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DISTRIBUTION OF ZEROS OF THE SOLUTIONS OF LINEAR THIRD-ORDER DIFFERENTIAL EQUATIONS

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Dedicated to Prof. Michal Greguš on his 60th birthday

1 Introduction

In [2] M. Greguš summarised the main results of the theory of linear differential equations of the third order.

V. Šeda [5] developed the theory of transformations of the linear differential equation of the n-th order.

J. Moravčík in [4] investigated the differential equations

$$y''' + p(x)y' + q(x)y = 0, \quad x \in I = (a, b)$$

belonging to the class $T_K(I)$ (see (2.2)), i.e. which can be transformed to a differential equation with constant coefficients and whose solutions can be expressed explicitly.

In [6] the sequences of zeros of the solutions of the oscillatory equation

$$y'' + a(x)y' + b(x)y = 0$$

are compared with the sequences of zeros of the solutions of the equation

$$y'' + A(x)y' + B(x)y = 0$$

when the coefficients possess certain higher monotonicity properties.

In this paper, using the above-mentioned results, we derive certain higher monotonicity properties of the sequence $\{x_k - X_k\}$, where $\{x_k\}$ and $\{X_k\}$ denote the sequence of consecutive zeros of an oscillatory solution of

$$y''' + p_1(x)y' + q_1(x)y = 0 \in T_K(I)$$

and

$$y''' + p_2(x)y' + q_2(x)y = 0 \in T_K(I),$$

respecitively.

The presented approach is the model for investigation of differential equations which cannot be transformed to a differential equation with constant coefficients.

2 Definitions and notation

Consider a differential equation

$$\begin{gathered} y''' + p(x)y' + q(x)y = 0, \\ x \in I = (a, b), \ -\infty \leqq a < b \leqq \infty, \ p(x) \in C_4(I), \ q(x) \in C_3(I). \end{gathered} \tag{2.1}$$

Let K be a real number. By [4] the differential equation (2.1) on an interval I is said to be in the class $T_K(I)$ if

$$\frac{p(x) + \dfrac{7}{9}\left(\dfrac{b'(x)}{b(x)}\right)^2 - \dfrac{2}{3}\dfrac{b''(x)}{b(x)}}{(b(x))^{2/3}} \equiv K, \quad x \in I, \tag{2.2}$$

where

$$b(x) = q(x) - \frac{1}{2}p'(x) \neq 0 \quad \text{on } I. \tag{2.3}$$

Suppose that the differential equation (2.1) is in the class $T_K(I)$ for $K > -\frac{3}{2}2^{1/3}$.

Put

$$h(x) = \int_{x_0}^{x} |b(s)|^{1/3}\,\mathrm{d}s,$$

where $x_0 \in I$. From this it follows that

$$h'(x) = |b(x)|^{1/3}. \tag{2.4}$$

Suppose that the integral $\int_a^b |b(s)|^{1/3}\,\mathrm{d}s = \infty$ and that the finite $\lim_{x \to a^+} \int_{x_0}^{x} |b(s)|^{1/3}\,\mathrm{d}s$ exists. The change of variables

$$\begin{gathered} y(x) = (h'(x))^{-1}u(t), \\ h(x) = t, \quad (h'(x))^{-1} = \dot{x}(t), \quad \left[\dot{x} = \frac{\mathrm{d}}{\mathrm{d}t}\right], \end{gathered} \tag{2.5}$$

where $h(I) = J = (\alpha, \infty)$, $\alpha = \lim_{x \to a^+} \int_{x_0}^{x} |b(s)|^{1/3} \mathrm{d}s$, transforms the differential equation (2.1) into

$$\dddot{u} + K\dot{u} + ru = 0, \quad t \in J, \tag{2.6}$$

where $|r| = 1$, $\operatorname{sgn} r = \operatorname{sgn} b(x)$ (see [4]).

