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ON A LINEAR EIGENVALUE PROBLEM

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We shall investigate the following linear eigenvalue problem

$$y''' + \{[f(x) + \lambda g(x)]y\}' = 0, \quad -a \leqq x \leqq a, \tag{1}$$

$$y(-a) = y(a) = 0, \tag{2}$$

$$\int_{-a}^{a} h(t)g(t)y(t)\,dt = 0, \tag{3}$$

where $a > 0$, the functions $f, g \in C^1([-a, a], R)$, $g(x) > 0$ in $[-a, a]$, $h \in C([-a, a], R)$ and $h(-a) = h(a) = 0$. The problem has been formulated in [6], [4], pp. 248—255, and it represents a mathematical model for deflection of a curved beam. M. Greguš in [2], [3] and [4] has shown that under some conditions this problem is equivalent to the problem (1), (2), (4) with

$$y''(-a) = 0 \tag{4}$$

and hence, the theory of the third order linear differential equation in [4] can be applied. Here we find a simple sufficient condition for the existence of a nontrivial solution of (1), (2), (3). As usual, each real or complex number λ for which there exists a nontrivial solution y of (1), (2), (3) will be called an *eigenvalue* of that problem and the function y is called the *eigenfunction* of that problem.

Remark. A necessary and sufficient condition that each initial value problem for (1) have a unique solution $y \in C^3([-a, a], R)$ is that $f, g \in C^1([-a, a], R)$. Thus the assumption $f, g \in C^1([-a, a], R)$ is not superfluous.

In order to solve the problem (1), (2), (3), let us notice that this problem is equivalent to the problem (1′), (2), (3) with

1.

$$y'' + [f(x) + \lambda g(x)]y = y''(-a), \quad -a \leqq x \leqq a \tag{1′}$$

as it follows by integrating (1) term-by-term from $-a$ to x, $-a \leqq x \leqq a$, and considering the condition $y(-a) = 0$. Consider, now, the problem

$$y'' + [f(x) + \lambda g(x)]y = 0, \quad -a \leqq x \leqq a, \tag{5}$$

$$y(-a) = y(a) = 0. \tag{2}$$

This is the famous Sturm—Liouville eigenvalue problem. In the sequel we shall use the following results concerning that problem which are collected in the following lemma.

Lemma 1 ([1], pp. 159, 160, 171, 175—177 and [5], p. 292).

(i) The set of all eigenvalues of the problem (5), (2) can be written in the form of a real increasing sequence

$$\lambda_0 < \lambda_1 < \ldots < \lambda_n < \ldots$$

such that $\lim\limits_{n \to \infty} \lambda_n = \infty$.

(ii) There exists a sequence $\{y_n\}_{n=0}^{\infty}$ of the eigenfunctions y_n of (5), (2) corresponding to λ_n which have the following properties:

a) Each y_n, $n = 0, 1, 2, \ldots$, has exactly n zeros in $(-a, a)$.

b) The sequence $\{y_n\}_{n=0}^{\infty}$ is orthonormal with respect to the weight function g, i. e.

$$\int_{-a}^{a} g(t)y_n(t)y_m(t)\,\mathrm{d}t = 0 \quad \text{for } n \neq m, \text{ and } \int_{-a}^{a} g(t)y_n^2(t)\,\mathrm{d}t = 1,$$

$n, m = 0, 1, \ldots$.

c) If $L_g^2([-a, a])$ is the vector space of all real measurable functions in $[-a, a]$ such that $g^{1/2}y \in L^2([-a, a])$ provided with the scalar product $(y, z)_g = \int_{-a}^{a} g(t)y(t)z(t)\,\mathrm{d}t$ for each $y, z \in L_g^2([-a, a])$, then $L_g^2([-a, a])$ is a Hilbert space, whereby both $L_g^2([-a, a])$, $L^2([-a, a])$ as vector spaces are equal to each other and their norms are mutually equivalent.

d) Each function $y \in L_g^2([-a, a])$ can be expanded into the Fourier series of the eigenfunctions y_n,

$$y(x) \sim \sum_{n=0}^{\infty} \gamma_n y_n(x) \tag{6}$$

where

$$\gamma_n = \int_{-a}^{a} y(t)g(t)y_n(t)\,\mathrm{d}t, \quad n = 0, 1, 2, \ldots,$$

and the series in (6) converges to the function y in the metric of the space $L_g^2([-a, a])$. Hence $\{y_n\}_{n=0}^{\infty}$ is complete in $L_g^2([-a, a])$.

e) If the function $y \in C([-a, a])$, y' is piecewise continuous on $[-a, a]$ and $y(-a) = y(a) = 0$, then the series in (6) converges absolutely and uniformly to y on $[-a, a]$.

