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GENERIC CHAOS CAN BE LARGE

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In the paper we shall consider continuous maps of a compact real interval I to itself. A concept of chaos was given in 1975 by Li and Yorke [3]. An equivalent definition can be found in [2] (cf. also [6]):

Definition 1. A map f is chaotic if there is an $\varepsilon > 0$ and a non-empty perfect set $S \subset I$ such that for any $x, y \in S$, $x \neq y$, and any p from the set $\mathrm{Per}(f)$ of periodic points of f,

$$\limsup_{n \to \infty} |f^n(x) - f^n(y)| > \varepsilon, \tag{1}$$

$$\liminf_{n \to \infty} |f^n(x) - f^n(y)| = 0, \tag{2}$$

$$\limsup_{n \to \infty} |f^n(x) - f^n(p)| > \varepsilon. \tag{3}$$

The set S is called a chaotic set for f.

In 1985 Piórek [5] introduced the notion of generic chaos for more general systems; for maps of the interval it reads as follows:

Definition 2. A map f is called generically chaotic if the set G of all $(x, y) \in I^2$ for which

$$\liminf_{n \to \infty} |f^n(x) - f^n(y)| = 0, \tag{4}$$

and

$$\limsup_{n \to \infty} |f^n(x) - f^n(y)| > 0, \tag{5}$$

is generic (i.e. residual) in I^2.

It can be proved that any generically chaotic map f is chaotic in the sense of Li and Yorke and that the converse implication is not true [8]. However, there is another, essential difference between these two concepts. Any scrambled set G, satisfying Definition 2, is a priori residual in I^2, but there is no scrambled set $S \subset I$, satisfying Definition 1 and residual in a subinterval $J \subset I$ [1].

It is known that a scrambled set of a map, satisfying Definition 1, can have a positive Lebesgue measure, see e.g. [4] among others. The main aim of this note is to exhibit a class of generically chaotic maps which have scrambled sets G of the full two-dimensional Lebesgue measure (Theorem 2).

In the sequel, a map f: $[0, 1] \to [0, 1] = I$ is called a full piecewise monotonic map (denoted $f \in FPM$) if there exist $r > 2$ and a sequence $a_0 = 0 < a_1 < a_2 < \dots < a_r = 1$ such that, for any $i = 1, \dots, r$,

$$g_i = f|[a_{i-1}, a_i] \text{ is continuous and differentiable in } (a_{i-1}, a_i) \tag{6}$$

$$g_i([a_{i-1}, a_i]) = I \tag{7}$$

$$\inf(g_i') \geqq q > 1 \text{ in } (a_{i-1}, a_i), \text{ for some } q. \tag{8}$$

A map f is called full piecewise linear ($f \in FPL$) if $f \in FPM$ and

$$g_i = f|[a_{i-1}, a_i] \text{ is linear for } i = 1, \dots, r. \tag{9}$$

Piórek [5] has proved that if f: $[0, 1] \to [0, 1]$ and $f \in FPM$, then f is generically chaotic. The next theorem gives a stronger result:

Theorem 1. Let f: $[0, 1] \to [0, 1] = I$ and $f \in FPM$. Then there exists a set $G \subset I^2$ residual in I^2 and such that for any $(x, y) \in G$

$$\liminf_{n \to \infty} |f^n(x) - f^n(y)| = 0 \tag{10}$$

$$\limsup_{n \to \infty} |f^n(x) - f^n(y)| = 1. \tag{11}$$

Proof: Let a_i be the critical points of f as in the definition of FPM and let $A_0 = \{a_0, a_1, \dots, a_r\}$, $A_{n+1} = A_n \cup f^{-1}(A_n)$. The set A_n divides I into r^{n+1} open intervals $I(n, 1)$, $I(n, 2)$, …, $I(n, r^{n+1})$. Clearly f^n is monotonic and continuous on every $I(n, i)$, and $f^n(I(n, i)) = I$ for every i.

