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ON A CERTAIN BOUNDARY VALUE THIRD ORDER PROBLEM WITH TWO PARAMETERS

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1. The boundary value problem of the form

$$y''' + [\mu f(x) + \lambda g(x)]y' + \lambda h(x)y = 0 \tag{1}$$

$$y(-a, \lambda, \mu) = y(a, \lambda, \mu) = 0, \quad a > 0 \tag{2}$$

$$\lambda \int_{-a}^{a} r(t, \mu)g((t)y(t, \lambda, \mu) + \int_{-a}^{t} (h(\tau) - g(\tau))y(\tau, \lambda, \mu)\,d\tau]\,dt =$$

$$= \mu \int_{-a}^{a} r(t, \mu)[f(t)y(t, \lambda, \mu) - \int_{-a}^{t} f(\tau)y'(\tau, \lambda, \mu)\,d\tau]\,dt, \tag{3}$$

where $f'(x)$, $g'(x)$, $h(x)$ are continuous functions on $\langle -a, a\rangle$, $r(x, \mu)$ is a suitable function of $x \in \langle -a, a\rangle$ and $-\infty < \lambda, \mu < \infty$ are parameters, will be studied.

The boundary condition (3) is in the integral form. For the first time such a condition was formulated in [1] and the problem (1), (2), (3) is a generalisation of the problem discussed in [1].

It will be shown that under certain conditions on the function $r = r(x, \mu)$ the problem can be solved by means of the problem (1), (4).

$$y(-a, \lambda, \mu) = y''(-a, \lambda, \mu) = y(a, \lambda, \mu) = 0. \tag{4}$$

In order to solve the problem (1), (4) the theory of the third order linear differential equation [2] can be applied.

2. Consider the problem (1), (2), (3). Let the functions f, g, h fulfil the conditions formulated in Section 1. Then the following theorem is true.

Theorem 1. A solution of the problem (1), (4) where the parameter μ is one of the eigenvalues and the function $r = r(, x, \mu)$ is the corresponding eigenfunction of the second order eigenvalue problem

$$r'' + \mu f(x) r = 0, \tag{5}$$

$$r(-a, \mu) = r(a, \mu) = 0 \tag{6}$$

is the solution of the problem (1), (2), (3), too.

Proof. Integrating the differential equation (1) written in the form

$$y''' + \{[\mu f(x) + \lambda g(x)]y\}' + \{-\mu f'(x) + \lambda[h(x) - g'(x)]\}y = 0$$

term by term from $-a$ to x, $x \leqq a$, and considering (2) we get

$$y'' + \mu f(x)y + \lambda g(x)y + \int_{-a}^{x} [-\mu f'(\tau) + \lambda(h(\tau) - g'(\tau))]y(\tau, \lambda, \mu)\, d\tau =$$
$$= y''(-a, \lambda, \mu).$$

Now suppose that $y''(-a, \lambda, \mu) = 0$, multiply the last equality by $r(x, \mu)$ and integrate it from $-a$ to a. We come to the equality

$$-\int_{-a}^{a} r(t, \mu)[y''(t, \lambda, \mu) + \mu f(t)y(t, \lambda, \mu)]\, dt =$$
$$= \int_{-a}^{a} r(t, \mu)\left\{\lambda g(t)y(t, \lambda, \mu) + \int_{-a}^{t} [-\mu f'(\tau) + \lambda(h(\tau) - g'(\tau))]y(\tau, \lambda, \mu)\, d\tau\right\} dt =$$
$$= \lambda \int_{-a}^{a} r(t, \mu)\left\{g(t)y(t, \lambda, \mu) + \int_{-a}^{t} [h(\tau) - g'(\tau)]y(\tau, \lambda, \mu)\, d\tau\right\} dt -$$
$$- \mu \int_{-a}^{a} r(t, \mu)\left\{f(t)y(t, \lambda, \mu) - \int_{-a}^{t} f(\tau)y'(\tau, \lambda, \mu)\, d\tau\right\} dt. \tag{7}$$

The right-hand side of (7) contains the expression which stands in the boundary condition (3). Therefore it is necessary to prove that the integral on the left-hand side of (7) is equal to zero.

Calculate this integral. Under the condition (2) it follows that

$$\int_{-a}^{a} [y''(t, \lambda, \mu) + \mu f(t)y(t, \lambda, \mu)]r(t, \mu)\, dt = y'(a, \lambda, \mu)r(a, \mu) -$$
$$-y'(-a, \lambda, \mu)r(-a, \mu) - \int_{-a}^{a} y'(t, \lambda, \mu)r'(t, \mu)\, dt + \mu \int_{-a}^{a} r(t, \mu)f(t)y(t, \lambda, \mu)\, dt =$$
$$= y'(a, \lambda, \mu)r(a, \mu) - y'(-a, \lambda, \mu)r(-a, \mu) + \int_{-a}^{a} [r'' + \mu f r]y\, dt.$$

This implies that the boundary condition (3) will be fulfilled if $y''(-a, \lambda, \mu) = 0$ and if the parameter μ is one of the eigenvalues and the function $r = r(x,$

μ) is the corresponding eigenfunction of the problem (5), (6). Thus the theorem is proved.