Now we shall consider the case that $\lambda \in R$, $\lambda \neq \lambda_n$, $n = 0, 1, 2, \ldots$. Then the problem (5), (2) has the Green function $G_\lambda(x, t)$, $-a \leqq x, t \leqq a$, such that the unique solution y_λ of the problem

$$y'' + [f(x) + \lambda g(x)]y = 1, \quad -a \leqq x \leqq a, \tag{7}$$

$$y(-a) = y(a) = 0, \tag{2}$$

can be represented in the form

$$y_\lambda(x) = \int_{-a}^{a} G_\lambda(x, t)\, \mathrm{d}t, \; -a \leqq x \leqq a. \tag{8}$$

Consider the functional Φ: $R - \{\lambda_n\}_{n=0}^{\infty} \to R$ defined by

$$\Phi(\lambda) = (h, y_\lambda)_g = \int_{-a}^{a} h(t)g(t)y_\lambda(t)\, \mathrm{d}t. \tag{9}$$

Denote

$$c_n = \left(\frac{1}{g}, y_n\right)_g = \int_{-a}^{a} y_n(t)\, \mathrm{d}t, \tag{10}$$

$$d_n = (h, y_n)_g = \int_{-a}^{a} h(t)g(t)y_n(t)\, \mathrm{d}t, \quad n = 0, 1, 2, \ldots. \tag{11}$$

Then the following lemma holds.

Lemma 2. If $\lambda \neq \lambda_n$, $n = 0, 1, 2, \ldots$, then

$$y_\lambda(x) = \sum_{n=0}^{\infty} \frac{c_n}{\lambda - \lambda_n} y_n(x), \quad -a \leqq x \leqq a, \tag{12}$$

whereby the series on the right-hand side of (12) converges uniformly and absolutely in $x \in [-a, a]$ and the functional Φ enjoys the properties:

$$\Phi(\lambda) = \sum_{n=0}^{\infty} \frac{c_n d_n}{\lambda - \lambda_n}, \quad \Phi'(\lambda) = -\sum_{n=0}^{\infty} \frac{c_n d_n}{(\lambda - \lambda_n)^2}. \tag{13}$$

Both series in (13) converge uniformly in λ on each compact interval which does not intersect the set $\{\lambda_n\}_{n=0}^{\infty}$. Hence $\Phi \in C^1$ on the open set $R - \{\lambda_n\}_{n=0}^{\infty}$.

Proof. By Lemma 1, the function y_λ can be expanded into a uniformly and absolutely convergent series

$$y_\lambda(x) = \sum_{n=0}^{\infty} b_n y_n(x), \quad -a \leqq x \leqq a.$$

We shall show that for each $n = 0, 1, 2, \ldots, b_n = c_n/(\lambda - \lambda_n)$ and (12) will be proved. Let n be an arbitrary but fixed nonnegative integer. Since y_n satisfies the condition (2) as well as the equation

$$y_n'' + [f(x) + \lambda g(x)]y_n = (\lambda - \lambda_n)g(x)y_n(x), \quad -a \leqq x \leqq a,$$

by the meaning of the Green function G_λ we get that

$$y_n(t) = \int_{-a}^{a} G_\lambda(t, x)(\lambda - \lambda_n)g(x)y_n(x)\,dx, \quad -a \leqq t \leqq a. \tag{14}$$

Further the Green function G_λ is symmetric, i.e. $G_\lambda(x, t) = G_\lambda(t, x)$, $-a \leqq t$, $x \leqq a$. Then with the help of (8), (14) and (10)

$$b_n = (y_\lambda, y_n)g = \int_{-a}^{a} y_\lambda(x)g(x)y_n(x)\,dx = \int_{-a}^{a}\left[\int_{-a}^{a} G_\lambda(x, t)\,dt\right].$$

$$.\,g(x)y_n(x)\,dx = \int_{-a}^{a}\left[\int_{-a}^{a} G_\lambda(t, x)g(x)y_n(x)\,dx\right]dt =$$

$$= \frac{1}{\lambda - \lambda_n}\int_{-a}^{a} y_n(t)\,dt = \frac{c_n}{\lambda - \lambda_n}$$

and hence (12) is proved.

By the uniform convergence of the series (12), on the basis of (9), (11),

$$\Phi(\lambda) = \int_{-a}^{a} h(t)g(t)\sum_{n=0}^{\infty}\frac{c_n}{\lambda - \lambda_n}y_n(t)\,dt = \sum_{n=0}^{\infty}\frac{c_n d_n}{\lambda - \lambda_n}$$

and the first equality in (13) is proved. If λ varies in a compact interval J in the set $R - \{\lambda_n\}_{n=0}^{\infty}$, then there is a $c > 0$ such that $|\lambda - \lambda_n| \geqq c$ for each $n = 0, 1, 2, \ldots$, and the series $\sum_{n=0}^{\infty} |c_n d_n|/c$, and $\sum_{n=0}^{\infty} |c_n d_n|/c^2$, majorize the first and the second series in (13), respectively. In view of Bessel's inequality,

$$\sum_{n=0}^{\infty} |c_n d_n| \leqq \frac{1}{2}\sum_{n=0}^{\infty}(c_n^2 + d_n^2) \leqq \frac{1}{2}\left(\frac{1}{g}, \frac{1}{g}\right)_g + \frac{1}{2}(h, h)_g. \tag{15}$$

Thus both series in (13) are uniformly convergent in J, and the first equality implies the second one in (13). The lemma is proved.

