \nabla_x L(x, u) = 0, \quad u \geq 0\} \tag{24}$$

$$L(x, u) = f(x) - \sum_{i=1}^{m} u_i g_i(x).$$

Moreover, all limit points of the sequence $\{(x^k, u^k)\}$ are finite optimal solutions to the dual problem (24), and at least one limit point exists.

**Proof** is analogous to that for barrier functions ([1], Theorem 26, p. 98) so we do not repeat it here.

**Remark.** From Theorem 12, by the weak duality theorem ([1], Theorem 22, p. 92), we get the following estimation for the optimal objective function value:

$$0 \leq f(x^k) - f(\hat{x}) \leq -r_k \sum_{i=1}^{m} \Gamma'[g_i(x^k)] g_i(x^k).$$

This estimation can be improved in the following way. By convexity of $L(x, u^k)$ and Theorem 12

$$L(\hat{x}, u^k) - L(x^k, u^k) \geq \nabla_x L(x^k, u^k)^T (\hat{x} - x^k) = 0,$$

which implies

$$0 \leq f(x^k) - f(\hat{x}) \leq \sum_{i=1}^{m} u_i^k [g_i(x^k) - g_i(\hat{x})] =$$

$$= \sum_{i \in \hat{I}} u_i^k g_i(x^k) + \sum_{i \notin \hat{I}} u_i^k [g_i(x^k) - g_i(\hat{x})],$$

where

$$\hat{I} = \{i \mid g_i(\hat{x}) = 0\}.$$

After substituting $u_i^k$ according to (23) we get

$$O \leq f(x^k) - f(\hat{x}) \leq r_k \hat{\omega}_k + r_k \omega_k^0,$$

where

$$\hat{\omega}_k = -\sum_{i \in \hat{I}} \Gamma'[g_i(x^k)] g_i(x^k) \geq 0$$

$$\omega_k^0 = -\sum_{i \notin \hat{I}} \Gamma'[g_i(x^k)][g_i(x^k) - g_i(\hat{x})].$$

Note that by assumption (Q)

$$\lim_{k \to \infty} \Gamma'[g_i(x^k)] \begin{cases} = \Gamma'[g_i(\hat{x})] & \text{for } i \notin \hat{I} \\ = -\infty & \text{for } i \in \hat{I}. \end{cases}$$

Hence

$$\lim_{k \to \infty} \omega_k^0 = 0.$$

By convexity and monotonicity of $\Gamma(\xi)$ we have

$$0 \leq -\xi\Gamma'(\xi) \leq \Gamma(0) - \Gamma(\xi),$$

which implies (in accordance to the superlinear convergence of quasibarrier type methods proved in Theorem 6)

$$\lim_{k\to\infty} \hat{\omega}_k = 0.$$

Note that in performing the SUMT procedure we can gradually identify all nonbinding constraints ($i \notin \hat{I}$) and omit them from the following iteration steps. Thus for sufficiently small $r > 0$ we can deal entirely with the binding constraints. In such a case the speed of convergence is determined by

$$\sum_{i\in\hat{I}} u_i^k g_i(x^k) = r_k \hat{\omega}_k = -r_k \sum_{i\in\hat{I}} \Gamma'[g_i(x^k)]g_i(x^k).$$

**Example.** For $h > 1$ let

$$\Gamma(\xi) = -\xi^{\left(1-\frac{1}{h}\right)}, \quad \text{or } \Gamma'(\xi) = -\left(1-\frac{1}{h}\right)\xi^{-\frac{1}{h}}.$$

Then

$$u_i(r) = -r\Gamma'[g_i(x(r))] = r\left(1-\frac{1}{h}\right)[g_i(x(r))]^{-\frac{1}{h}},$$

which implies

$$g_i(x(r)) = \left[\frac{r}{u_i(r)}\right]^h\left(1-\frac{1}{h}\right)^h.$$

Hence

$$r\,.\,\hat{\omega}(r) = \sum_{i\in\hat{I}} u_i(r)\,.\,g_i(x(r)) = r^h\left(1-\frac{1}{h}\right)^h \sum_{i\in\hat{I}} \frac{1}{[u_i(r)]^{h-1}}.$$