Remark 1. If $f(x) = 1$, $h(x) = g'(x)$, $a = \frac{\pi}{2}$ and $\mu = 1$, then we have the following special problem

$$y''' + \{[1 + \lambda g(x)]y\}' = 0 \tag{1'}$$

$$y\left(-\frac{\pi}{2}, \lambda\right) = y\left(\frac{\pi}{2}, \lambda\right) = 0 \tag{2'}$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\cos t - \cos\frac{\pi}{2}\right) g(t) y(t, \lambda)\,\mathrm{d}t = 0. \tag{3'}$$

This is the case of the paper [1] for $a = \frac{\pi}{2}$.

Remark 2. If $f(x) = 1$, $h(x) = g'(x)$ and $a = \frac{\pi}{n}$, $n \geqq 2$ being a natural number, then the parameter μ can be for example equal to $\frac{n^2}{4}$ and we obtain

$$y''' + \left\{\left[\frac{n^2}{4} + \lambda g(x)\right]y\right\}' = 0 \tag{1''}$$

$$y\left(-\frac{\pi}{n}, \lambda\right) = y\left(\frac{\pi}{n}, \lambda\right) = 0 \tag{2''}$$

$$\int_{-\frac{\pi}{n}}^{\frac{\pi}{n}} g(t) y(t, \lambda) \cos\frac{n}{2} t\,\mathrm{d}t = 0. \tag{3''}$$

Corollary 1. The problem (1″), (2″), (3″) and for $n = 2$ also the problem (1′), (2′), (3′) can be solved by means of the problem (1″), (4″), where

$$y\left(-\frac{\pi}{n}, \lambda\right) = y''\left(-\frac{\pi}{n}, \lambda\right) = y\left(\frac{\pi}{n}, \lambda\right) = 0. \tag{4''}$$

Remark 3. Suppose that $g(x) > 0$ for $x \in \left\langle -\frac{\pi}{n}, \frac{\pi}{n}\right\rangle$. Then the problem (1″), (4″) is clearly equivalent to the Sturm-Liouville problem

$$y'' + \left[\frac{n^2}{4} + \lambda g(x)\right]y = 0$$

$$y\left(-\frac{\pi}{n}, \lambda\right) = y\left(\frac{\pi}{n}, \lambda\right) = 0.$$

3. In this section the conditions on the coefficients of the equation (1) for the solution of the problem (1), (4) will be stated.

First of all we introduce some auxiliary results of the theory of the differential equation of the third order [2] of the form

$$y''' + 2A(x, \lambda)y' + [A'(x, \lambda) + b(x, \lambda)]y = 0. \tag{a}$$

Lemma 1. Let $A = A(x, \lambda)$, $A' = \frac{\partial A}{x}(x, \lambda)$ and $b(x, \lambda)$ be continuous functions of $x \in \langle a, \infty)$, a being a real number and $\lambda \in (\Lambda_1, \Lambda_2)$ and let $b(x, \lambda) \geqq 0$ for $x \in (a, \infty)$ and $\lambda \in (\Lambda_1, \Lambda_2)$ and $b(x, \lambda) \equiv 0$ do not hold in any subinterval of (a, ∞) for any λ. Let $y(x, \lambda)$ be a non-trivial solution of (a) with $y(a, \lambda) = 0$. Then the zeros of $y(x, \lambda)$ in (a, ∞) (if they exist) are continuous functions of the parameter $\lambda \in (\Lambda_1, \Lambda_2)$.

Proof. (See Lemma 4.2 in [2]).

Theorem A (Oscillation Theorem). Let the supposition of Lemma 1 on the coefficients of (a) be satisfied. Let further $\lim_{\lambda \to \Lambda_2} A(x, \lambda) = \infty$ uniformly for all $x \in \langle a, \infty)$. Let $a < b < \infty$ and let $y(x, \lambda)$ be a non-trivial solution of (a) with $y(a, \lambda) = 0$. With increasing $\lambda \to \Lambda_2$ the number of zeros of y in $\langle a, b\rangle$ increases to infinity and at the same time the distance between any two neighbouring zeros of y converges to zero.

Proof. (See Theorem 4.5,b) in [2]).

Theorem B. Let the suppositions of Lemma 1 be satisfied and let for some $\lambda = \bar{\lambda} \in (\Lambda_1, \Lambda_2)$ the differential equation of the second order

$$u'' + \frac{1}{2} A(x, \bar{\lambda})u = 0$$

be oscillatory in (a, ∞).

Then every solution y of the differential equation (a) with the property $y(\alpha, \lambda) = 0$, $\alpha \geqq a$ oscillates in (a, ∞).

Proof. (See Corollary 2.3, [2]).

Consider now the differential equation (1) and suppose (without loss of generality) that the functions $f'(x)$, $g'(x)$, $h(x)$ are continuous on $\langle -a, \infty)$, $0 < a < \infty$. Let further $f(x) > 0$, $g(x) > 0$, $f'(x) \leqq 0$ and let $h(x) - \frac{1}{2}g'(x) \geqq 0$ for $x > -a$ and let $h(x) - \frac{1}{2}g'(x) \equiv 0$ not hold in any subinterval of $\langle -a, \infty)$.