In [4] it is shown that

$$\begin{aligned} u_1(t) &= \exp(\lambda_1 t) \\ u_2(t) &= \exp\left(-\frac{1}{2}\lambda_1 t\right)\cos \varkappa t, \\ u_3(t) &= \exp\left(-\frac{1}{2}\lambda_1 t\right)\sin \varkappa t \end{aligned} \tag{2.7}$$

is a fundamental system of the solutions of (2.6) (λ_1 and $\varkappa$ are suitable real numbers) and

$$y_i(x) = (h'(x))^{-1} u_i(h(x)), \quad i = 1, 2, 3 \tag{2.8}$$

is a fundamental system of the solutions of (2.1).

If c_1, c_2, c_3 are real numbers ($c_2^2 + c_3^2 > 0$) and $b(x) > 0$, $x \in I$, then oscillatory solutions of the differential equation (2.1) form a system

$$S_{1,2,3} = \{c_1 y_1 + c_2 y_2 + c_3 y_3\}.$$

If $b(x) < 0$, $x \in I$, then oscillatory solutions of the differential equation (2.1) form a system

$$S_{2,3} = \{c_2 y_2 + c_3 y_3\}.$$

Let n be a nonnegative integer and I be an open interval.

We say that $f(x) \in M_n(I)$ ($f(x) \in M_n^*(I)$) if

$$(-1)^i f^{(i)}(x) \geqq 0 \quad ((-1)^i f^{(i)}x) > 0) \quad \text{for } x \in I, \quad i = 0, 1, \dots, n.$$

A sequence $\{x_k\} \in M_n$ ($\{x_k\} \in M_n^*$) if $(-1)^i \Delta^i x_k \geqq 0$ ($(-1)^i \Delta^i x_k > 0$) for $i = 0, 1, \dots, n$; $k = 1, 2, \dots$.

3 New results

In this section we consider the differential equations

$$y''' + p_j(x)y' + q_j(x)y = 0, \quad j = 1, 2, \tag{3.1$_j$}$$

where $x \in I$, $p_1(x), p_2(x) \in C_{m+1}(I)$, $q_1(x), q_2(x) \in C_m(I)$, $m \geqq 3$.

Let $b_j(x)$ and $h_j'(x)$ be determined from the coefficients of equations (3.1_j), $j = 1, 2$ by formulae analogous to (2.3) and (2.4). Suppose that the differential equations (3.1_j), $j = 1, 2$ are in the class $T_K(I)$ for $K > -\frac{3}{2}2^{1/3}$, $b_j(x) < 0$, $x \in I$, $j = 1, 2$.

The following theorem compares the sequences of zeros of the solutions of two differential equations. It describes the sequence of differences of corresponding zeros of the solutions of equations (3.1_1) and (3.1_2).

Theorem 3.1. Let $n \geqq 0$ be an integer and $m \geqq \max(n, 3)$. Let the differential equations (3.1_j) be in the class $T_K(I)$ for $K > -\frac{3}{2}2^{1/3}$ and $b_j(x) < 0$ on $I, j = 1, 2$. Let the integrals

$$\int_a^b |b_j(s)|^{1/3}\, \mathrm{d}s = \infty, \quad j = 1, 2$$

and let the finite

$$\lim_{x \to a^+} \int_{x_0}^{x} |b_j(s)|^{1/3}\, \mathrm{d}s, \quad j = 1, 2$$

exist. Suppose that for $x \in I$

$$(-1)^i((h_2'(x))^{-1})^{(i)} \geqq (-1)^i((h_1'(x))^{-1})^{(i)} > 0, \quad i = 0, 1, \ldots, n, \tag{3.2}$$

where the n-th derivatives exist in the open interval I, and the lower derivatives are continuous in its closure I^*. Suppose $y(x)$ is an oscillatory solution of (3.1_1) and $Y(x)$ is an oscillatory solution of (3.1_2). Then, if $x_1 > X_1$, we have