Let, for $\varepsilon > 0$ and $n = 1, 2, \dots$

$$L_{n,\varepsilon} = \{(x, y) \in I^2;\ \inf_{k \geq n} |f^k(x) - f^k(y)| < \varepsilon\},$$

$$U_{n,\varepsilon} = \{(x, y) \in I^2;\ \sup_{k \geq n} |f^k(x) - f^k(y)| > 1 - \varepsilon\},$$

$$G = \{(x, y) \in I^2;\ \liminf_{n \to \infty} |f^n(x) - f^n(y)| = 0,$$

$$\limsup_{n \to \infty} |f^n(x) - f^n(y)| = 1\}.$$

We show that each of the sets $L_{n,\varepsilon}$, $U_{n,\varepsilon}$ for $n = 1, 2, \dots$ $\varepsilon > 0$ is open and dense in I^2. This will imply the genericity of G, since $G \supset \bigcap_{n=1}^{\infty} L_{n,1/n} \cap U_{n,1/n}$. A proof of the fact that $L_{n,\varepsilon}$ is dense in I^2 can be found in [5].

To see that $U_{n,\varepsilon}$ and $L_{n,\varepsilon}$ are both open in I^2 it is sufficient to realize that $h(x, y) = |f^k(x) - f^k(y)|$ is continuous for every k and every point $(x, y) \in I^2$.

The proof of density of $U_{n,\varepsilon}$ is given in the following.

Lemma 1. $U_{n,\varepsilon}$ is dense in I^2.

Proof: Fix $(x_0, y_0) \in I^2$, $\varepsilon > 0$, $\sigma > 0$. Choose $p \geq n$ so large that each of the intervals $I(p, j)$ has the length less than σ (this is possible because of (8)). Next find s and t with $x_0 \in \overline{I(p, s)}$, $y_0 \in \overline{I(p, t)}$ (here $\bar{B}$ denotes the closure of a set B).

Choosing $x \in I(p, s)$ and $y \in I(p, t)$ so that $f^p(x) < \frac{\varepsilon}{2}$ and $f^p(y) > 1 - \frac{\varepsilon}{2}$ we obtain

$$|x - x_0| < \sigma, \ |y - y_0| < \sigma \quad \text{and}$$

$$|f^p(x) - f^p(y)| > 1 - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} = 1 - \varepsilon. \qquad \square$$

Before stating the main result, we prove the following.

Lemma 2. Let f: $[0, 1] \to [0, 1]$, $f \in FPL$, $I \times J \subset (0, 1) \times (0, 1)$ be an open interval, M an infinite set of natural numbers and λ the Lebesgue measure in the plane. Then there exists a G_δ set $B = B(I \times J, M) \subset (0, 1) \times (0, 1)$ such that $\lambda(B) = 1$ and for any $(x, y) \in B$ there is an infinite set $S \subset M$ with

$$(f^n(x), f^n(y)) \in I \times J \quad \text{for } n \in S.$$

Proof: Denote $f^{-n}(I \times J) = \{(x, y); \ (f^n(x), f^n(y)) \in I \times J\}$ and $C_{m,n} = \bigcup \{f^{-i}(I \times J), \ i \in M, \ m \leqq i < n\}$ for any $m < n$. If there exists a sequence $n(1) < n(2) < \ldots$ of natural numbers such that $\lambda(C_{n(i),n(i+1)}) = 1$ for any i, then $B = \bigcap^{\infty} C_{n(i),n(i+1)}$ has the desired properties.

If such a sequence does not exist there must be $n(1) \in M$ such that

$$0 < \lambda(C_{n(1),m}) < 1 \text{ for any } m > n(1). \tag{12}$$

Denote $B_1 = f^{-n(1)}(I \times J)$. Since f is full piecewise linear $\lambda(B_1) = d = \lambda(I \times J)$ (clearly, $d > 0$). Now assume by induction that sets $B_1, \ldots, B_k$ and integers $n(1), \ldots, n(k)$ from M are defined such that

$$\text{any } B_i \text{ is open and } B_{i-1} \subset B_i \subsetneqq (0, 1) \times (0, 1) \tag{13}$$

$$\lambda(B_i \backslash B_{i-1}) > \frac{d}{2}(1 - \lambda(B_{i-1})) \tag{14}$$

$$(f^{n(i)}(x), f^{n(i)}(y)) \in I \times J \text{ for any } (x, y) \in B_i \backslash B_{i-1}. \tag{15}$$

For $n > n(k)$ define a system G_n of open intervals $I(n, j) \times I(n, i)$ $i, j = 1, 2, \ldots$

which are contained in $(0, 1) \times (0, 1)\backslash B_k$. (Here intervals $I(n, i)$ are the same as in the proof of Theorem 1.)