Under these conditions the following theorem holds.

Theorem 2. Let the coefficients of the differential equation (1) satisfy the above assumptions. Let μ be one of the positive eigenvalues and $r(x, \mu)$ the corresponding eigenfunction of the problem (5), (6). Then there exist such a number p and a sequence $\{\lambda_{p+v}\}_{v=1}^{\infty}$ of value S of the parameter λ and such a sequence of functions $\{y_{p+v}\}_{v=1}^{\infty}$ that $y_{p+v} = y(x, \lambda_{p+v}, \mu)$ is a solution of (1) satisfying the boundary conditions (4), $y(x, \lambda_{p+v}, \mu)$ having exactly $p + v - 1$ zeros in $(-a, a)$.

Proof. First of all we should observe that the coefficients of (1) satisfy the hypotheses of Lemma 1 and Theorem A. The equation (1) can be written in the form (a), that is

$$y''' + [\mu f(x) + \lambda g(x)]y' + \left[\frac{1}{2}(\mu f'(x) + \lambda g'(x)) + \lambda(h(x) - \frac{1}{2}g'(x)) - \right.$$

$$\left. - \frac{1}{2}\mu f'(x)\right]y = 0.$$

It is easy to see that the hypotheses of Lemma 1 and Theorem A are satisfied if $\lambda \in (0, \infty)$ is a positive parameter and μ is a positive eigenvalue of the problem (5), (6). Such eigenvalue μ clearly exists.

In virtue of Theorem B to the eigenvalue μ there exists such a $\lambda = \bar{\lambda} \in (0, \infty)$ that the differential equation

$$u'' + \frac{1}{4}[\mu f(x) + \lambda g(x)]u = 0$$

oscillates in $(-a, \infty)$ for $\lambda \geqq \bar{\lambda}$ and therefore the solution $y = y(x, \lambda)$ of the differential equation (1) with the property $y(-a, \lambda) = y''(-a, \lambda) = 0$ oscillates in $(-a, \infty)$ for $\lambda \geqq \bar{\lambda}$. Denote by $x_n(\bar{\lambda})$, $n = 1, 2, \ldots$ the zeros to the right of $-a$, of $y(x, \bar{\lambda})$ satisfying $y(-a, \bar{\lambda}) = y''(-a, \bar{\lambda}) = 0$. Let for $n = p$ be $x_p(\bar{\lambda}) < a$ and $x_{p+1}(\bar{\lambda}) \geqq a$. According to Lemma 1 $x_{p+1}(\lambda)$ is a continuous function of λ and hence in virtue of Theorem A there is $\bar{\bar{\lambda}} > \bar{\lambda}$ such that $x_{p+1}(\bar{\bar{\lambda}}) < a$, and from the continuity of $x_{p+1}(\lambda)$ there is $\bar{\lambda} \leqq \lambda_{p+1} < \bar{\bar{\lambda}}$ such that $x_{p+1}(\lambda_{p+1}) = a$, and $y(x, \lambda_{p+1}, \mu)$ satisfies the conditions (4) and has exactly p zeros in $(-a, a)$. Proceeding in this way we prove the existence of the sequences $\{\lambda_{p+v}\}_{v=1}^{\infty}$ and $\{y_{p+v}\}_{v=1}^{\infty}$.

Thus the theorem is proved.

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SÚHRN

O ISTOM OKRAJOVOM PROBLÉME TRETIEHO RÁDU S DVOMA PARAMETRAMI

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Práca sa zaoberá lineárnym dvojbodovým okrajovým problémom tretieho rádu (1), (2), (3), v ktorom koeficienty lineárnej diferenciálnej rovnice tretieho rádu sú funkciami nezávisle premennej a dvoch parametrov λ, n. Naviac jedna okrajová podmienka je daná v integrálnom tvare. Ukáže sa, že riešenie problému (1), (4) je tiež riešením okrajovému problému (1), (2), (3), kde okrajové podmienky (4) nie sú v integrálnom tvare. Pomocou teórie lineárnej rovnice tretieho rádu sa potom dokáže existencia vlastných hodnôt a vlastných funkcií, ktoré sú funkciami vlastných hodnôt istého lineárneho okrajového problému druhého rádu.

РЕЗЮМЕ

О НЕКОТОРОЙ КРАЕВОЙ ЗАДАЧЕ ТРЕТЬЕГО ПОРЯДКА С ДВУМА ПАРАМЕТРАМИ

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В работе изучается линейная двухточечная краевая задача третьего порядка (1), (2), (3), в которой коэффициенты линейного дифференциального уравнения третьего порядка являются функциями независимой переменной и двух параметров λ, μ. Более того, одно краевое условие задано в интегральной форме. Показано, что решение задачи (1), (4), в которой краевые условия уже не выражаются в интегральной форме, является решением задачи (1), (2), (3). С помощью теории линейного уравнения третьего порядка доказывается существование собственных значений и собственных функций, которые, в свою очередь, являются функциями собственных значений некоторой линейной краевой задачи второго порядка.