$$(-1)^i \Delta^i (x_k - X_k) > 0, \quad i = 0, 1, \ldots, n; \quad k = 1, 2, \ldots, \tag{3.3}$$

for the zeros x_k, X_k of $y(x)$ and $Y(x)$, respectively, in the intersection $I^* \cap \bar{I}$, where $\bar{I}$ is an interval defined to be either (X_1, ξ_0) or (X_1, ∞) according as there does or does not exist a solution ξ for the equation

$$\int_{x_1}^{\xi} h_1'(s)\, \mathrm{d}s = \int_{X_1}^{\xi} h_2'(s)\, \mathrm{d}s, \tag{3.4}$$

ξ_0 being the least such solution.

All of the above remains true if the factor $(-1)^i$ is deleted simultaneously from (3.2) and (3.3).

Proof. The transformations

$$\begin{aligned} y(x) &= \dot{x}(t)u(t), \quad \dot{x}(t) = (h_1'(x))^{-1}, \\ Y(X) &= \dot{X}(t)U(t), \quad \dot{X}(t) = (h_2'(X))^{-1}, \end{aligned} \tag{3.5}$$

transform the equations (3.1_1), (3.1_2) into the equations

$$\dddot{u} + K\dot{u} - u = 0,$$
$$\dddot{U} + K\dot{U} - U = 0, \quad t \in J, \tag{3.6}$$

respectively.

Since

$$\dot{x}(t) = (-b_1(x))^{-1/3} > 0, \quad \dot{X}(t) = (-b_2(x))^{-1/3} > 0, \quad x \in I,$$

there is a one-to-one correspondence between the zeros of $y(x)$ and $u(t)$ and between those of $Y(X)$ and $U(t)$, respectively. But all oscillatory solutions of (3.6) are of the form

$$\exp\left(-\frac{1}{2}\lambda_1 t\right) d_1 \cos(\varkappa t - d_2)$$

where d_1 and d_2 are constants. So that

$$\Delta t_k = \Delta T_k = \frac{\pi}{\varkappa}, \quad k = 1, 2, \ldots,$$

where $t_1, t_2, \ldots$ are the zeros of $u(t)$ corresponding, respectively, to $x_1, x_2, \ldots$, and $T_1, T_2 \ldots$ are the zeros of $U(t)$ corresponding, respectively, to $X_1, X_2, \ldots$.

Since, by (3.5),

$$t = \int_{x_1}^{x} h_1(s)\,\mathrm{d}s = \int_{X_1}^{X} h_2(s)\,\mathrm{d}s$$

we have $t_1 = T_1 = 0$, so that $t_k = T_k = (k-1)\dfrac{\pi}{\varkappa}$, $k = 1, 2, \ldots$ and

$$(-1)^i \Delta^i (x_k - X_k) = (-1)^i \Delta^i_{\frac{\pi}{\varkappa}} (x(t_k) - X(t_k)). \tag{3.7}$$

Applying a mean-value theorem for higher differences and derivatives [1] to the expression (3.7) we obtain

$$(-1)^i \Delta^i (x_k - X_k) = \left(-\frac{\pi}{\varkappa}\right)^i \left(x^{(i)}\left(t_k + i\Theta\frac{\pi}{\varkappa}\right) - X^{(i)}\left(t_k + i\Theta\frac{\pi}{\varkappa}\right)\right),$$
$$0 < \Theta < 1, \quad i = 0, 1, \ldots, n; \quad k = 1, 2, \ldots. \tag{3.8}$$

However,

$$(-1)^i x^{(i)}(t) > (-1)^i X^{(i)}(t), \quad i = 0, 1, \ldots, n, \tag{3.9}$$

for t in the appropriate interval (see (3.4) and [3], p. 67). So, the assertion (3.3) follows immediately from (3.8) and (3.9).

The last assertion in the statement of Theorem 3.1 follows on making obvious changes in the above proof.