Suppose that for $i \in \hat{I}$ (Theorem 12)

$$\lim_{r\downarrow 0} u_i(r) = \hat{u}_i > 0.$$

Then for some constant $c > 0$ we have

$$r\,.\,\hat{\omega}(r) \approx c\,.\,r^h, \quad (h > 1)$$

or

$$0 \leq f[x(r)] - f(\hat{x}) \lesssim c\,.\,r^h$$

**Remark.** For $h = 2$ we have $\Gamma(\xi) = -\sqrt{\xi}$. So the convergence of the "squareroot quasibarrier method" is quadratic with regard to the parameter $r > 0$ (provided that $\hat{u}_i > 0$ for $i \in \hat{I}$ and the nonbinding constraints are omitted).

**Remark.** In general, the value of the limit

$$\lim_{k \to \infty} \frac{\omega_k^0}{\hat{\omega}_k} = \lim_{r \downarrow 0} \frac{\omega^0(r)}{\hat{\omega}(r)}$$

can be any nonnegative number (including $+\infty$) as is demonstrated by the following example. Let us consider

$$\operatorname{Min}\{(x_1 - 1)^p + 2x_2 \mid x_1 \geq 0,\ x_2 \geq 0\}, \quad (p \geq 1).$$

This is a convex problem with a unique optimal solution $\hat{x} = (1, 0)$. Let $\Gamma(\xi) = -\sqrt{\xi}$. Then $\nabla_x T(x, r) = 0$ gives a unique trajectory $x(r)$:

$$2p\sqrt{x_1}(x_1 - 1)^{p-1} = r, \quad 4\sqrt{x_2} = r, \quad \text{or} \quad \sqrt{x_2} = \frac{p}{2}\sqrt{x_1}(x_1 - 1)^{p-1}.$$

Thus we have

$$\omega^0 = \frac{1}{2\sqrt{x_1}}(x_1 - 1); \quad \hat{\omega} = \frac{1}{2}\sqrt{x_2} = \frac{p}{4}\sqrt{x_1}(x_1 - 1)^{p-1};$$

and

$$\lim_{r \downarrow 0} \frac{\omega^0(r)}{\hat{\omega}(r)} = \lim_{x_1 \to 1} \frac{2}{px_1}(x_1 - 1)^{2-p} \begin{cases} = 0 & \text{if } 1 \leq p < 2 \\ = 1 & \text{if } p = 2 \\ = +\infty & \text{if } p > 2. \end{cases}$$

## 6 The Relation between the PIPM and Kuhn-Tucker Conditions

In this part we shall prove a theorem on perturbed Kuhn-Tucker conditions which formally unifies both the theory of barrier and quasibarrier methods and at the same time shows the close relation between the theory of parametric interior point methods and the Kuhn-Tucker optimality conditions. We begin with some intuitive considerations which will lead us to a natural problem formulation.

Consider the convex programming problem (1) with differentiable convex functions $f$, $-g_i$, the nonempty set $K^0$ (1b) and an optimal solution $\hat{x}$. The related Kuhn-Tucker (necessary and sufficient) optimality conditions are:

$$\nabla f(\hat{x}) - \sum_{i=1}^{m} \hat{u}_i \nabla g_i(\hat{x}) = \bar{0} \tag{25a}$$

$$(i = 1, 2, \ldots, m) \begin{cases} g_i(\hat{x}) \geq 0 & \text{(25b)} \\ \hat{u}_i \cdot g_i(\hat{x}) = 0 & \text{(25c)} \\ \hat{u}_i \geq 0. & \text{(25d)} \end{cases}$$

If $T(x, r)$ is some convex and differentiable (interior point) transformation function of problem (1), then its minimum point $x(r) \in K^0$ is determined by

$$\nabla_x T(x, r) = \bar{0}. \tag{26}$$

If $x(r) \to \hat{x}$, or at least $f[x(r)] \to f(\hat{x})$ (as $r \downarrow 0$), then the relation (26) can be interpreted as some approximation of the Kuhn-Tucker conditions (25) (as will be made more clear below).