Let $D_n = \bigcup G_n$. The set D_n is nonempty because B_k consists of a finite number of two-dimensional intervals. Since $D_n \cap B_k = \emptyset$, we can choose an $n = n(k + 1) \in M$ such that

$$\lambda(D_n) > \frac{1}{2}(1 - \lambda(B_k)).$$

Now put

$$B_{k+1} = B_k \cup (f^{-n(k+1)}(I \times J) \cap D_n).$$

One can easily check that conditions (13), (14), (15) hold. Denote $\lim_{\to \infty} \lambda(B_k) = b$. Suppose $b < 1$. Then we can write (see (13) and (14)) $\lambda(B_{k+1}\backslash B_k) > \frac{d}{2}(1 - b) > 0$ and thus $\lim_{k \to \infty} \lambda(B_k) = \infty$, which is impossible. Therefore $b = 1$.

Denote $E_0 = \bigcup_{k=1}^{\infty} B_k$. Repeating of the above construction with condition $n(1) > i$, (this can always be done because of (12)) yields sets E_i with $\lambda(E_i) = 1$ for $i = 1, 2, \ldots$. Now the set $B = \bigcap_{i=1}^{\infty} E_i$ has the required properties. □

Theorem 2. Let f: $[0, 1] \to [0, 1]$ be full piecewise linear. Then f is generically chaotic and the corresponding scrambled set G (according to the Definition 2) has the full Lebesgue measure. Moreover, there exists a subset $A \subset G$ which has full measure such that for any $(x, y) \in A$

$$\limsup_{n \to \infty} |f^n(x) - f^n(y)| = 1.$$

Proof: Denoting by N the set of positive integers, let

$$A_k = B((0, 1/k) \times (0, 1/k), N)), \; A^k = B((0, 1/k) \times (1 - 1/k, 1), N)).$$

According to the Lemma 3, $A = \bigcap_{k=1}^{\infty} A^k \cap A_k$ has desired properties. □

Remark. It is easy to see that every full piecewise linear map f is chaotic in the sense of Li and Yorke, but every scrambled set S of such an f (according to Definition 1) if measurable, must have zero Lebesgue measure. To see it, we can slightly modify the argument from [7].

Assume the contrary: Let A be a scrambled set for f with $\lambda(A) = \alpha$. (Here λ is the Lebesgue measure on the line and $\alpha > 0$.) Recall that $I(m, i)$ are the maximal open intervals with the property that $f^{m+1}|I(m, i)$ is linear and r is the number of laps of f.

Denote $h(m) = \sup_i \{|I(m, i)|\}$. Fix m such that $\alpha/h(m) > 1$. Then f^{m+1} maps each of the intervals $I(m, 1), \ldots, I(m, r^{m+1})$ linearly onto $(0, 1)$. Put $A_i = A \cap I(m, i)$. Then $\sum_i \lambda(f^{m+1}(A_i)) \geq \sum_i 1/h(m)\lambda(A_i) = \alpha/h(m) > 1$, hence for some $i \neq j$ we have $f^{m+1}(A_i) \cap f^{m+1}(A_j) \neq \emptyset$. Consequently, there are two different points $x \in A_i$, $y \in A_j$ with $f^n(x) = f^n(y)$ for every $n \geqq m + 1$, contrary to (1).

Therefore, Theorem 2 is interesting also from this point of view.

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SÚHRN

GENERIC CHAOS CAN BE LARGE

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V práci sa skúmajú metrické vlastnostiętgenericky chaotických množín.

Našli sme triedu funkcií zobrazujúcich reálny kompaktný interval do seba, ktorých genericky chaotické množiny majú plnú mieru.

РЕЗЮМЕ

ГЕНЕРИЧЕСКИЙ ХАОС МОЖЕТ ОКАЗАТЬСЯ БОЛЬШИМ

Томаш Гедеон, Братислава

В предложенной работе мы исследуем метрические свойства генерически хаотических множеств.

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