This completes the proof of Theorem 3.1.

Remark 3.1. The conclusion of Theorem 3.1 remains true if the hypothesis $b_j(x) < 0$ on I, $j = 1, 2$ is replaced by $b_j(x) > 0$ on I, $j = 1, 2$ and $y(x)$, $Y(x)$ are of the form $y = c_2 y_{12} + c_3 y_{13}$, $Y = c_2 y_{22} + c_3 y_{23}$, where $y_{ji} = (h_j')^{-1} u_i(h_j)$, $i = 2, 3$; $j = 1, 2$.

Example 3.1. Consider the differential equations

$$y''' + (MN_j^{2/3} e^{2x} - 1)y' + (MN_j^{2/3} e^{2x} + N_j e^{3x})y = 0,$$

$$x \in I = (0, \infty), \quad j = 1, 2, \tag{3.10$_j$}$$

where $M > -\dfrac{3}{2} 2^{1/3}$ and $N_2 \leqq N_1 < 0$.

In the case of the differential equations (3.10$_j$) the coefficients $p_j(x)$ and $q_j(x)$ have the form

$$p_j(x) = MN_j^{2/3} e^{2x} - 1, \quad q_j(x) = MN_j^{2/3} e^{2x} + N_j e^{3x}, \quad j = 1, 2.$$

Since, by (2.3),

$$b_j(x) = N_j e^{3x}, \quad j = 1, 2$$

and by (2.2)

$$K = M,$$

we have

$$(3.10_j) \in T_M(I), \quad j = 1, 2.$$

Further, by (2.4),

$$(h_j'(x))^{-1} = (-N_j)^{-1/3} e^{-x}, \quad j = 1, 2,$$

so that (3.2) holds.

It is obvious that

$$\int_0^{\infty} (-N_j)^{1/3} e^{s} \, \mathrm{d}s = \infty$$

and the finite

$$\lim_{x \to 0^+} \int_{x_0}^{x} (-N_j)^{1/3} e^{s} \, \mathrm{d}s, \quad j = 1, 2$$

exist.

Thus, the hypotheses of Theorem 3.1 are fulfilled and (3.3) holds.

Consider an oscillatory solution $y(x)$ $(Y(x))$ of $(3.1_1)((3.1_2))$. By [2] the function $y(x)$ $(Y(x))$ is a solution of the equation of the band of solutions of the differential equation (3.1_j) on the whole interval I

$$(L_j(y)) = y'' - \frac{w_j'}{w_j}y' + \left(p_j + \frac{w_j''}{w_j}\right)y = 0, \qquad (3.11_j)$$

where
$w_j = \varkappa(h_j')^{-1}\exp(-\lambda_1 h_j)$, $j = 1$, $(j = 2)$.

It is known in [2] that the substitution

$$y(x) = (h_j'(x))^{-1/2}\exp(-\lambda_1 h_j(x)/2)v(x) \qquad (3.12_j)$$

transforms (3.11_j) into the equation

$$v'' + f_j(x)v = 0, \quad x \in I, \qquad (3.13_j)$$

where

$$f_j = \frac{1}{2}\frac{h_j'''}{h_j'} - \frac{3}{4}\left(\frac{h_j''}{h_j'}\right)^2 + \varkappa^2 h_j'^2, \quad j = 1, 2. \qquad (3.14_j)$$

Using this relation between the equations of the second and of the third order we derive further sufficient conditions for the higher monotonicity properties of solutions of certain 2nd order differential equations.

Theorem 3.2. Let the conditions of Theorem 3.1 be satisfied. If $y(x))$ and $Y(x)$ are solutions of the equation of the band of solutions (3.11_1), (3.11_2) of the differential equation (3.1_1), (3.1_2), respectively, then (3.3) holds.