**Problem.** Is it possible to proceed backwards?

Or in other words, are there any reasonable approximations of the K-T conditions (25) from which we can derive some appropriate transformation functions? Is there any simple way to derive all transformation functions of the form (18) by modifying the K-T conditions (25)?

**Example.** Fiacco-McCormick ([1] p. 40) replace the complementarity condition (25c) by the perturbed one

$$u_i g_i(x) = r, \quad (r > 0) \tag{25r}$$

and then substitute for $u_i$ into (25a) to get

$$\nabla f(x) - r \sum_{i=1}^{m} \frac{1}{g_i(x)} \nabla g_i(x) = \bar{0}$$

or

$$\nabla \left\{ f(x) - r \sum_{i=1}^{m} \ln g_i(x) \right\} = \bar{0}.$$

Hence the perturbation (25r) generates the Logarithmic barrier function.

Now, following the basic idea of the perturbation (25r), we shall present a general scheme of modifying the K-T conditions, which will enable us to generate all barrier and quasibarrier transformation functions of the form (18).

Consider a function $\Pi$: $R_+ \to R$ such that $\Pi(0) = 0$, $\Pi(\xi) > 0$ for $\xi > 0$. Then the complementarity condition (25c) is equivalent to

$$u_i \Pi\,[g_i(x)] = 0. \tag{27}$$

Now instead of perturbing (25c) we shall perturbe (27), i.e.

$$u_i \Pi[g_i(x)] = r, \quad (r > 0). \tag{27r}$$

Substituting for $u_i$ into (25a) we get

$$\nabla f(x) - r \sum_{i=1}^{m} \frac{1}{\Pi[g_i(x)]} \nabla g_i(x) = 0$$

or

$$\nabla \left\{ f(x) - r \sum_{i=1}^{m} \int \frac{\mathrm{d}g_i(x)}{\Pi[g_i(x)]} \right\} = 0. \tag{28}$$

If we define

$$\Gamma(\xi) = -\int \frac{d\xi}{\Pi(\xi)}, \quad \text{or } \Gamma'(\xi) = -\frac{1}{\Pi(\xi)}, \tag{29}$$

then the relation (28) becomes

$$\nabla_x T(x, r) = 0, \tag{30a}$$

where

$$T(x, r) = f(x) + r \sum_{i=1}^{m} \Gamma[g_i(x)], \tag{31}$$

i.e. a function of the required type (18). Hence the original K-T conditions (25) have been transformed to (30a) and

$$(i = 1, 2, \ldots, m) \quad g_i(x) > 0. \tag{30b}$$

If $T(x, r)$ is convex, the conditions (30) are equivalent to the

$$\mathrm{Min}\{T(x, r) | x \in K^0\}. \tag{32}$$

But the basic question of the "Existence property" (E) of the function (31) still remains open. Hence, in the next step we shall make clear under what conditions (for the function $\Pi$: $R_+ \to R$, $\Pi(0) = 0$, $\Pi(\xi) > 0$ if $\xi > 0$) the resulting function $T(x, r)$ (31) will be a *convex barrier* or a *convex quasibarrier* transformation function.

Recall that the desired properties of $\Gamma(\xi)$ (29) are (CX), (A), (B) or (CX), (A), (Q) (Theorem 11). We shall express these properties in terms of the derivative function $\Gamma'(\xi)$.

The convexity property (CX) is equivalent to the monotonicity of $\Gamma'(\xi)$. Hence, instead of (CX) we have

(CX′) $\quad \Gamma'(\xi)$ is nondecreasing.

