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ON A TYPICAL PROPERTY OF SOME FUNCTION SPACES

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In the present paper we shall point out a typical property of the spaces of real functions which are determined by the n-th symmetric difference ($n = 1, 2, \ldots$). For a function $f\colon (a, b) \to R$ (R — the real line, $-\infty \leq a < b \leq +\infty$), we define the n-th symmetric difference of f at x to be

$$\Delta^n f(x, h) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(x + (n - 2i)h).$$

We shall deal with functions from a local point of view. For every $n = 1, 2, \ldots$ let LS_n and SC_n be classes of functions defined in the following way:

$$LS_n = \left\{f\colon \underset{x\in(a,b)}{\forall}\ \underset{\delta_x>0}{\exists}\ \underset{h:0<h<\delta_x}{\forall}\ \Delta^n f(x, h) = 0\right\},$$

$$SC_n = \left\{f\colon \underset{x\in(a,b)}{\forall}\ \lim_{h\to 0} \Delta^n f(x, h) = 0\right\}.$$

Note that some properties of the functions of classes LS_1, LS_2, SC_1 and of the measurable functions of SC_2 are known (locally symmetric functions [F], locally Jensen functions [Ko], symmetrically continuous and symmetric functions [L]). The n-th symmetric difference $\Delta^n f(x, h)$ is investigated also from a global point of view. A function $f\colon R \to R$ fulfiling the functional equation $\Delta^n f(x, h) = 0$ is called a polynomial function of the n-th order (see e.g. [Ku]).

First we shall investigate the relations between the classes LS_n and SC_n ($n = 1, 2, \ldots$).

Theorem 1. (a) For every $n = 1, 2, \ldots$ the class LS_n is a proper subset of the class SC_n;

(b) We have $SC_1 \subset SC_{2k-1}$, $SC_2 \subset SC_{2k}$, $LS_1 \subset LS_{2k-1}$ and $LS_2 \subset LS_{2k}$ for every $k = 1, 2, \ldots$;

(c) Inequalities $LS_{2m-1} - SC_{2k} \neq \emptyset$ and $LS_{2m} - SC_{2k-1} \neq \emptyset$ hold for each $m = 1, 2, \ldots$ and $k = 1, 2, \ldots$.

Proof. (a) Let n be fixed. The inclusion $LS_n \subset SC_n$ is an easy consequence of the definition of classes LS_n and SC_n. We prove $SC_n - LS_n \neq \emptyset$. First we show that $f \in SC_n$ holds for each continuous function $f: (a, b) \to R$. Continuity of f at x implies that $f(x + (n - 2i)h) = f(x) + \varepsilon_i(h)$ for each $i = 0, 1, \ldots, n$, and $\varepsilon_i(h) \to 0$ whenever $h \to 0$. Then $\Delta^n f(x, h) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (f(x) + \varepsilon_i(h)) = f(x) \sum_{i=0}^{n} (-1)^i \binom{n}{i} + \sum_{i=0}^{n} (-1)^i \binom{n}{i} \varepsilon_i(h)$. Hence $\lim_{h \to 0} \Delta^n f(x, h) = 0$ and $f \in SC_n$. To prove $SC_n - LS_n \neq \emptyset$, it is sufficient to show that there is a continuous function $g: (a, b) \to R$ such that $g \notin LS_n$. Put $g(t) = c^t$, $c > 0$, $c \neq 1$. Using the binomial formula we have:

$$c^x (c^{2h} - 1)^n = c^{nh} \sum_{i=0}^{n} (-1)^i g(x + (n - 2i)h) = c^{nh} \Delta^n g(x, h), \text{ hence}$$

$$\Delta^n g(x, h) = c^{x - nh} (c^{2h} - 1)^n.$$

For every $x \in (a, b)$ and $h \neq 0$ we have $\Delta^n g(x, h) \neq 0$. Consequently $g \notin LS_n$.

(b) In this part of the proof of Theorem 1 we shall use the identity

$$\Delta^{n+2} f(x, h) = \Delta^n f(x + 2h, h) - 2\Delta^n f(x, h) + \Delta^n f(x - 2h, h), \tag{I}$$

which is an easy consequence of the identity

$$\Delta^n f(x, h) = \Delta^{n-1} f(x + h, h) - \Delta^{n-1} f(x - h, h)$$

(the last identity can be proved by a straightforward verification).