Proof. By a direct calculation (by (2.5), (2.8) and (3.12_j)) we can show that the equation (3.13_j) possesses a pair of linearly independent solutions

$$v_{j2}(x) = ((\varkappa h_j'(x))^{1/2}\exp(\lambda_1 h_j(x)/2)y_{j2}(x) = (\varkappa h_j'(x))^{-1/2}\cos\varkappa h_j(x),$$
$$v_{j3}(x) = \; = (\varkappa h_j'(x))^{1/2}\exp(\lambda_1 h_j(x)/2)y_{j3}(x) = (\varkappa h_j'(x))^{-1/2}\sin\varkappa h_j(x)$$

such that the Wronskian $w(v_{j2}, v_{j3}) = 1$ and

$$p_j(x) = v_{j2}^2(x) + v_{j3}^2(x) = (\varkappa h_j'(x))^{-1}, \quad j = 1, 2,$$

by (3.2), satisfy

$$(-1)^i p_2^{(i)}(x) \geqq (-1)^i p_1^{(i)}(x) > 0, \quad i = 0, 1, \ldots, n$$

for x in the appropriate interval.

So, the conditions of Theorem 5.1 in [3] are fulfilled. By this theorem (3.3) holds.

Remark 3.2. Proof of Theorem 3.2 follows also from Theorem 9 in [5], because by this theorem the transformations (3.5) transform the equations $(h_j')^{-2}L_j(y) = 0, j = 1, 2$, respectively, into the equation

$$\ddot{u} + \lambda_1 \dot{u} + (\lambda_1^2 + K)u = 0, \quad t \in J$$

the fundamental system of solutions of which is formed by the functions $u_2(t)$, $u_3(t)$ from (2.7).

Remark 3.3. The equation of the band of solutions (3.11_1) of the differential equation (3.10_1) does not fulfil the hypothesis $f_1'(x) \in M_{n+1}(I)$ of Theorem 1.1 in [6] because the function $f_1(x)$ defined by (3.14_1) has the form $f_1(x) = -\frac{1}{4} + {}$ $+ \varkappa^2 N_1^{2/3} e^{2x}$.

But one can show that the equations of the band of solutions (3.11_1) and (3.11_2) of the differential equations (3.10_1) and (3.10_2), respectively, fulfil the hypotheses of Theorem 3.2. By this theorem we have $\{x_k - X_k\} \in M_n^*$.

So, this result does not follow from Theorem 1.1 in [6].

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SÚHRN

O ROZLOŽENÍ NULOVÝCH BODOV RIEŠENÍ LINEÁRNYCH DIFERENCIÁLNYCH ROVNÍC TRETIEHO RÁDU

Elena Pavlíková, Žilina

V práci sa porovnáva postupnosť nulových bodov $\{x_k\}$ oscilatorického riešenia rovnice

$$y''' + p_1(x)y' + q_1(x)y = 0 \in T_K(I)$$

s postupnosťou nulových bodov $\{X_k\}$ oscilatorického riešenia rovnice

$$y''' + p_2(x)y' + q_2(x)y = 0 \in T_K(I).$$

Vo vete 3.1 sú uvedené postačujúce podmienky na to, aby postupnosť $\{x_k - X_k\}$ bola monotónna n-tého rádu.

РЕЗЮМЕ

О РАСПРЕДЕЛЕНИИ НУЛЕЙ РЕШЕНИЙ ЛИНЕЙНЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ТРЕТЬЕГО ПОРЯДКА

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В работе сравнивается последовательность нулей $\{x_k\}$ осцилляционного решения уравнения

$$y''' + p_1(x)y' + q_1(x)y = 0 \in T_K(I)$$

с последовательностью нулей $\{X_k\}$ осцилляционного решения уравнения

$$y''' + p_2(x)y' + q_2(x)y = 0 \in T_K(I).$$

В теореме 3.1 приведены достаточные условия для того, чтобы последовательность $\{x_k - X_k\}$ являлась монотонной порядка n.