The asymptotic property (A) is equivalent to

(A′) $$\lim_{\xi \to \infty} \Gamma'(\xi) = 0.$$

It can be easily shown that the barrier property (B) together with the quasi-barrier property (Q) is equivalent to

(BQ′) $$\lim_{\xi \downarrow 0} \Gamma'(\xi) = -\infty.$$

Now through the transformation (29), i.e.

$$\Pi(\xi) = -\frac{1}{\Gamma'(\xi)} \tag{33}$$

the properties (CX′), (A′), (BQ′) correspond to the following properties for the function $\Pi(\xi)$:

(CX*) $\qquad \Pi(\xi)$ is nondecreasing;

(A*) $\qquad \lim_{\xi \to \infty} \Pi(\xi) = +\infty;$

(BQ*) $\qquad \lim_{\xi \downarrow 0} \Pi(\xi) = 0.$

Note that the requirement $\Pi(0) = 0$ and (BQ*) imply the continuity of $\Pi(\xi)$ at $\xi = 0$.

Hence, we have proved the following *perturbation theorem* (as a consequence of the theory of barrier and quasibarrier functions).

**Theorem 13.** Given the convex programming problem (1) with differentiable convex functions $f$, $-g_i$, the nonempty set $K^0$ (1b) and the bounded set $\hat{K} \neq \emptyset$ of optimal solutions. Let $\Pi\colon R_+ \to R$ be such that $\Pi(0) = 0$, $\Pi(\xi) > 0$ for $\xi > 0$ and satisfying the conditions (CX*), (A*), (BQ*). Then for any $r > 0$ the perturbed system of the Kuhn-Tucker conditions

$$\begin{aligned} \nabla f(x) - \sum_{i=1}^{m} u_i \nabla g_i(x) &= 0 \\ g_i(x) &\geq 0 \\ u_i \Pi[g_i(x)] &= r \\ u_i &\geq 0 \end{aligned}$$

has a solution $x(r)$, $u(r)$ and for some $\hat{x} \in \hat{K}$

$$\lim_{r \downarrow 0} f[x(r)] = f(\hat{x}).$$

**Examples.**

$\Pi_1(\xi) = \xi^2 \qquad \Rightarrow \Gamma_1(\xi) = \dfrac{1}{\xi}$

$\Pi_2(\xi) = \xi \qquad \Rightarrow \Gamma_2(\xi) = -\ln \xi$

$\Pi_3(\xi) = 2\sqrt{\xi} \qquad \Rightarrow \Gamma_3(\xi) = -\sqrt{\xi}$

$\Pi_4(\xi) = \dfrac{h}{h-1}\,\xi^{\frac{1}{h}}$ (for $h > 1$) $\Rightarrow \Gamma_4(\xi) = -\xi^{\left(1-\frac{1}{h}\right)}.$

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## SÚHRN

## VŠEOBECNÝ PRÍSTUP K TRANSFORMAČNÝM METÓDAM VNÚTORNÉHO BODU PRE MATEMATICKÉ PROGRAMOVANIE


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Článok je venovaný novej triede parametrických metód vnútorného bodu, ktoré nemajú obvyklú bariérovú vlastnosť, avšak ich ostatné vlastnosti sú analogické ako v bariérových metódach.

Ukazuje sa, že táto nová trieda parametrických metód, ktoré nazývame kvázibariérovými metódami, je v určitom zmysle komplementárna k triede bariérových metód. Kvázibariérové metódy sú zaujímavé aj z výpočtového hľadiska, pretože ich rýchlosť konvergencie (vzhľadom na parameter) je lepšia ako v bariérových metódach.

## РЕЗЮМЕ


## ОБЩИЙ ПОДХОД К МЕТОДАМ ШТРАФНЫХ ФУНКЦИЙ ТИПА ВНУТРЕННЕЙ ТОЧКИ С ПАРАМЕТРОМ ДЛЯ ЗАДАЧИ МАТЕМАТИЧЕСКОГО ПРОГРАММИРОВАНИЯ

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Статья посвящена новому классу методов штрафных функций типа внутренней точки с параметром, которые не обладают обычным свойством барьера, но остальные их свойства аналогичны методам барьерных функций. Оказывается, что этот новый класс методов — которые мы называем методами квазибарьерных функций — в определенном смысле дополнительный к классу методов барьерных функций. Методы квазибарьерных функций интересны также с вычислительной точки зрения, потому что скорость их сходимости (относительно параметра) лучше, чем в методах барьерных функций.