The principle of induction will be used to prove the inclusion $SC_1 \subset SC_{2k-1}$. We show $SC_1 \cap SC_n \subset SC_{n+2}$ for $n = 2k - 1$, $k = 1, 2, \ldots$. We have $\Delta^n f(x + 2h, h) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(x + (n + 2 - 2i)h)$ and $\Delta^n f(x - 2h, h) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(x + (n - 2 - 2i)h)$. If we put $j = n - i$, then the last equality can be expressed in the form $\Delta^n f(x - 2h, h) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{n-j} f(x - (n + 2 - 2j)h) = \sum_{j=0}^{n} (-1)^{j+1} \binom{n}{j} f(x - (n + 2 - 2j)h)$. Hence $\Delta^n f(x + 2h, h) + \Delta^n f(x - 2h, h) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} [f(x + (n + 2 - 2i)h) - f(x - (n + 2 - 2i)h)]$. The identity (I) and $f \in SC_1 \cap SC_n$ imply $|\Delta^{n+2} f(x, h)| \leq |\Delta^n f(x + 2h, h) +$

$+ \Delta^n f(x - 2h, h)| + 2|\Delta^n f(x, h)| \leq \sum_{i=0}^{n} \binom{n}{i} |f(x + (n + 2 - 2i)h) - f(x - (n +$ $+ 2 - 2i)h)| + 2|\Delta^n f(x, h)| \to 0$ whenever $h \to 0$. Consequently, $f \in SC_{n+2}$.

Analogously, the inclusion $SC_2 \cap SC_n \subset SC_{n+2}$ for $n = 2k$, $k = 1, 2, \ldots$, is sufficient to $SC_2 \subset SC_{2k}$. We have $\Delta^n f(x + 2h, h) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(x + (n + 2 - 2i)h)$ and $\Delta^n f(x - 2h, h) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(x + (n - 2 - 2i)h)$. If we put $j = n - i$, then the last equality can be expressed in the form $\Delta^n f((x - 2h, h) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} f(x - (n + 2 - 2j)h)$. Then, using $\sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0$, we have $\Delta^n f(x + 2h, h) + \Delta^n f(x - 2h, h) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} [f(x + (n + 2 - 2i)h) - 2f(x) + f(x - (n + 2 - 2i)h)] = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \Delta^2 f\left(x, \left(\frac{n}{2} + 1 - i\right)h\right)$. The identity (I) and $f \in SC_2 \cap SC_n$ imply $|\Delta^{n+2} f(x, h)| \leq |\Delta^n f(x + 2h, h) + \Delta^n f(x - 2h, h)| + 2|\Delta^n f(x, h)| \leq \sum_{i=0}^{n} \binom{n}{i} \left|\Delta^2 f\left(x, \left(\frac{n}{2} + 1 - i\right)h\right)\right| + 2|\Delta^n f(x, h)| \to 0$ whenever $h \to 0$. Hence $f \in SC_{n+2}$.

Proofs of inclusions $LS_1 \subset LS_{2k-1}$ and $LS_2 \subset LS_{2k}$ are obvious modifications of the above ones.

(c) Choose $x \in (a, b)$. Let χ_x be the characteristic function of the set $\{x\}$, i.e. $\chi_x(x) = 1$ and $\chi_x(t) = 0$ for $t \in (a, b) - \{x\}$. The fact $\chi_x \in LS_{2m-1}$ follows from $\chi_x \in LS_1$ and $LS_1 \subset LS_{2m-1}$ (Theorem 1 (b)). Further we have $\Delta^{2k} \chi_x(x, h) = \sum_{i=0}^{2k} (-1)^i \binom{2k}{i} \chi_x(x + (2k - 2i)h) = (-1)^k \binom{2k}{k} \neq 0$, hence $\chi_x \notin SC_{2k}$.

Let $x \in (a, b)$. Define the function $g_x : (a, b) \to R$ as follows: $g_x(t) = 1$ for $t > x$, $g_x(t) = -1$ for $t < x$ and $g_x(x) = 0$. Obviously $g_x \in LS_{2m}$. A straightforward verification gives $\Delta^{2k-1} g_x(x, h) = 2 \sum_{i=0}^{k-1} (-1)^i \binom{2k-1}{i} = 2(-1)^{k-1} \binom{2k-2}{k-1} \neq$ $\neq 0$ (the last equality is a consequence of a combinatoric identity $\sum_{i=0}^{n} (-1)^i \binom{x}{i} = (-1)^n \binom{x-1}{n}$, see e.g. [Ka], p. 42). Hence $g_x \in LS_{2m} - SC_{2k-1}$.