Remark. By the Weierstrass theorem the functional $\Phi(\lambda)$ is analytic in $\lambda \in (R - \{\lambda_n\}_{n=0}^{\infty})$.

Lemma 3. If $|c_k d_k| > 0$ for a $k \geqq 0$, then

$$\lim_{\lambda \to \lambda_k -} \Phi(\lambda) = (-\infty)\,\mathrm{sgn}\,(c_k d_k), \quad \lim_{\lambda \to \lambda_k +} \Phi(\lambda) = \infty\,.\,\mathrm{sgn}\,(c_k d_k)$$

Proof. On the basis of (13), (15),

$$\Phi(\lambda) = \frac{c_k d_k}{\lambda - \lambda_k} + \sum_{\substack{n=0 \\ n \neq k}}^{\infty} \frac{c_n d_n}{\lambda - \lambda_n} = \frac{c_k d_k}{\lambda - \lambda_k} + \Phi_k(\lambda),$$

whereby $\Phi_k(\lambda)$ is bounded in a neighbourhood of λ_k. This implies the statement of the lemma.

Theorem 1. (i) If $d_k = 0$ or $c_k = 0$ for an integer $k \geqq 0$, then λ_k is an eigenvalue of the problem (1), (2), (3) whereby in the case $d_k = 0$ y_k is a corresponding eigenfunction of that problem.

(ii) If $c_k d_k c_{k+1} d_{k+1} > 0$, then in $(\lambda_k, \lambda_{k+1})$ there exists at least one eigenvalue of the problem (1), (2), (3).

Proof. (i) If $d_k = 0$, then clearly y_k is an eigenfunction and λ_k is an eigenvalue of the problem (1), (2), (3).

Suppose now that $d_k \neq 0$ and $c_k = 0$. Consider the equation

$$y'' + [f(x) + \lambda_k g(x)]y = 1. \tag{16}$$

If z is the solution of the corresponding homogeneous equation which satisfies $z(-a) = 1$, $z'(-a) = 0$, then the Wronskian of the solutions y_k, z satisfies the identity $w(y_k, z)(x) \equiv -y_k'(-a)$ in $[-a, a]$ and by the variation of constants formula an arbitrary solution y of (16) is of the form

$$y(x) = \bar{c}_1 y_k(x) + \bar{c}_2 z(x) - \frac{1}{y_k'(-a)} \int_{-a}^{x} [z(x) y_k(t) - y_k(x) . z(t)] \, \mathrm{d}t,$$

$-a \leqq x \leqq a,$

where $\bar{c}_1, \bar{c}_2 \in R$. Since $c_k = 0$, y satisfies (2) iff $\bar{c}_2 = 0$. Hence the problem (16), (2) is satisfied by the functions

$$y(x) = \bar{c}_1 y_k(x) - \frac{1}{y_k'(-a)} \int_{-a}^{x} [z(x) y_k(t) - y_k(x) z(t)] \, \mathrm{d}t = \bar{c}_1 y_k(x) + \bar{y}_0(x),$$

$$-a \leqq x \leqq a.$$

Here $\bar{y}_0$ is the solution of (16) which satisfies

$$y(-a) = 0, \; y'(-a) = 0. \tag{17}$$

Consider the condition (3). We have that

$$\int_{-a}^{a} h(t) g(t) y(t) \, \mathrm{d}t = \bar{c}_1 d_k + (h, \bar{y}_0)_g \tag{18}$$

and in view of $d_k \neq 0$ there exists a unique $\bar{c}_1 = C_1$ for which $C_1 d_k + (h, \bar{y}_0)_g =$

$= 0$. Hence λ_k is an eigenvalue of (1), (2), (3) and $C_1 y_k + \bar{y}_0$ is the corresponding eigenfunction of that problem.

If $c_k d_k > 0$, $c_{k+1} d_{k+1} > 0$ (the case $c_k d_k < 0$, $c_{k+1} d_{k+1} < 0$ would be proceeded in a similar way), then by Lemma 3 $\lim\limits_{\lambda \to \lambda_k +} \Phi(\lambda) = \infty$, while $\lim\limits_{\to \lambda_{k+1} - \lambda} \Phi(\lambda) = -\infty$, and Lemma 2 gives that Φ is coninuous. Hence there is a $\bar{\lambda} \in (\lambda_k, \lambda_{k+1})$ such that $\Phi(\bar{\lambda}) = 0$. This means that the solution $y_{\bar{\lambda}}$ of (7), (2) with $\lambda = \bar{\lambda}$ satisfies the condition (3) too.

Remark. Clearly, to any eigenvalue of the problem (1), (2), (3) exists at least one dimensional space of eigenfunctions (of course without the null solution). If they are two linearly independent eigenfunctions y, z of that problem belonging to the same eigenvalue λ, then without loss of generality we may assume that they both are solutions of the same equation (7) and hence their difference is a nontrivial solution of the corresponding homogeneous equation which satisfies the conditions (2). Thus there is a $k \geqq 0$ such that $\lambda = \lambda_k$, $y - z = \alpha y_k$ with $\alpha \neq 0$ and hence $d_k = 0$. At the same time both functions y, z satisfy (16). It can be shown that $c_k = 0$ is also a necessary condition for the existence of a solution to (16), (2). By (18) we have that $(h, \bar{y}_0)_g = 0$. Hence $d_k = 0$, $c_k = 0$ and $(h, \bar{y}_0)_g = 0$ is a necessary condition for the existence of two linearly independent eigenfunctions belonging to the same eigenvalue. But this is also a sufficient condition, since under this condition all functions $\bar{c}_1 y_k + \bar{y}_0$ satisfy the problem (16), (2), (3) and thus the functions $\bar{c}_1 y_k + \bar{c}_2 \bar{y}_0$ satisfy (1), (2), (3). The result can be summoned up in the theorem.

Theorem 2. With the exception of the case $c_k = 0$, $d_k = 0$, $(h, \bar{y}_0)_g = 0$ where there is a two dimensional vector space of eigenfunctions of (1), (2), (3) belonging to the eigenvalue λ_k, in the other cases the space of eigenfunctions of the problem (1), (2), (3) corresponding to the same eigenvalue λ is one-dimensional.

Further the following theorem is true.

Theorem 3. If $h(x) \geqq 0$, $h(x) \not\equiv 0$ in $[-a, a]$ (or $h(x) \leqq 0$, $h(x) \not\equiv 0$ in $[-a, a]$), then no $\lambda \leqq \lambda_0$ is an eigenvalue of the problem (1), (2), (3).

Proof. Consider only the case that $h(x) \geqq 0$ in $[-a, a]$. The other case can be investigated in a similar way. Let $\lambda < \lambda_0$. Since the eigenfunction y_0 of (5), (2) corresponding to λ_0 is different from 0 in $(-a, a)$, by the Sturm comparison theorem the equation (5) is disconjugate in $[-a, a]$ and hence the problem (5), (2) has no nontrivial solution. At the same time, the Green function $G_\lambda(x, t) < 0$ for $-a < x$, $t < a$ and therefore, $y_\lambda(x) < 0$ in $(-a, a)$. This implies that $\Phi(\lambda)$ determined by (9) is negative and λ is no eigenvalue of (1), (2), (3).

If $\lambda = \lambda_0$, then on the basis of the constant sign of the eigenfunction y_0, y_0 cannot satisfy the equality (3). If y is a solution of (7), (2), then multiplying the equation (7) by y_0 and integrating by parts we come to the equality

$$-\int_{-a}^{a} y_0(x)\,\mathrm{d}x = \int_{-a}^{a} y''(x)y_0(x)\,\mathrm{d}x + \int_{-a}^{a} [f(x) + \lambda_0 g(x)]\,.\,y(x)y_0(x)\,\mathrm{d}x =$$

$$= -\int_{-a}^{a} [f(x) + \lambda_0 g(x)]y(x)y_0(x)\,\mathrm{d}x + \int_{-a}^{a} [f(x) + \lambda_0 g(x)]y(x)y_0(x)\,\mathrm{d}x = 0$$

which contradicts the constant sign of y_0 in $(-a, a)$. Hence there is no nontrivial solution of (1), (2), (3) for $\lambda = \lambda_0$ and the theorem is proved.

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SÚHRN

O LINEÁRNOM VLASTNOM PROBLÉME

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V práci sa skúma vlastná úloha (1), (2), (3). Nájdená je postačujúca podmienka, aby existovalo netriviálne riešenie tejto úlohy.

РЕЗЮМЕ

О ЛИНЕЙНОЙ ЗАДАЧЕ НА СОБСТВЕННЫЕ ЗНАЧЕНИЯ

Вальтер Шеда, Братислава

В работе исследуется задача (1), (2), (3). Найдено достаточное условие для того, чтобы существовало нетривиальное решение этой задачи.