Theorem 2. Let (X, ϱ) be a metric space. Let F be a linear space of bounded functions $f: X \to R$ furnished with the sup norm $\|f\| = \sup_{x \in X} \{|f(x)|\}$. Suppose that in F there exists a function h such that the set of all its discontinuity points $D(h)$ is uncountable. Then the family

$$G = \{f \in F\colon D(f) \text{ is uncountable}\}$$

is a residual open set in (F, d), $d(f, g) = \|f - g\|$.

Proof. Let $C(f)$ be the set of all continuity points of the function f. If $f_n \in F - G$, $n = 1, 2, \ldots$ (i.e. each of the sets $D(f_n)$ is countable) and $d(f_n, f) \to 0$, then obviously $\bigcap_{n=1}^{\infty} C(f_n) \subset C(f)$. We have $D(f) \subset \bigcup_{n=1}^{\infty} D(f_n)$ and $f \in F - G$ — the set G is open in (F, d). The fact that G is residual in (F, d) is an easy consequence of the structure of the linear space F (see e.g. [NSŠ], Lemma 1).

Let bSC_n (bSC_n^L) stand for the class of all bounded (bounded Lebesgue measurable) functions of SC_n, $n = 1, 2, \ldots$. (Note that each function in SC_1 is Lebesgue measurable [P]. Since each additive function belongs to SC_2 this does not hold for the class SC_2.)

Corollary. Let F be one of the introduced classes bSC_n, or bSC_n^L, $n = 1, 2, \ldots$, furnished with the sup norm. Then F is a Banach space and

$$G = \{f \in F\colon D(f) \text{ is uncountable}\}$$

is a residual open set of the second category in (F, d).

Proof. F is a linear space closed with respect to the uniform convergence, i.e. F is a Banach space. In the paper [P] ([T]) there is constructed such a function $h \in SC_1$ ($h \in SC_2$, h — Lebesgue measurable), that $D(h)$ is an uncountable set. Obviously, we can arrange $h \in bSC_1$ ($h \in bSC_2^L$). The statement of the Corollary is a consequence of Theorem 1 (b) and Theorem 2.

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SÚHRN

O JEDNEJ TYPICKEJ VLASTNOSTI NIEKTORÝCH FUNKCIONÁLNYCH PRIESTOROV

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Práca sa zaoberá reálnymi funkciami definovanými na reálnom otvorenom intervale (a, b). Ak označíme symbolom $\Delta^n f(x, h)$ n-tú symetrickú diferenciu ($n = 1, 2, \ldots$) funkcie f v bode x (t.j. $\Delta^n f(x, h) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(x + (n - 2i)h)$), tak môžeme definovať triedy funkcií LS_n a SC_n nasledujúcim spôsobom:

$$LS_n = \left\{f: \forall_{x \in (a, b)} \exists_{\delta_x > 0} \forall_{h: 0 < h < \delta_x} \Delta^n f(x, h) = 0\right\},$$

$$SC_n = \left\{f: \forall_{x \in (a, b)} \lim_{h \to 0} \Delta^n f(x, h) = 0\right\}.$$

V práci sa študujú vlastnosti tried LS_n a SC_n. Napríklad je dokázané, že v normovanom priestore všetkých ohraničených (ohraničených lebesgueovsky merateľných) funkcií z SC_n so suprémovou normou je množina všetkých funkcií s nespočitateľnou množinou bodov nespojitosti reziduálnou otvorenou množinou druhej Baireovej kategórie.

РЕЗЮМЕ

ОБ ОДНОМ ТИПИЧНОМ СВОЙСТВЕ НЕКОТОРЫХ ПРОСТРАНСТВ ФУНКЦИЙ

Павел Костырко, Братислава

Работа занимается действительными функциями определенными на действительном открытом интервале (a, b). Если обозначим символом $\Delta^n f(x, h)$ n-тую симметрическую разность ($n = 1, 2, \ldots$) функции f в точке x (т.е. $\Delta^n f(x, h) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(x + (n - 2i)h)$), то можем определить классы функций LS_n и SC_n следующим образом:

$$LS_n = \left\{f: \forall_{x \in (a, b)} \exists_{\delta_x > 0} \forall_{h: 0 < h < \delta_x} \Delta^n f(x, h) = 0\right\},$$

$$SC_n = \left\{f: \forall_{x \in (a, b)} \lim_{h \to 0} \Delta^n f(x, h) = 0\right\}.$$

В работе изучаются свойства классов LS_n и SC_n. Доказано например, что в пространстве всех ограниченных (ограниченных измеримых по Лебегу) функций принадлежащих SC_n с нормой равномерной сходимости система всех функций, множество точек разрыва которых несчетно является открытым резидуальным множеством второй категории Бэра.