UNIVERSITAS COMENIANA
ACTA MATHEMATICA UNIVERSITATIS COMENIANAE
LIV—LV—1988

# THE MAXIMAL FAMILIES WITH RESPECT TO THE COMPOSITION OF FUNCTIONS WITH A CLOSED GRAPH

ROBERT MENKYNA, Liptovský Mikuláš

In the paper we shall consider functions whose domain $X$ and range $Y$ are topological spaces and which have closed graphs in $X \times Y$. Let $U(X, Y)$ denote the family of all functions with a closed graph and $C(X, Y)$ the family of all continuous functions.

A. M. Bruckner in the monograph [1] has defined the maximal additive (multiplicative) family. The paper [7] deals with maximal additive and maximal multiplicative families for functions with a closed graph. In a similar way we shall deal with the maximal family with respect to the inner (outer) component of the composition of functions with a closed graph.

**Definition 1.** Let $X$, $Y$, $Z$ be topological spaces. The family of functions $M^{out}(X, Y, Z) \subset U(Y, Z)$ is called the maximal family in $U(Y, Z)$ with respect to the outer component of the composition of functions, if $M^{out}(X, Y, Z)$ is the family of all functions $f \in U(Y, Z)$ such that $f(g) \in U(X, Z)$ for every $g \in U(X, Y)$.

Similarly we may define $M^{in}(X, Y, Z) \subset U(X, Y)$, the maximal family in $U(X, Y)$ with respect to the inner component of the composition of functions.

The paper [6] brings the following theorem.

**Theorem 1.** let $X$, $Y$, $Z$ be topological spaces. For every function $g \in C(X, Y)$ and $f \in U(Y, Z)$ it holds that $f(g) \in U(X, Z)$.

**Corollary.** $C(X, Y) \cap U(X, Y) \subset M^{in}(X, Y, Z)$.

**Theorem 2.** If $Y$ is a Hausdorff compact topological space, then $M^{in}(X, Y, Z) = U(X, Y)$ and $M^{out}(X, Y, Z) = U(Y, Z)$.

**Proof.** If a topological space $Y$ is Hausdorff and compact, then from Theorem 3 [5] it follows that $C(X, Y) \subset U(X, Y)$ and from Theorem 4 [5] $U(X, Y) \subset C(X, Y)$. That is, $C(X, Y) = U(X, Y)$. Because according to Theorem 1 for every function $f \in C(X, Y)$ and every $g \in U(Y, Z)$ it holds that $g(f) \in U(X, Z)$, then $M^{out}(X, Y, Z) = U(Y, Z)$ and $M^{in}(X, Y, Z) = U(X, Y)$.

In the following we shall deal with the cases where a topological space $Y$ is not compact.

**Definition 2** [4]. A compactification of a topological space $X$ is defined to be a pair $(\alpha, X_\alpha)$, where $X_\alpha$ is a compact topological space and $\alpha$ is a homeomorphism of $X$ onto a dense subspace of $X_\alpha$.

**Definition 3.** Let $(\alpha, Y_\alpha)$ be a compactification of a topological space $Y$. We say that the function $f \in U(Y, Z)$ is of the family $M(Y_\alpha, Z)$ if and only if $f$ is constant or satisfies the next condition: If a net $\{\alpha(y_i), i \in I\}$, $y_i \in Y$ converges to any point $\tilde{y} \in Y_\alpha - \alpha(Y)$, then the net $\{f(y_i), i \in I\}$ does not converge in $Z$.

In the general case correspondent families $M(Y_\alpha, Z)$ for different compactifications $(\alpha, Y_\alpha)$ of a topological space $Y$ may by different.

**Theorem 3.** Let $(\gamma, Y_\gamma)$, be compactification of a topological space $Y$. If $Y_\gamma$ is a Hausdorff topological space, then for every compactification $(\alpha, Y_\alpha)$ $M(Y_\alpha, Z) \subset M(Y_\gamma, Z)$ holds.

**Proof.** Let the function $f \notin M(Y_\gamma, Z)$. There is a net $\{y_i, i \in I\}$, $y_i \in Y$ such that the net $\{\gamma(y_i) \to \tilde{y} \in Y_\gamma - - \gamma(Y)$ and the net $\{f(y_i), i \in I\}$ converges to a point $z \in Z$. The topological space $Y_\gamma$ is Hausdorff, then any subnet of the net $\{y_i, i \in I\}$ does not converge in $Y$. However there is a subnet $\{y_{i_j}, j \in J\}$ such that $\alpha(y_{i_j})$ converges to the point $\tilde{y}_0 \in Y_\alpha - \alpha(Y)$. The net $\{f(y_{i_j}), j \in J\}$ converges to the point $z \in Z$, i.e. the function $f \notin M(Y_\alpha, Z)$, which finishes the proof.

**Remark.** If $\varphi$: $Y_1 \to Y_2$ is a homeomorphism of topological spaces $Y_1$, $Y_2$, then from Theorem 1 it is clear that $f(\varphi) \in U(Y_1, Z)$ if and only if $f \in U(Y_2, Z)$. Therefore if below $(\alpha, Y_\alpha)$ will be a compactification of a topological space $Y$, without loss of generality we can assume that $Y \subset Y_\alpha$.

**Lemma 1.** Let $(\alpha, Y_\alpha)$ be a compactification of a topological space $Y$. If the function $f \in U(X, Y)$ is discontinuous at a point $x_o \in X$, then there is a net $\{x_i, i \in I\}$, $x_i \to x_o$ such that $f(x_i) \to \tilde{y} \in Y_\alpha - Y$.

**Proof.** If $f$ is discontinuous at a point $x_o$, then there is a net $\{x_j, j \in J\}$, $x_j \to x_o$ for which $f(x_j) \not\to f(x_o)$ and an open neighborhood $V$ of the point $f(x_o)$ such that the net $\{f(x_j), j \in J\}$ is frequently in $Y_\alpha - V$. Since $Y_\alpha - V$ is compact, we may choose a convergent subnet $\{f(x_{j_i}), i \in I\}$, $f(x_{j_i}) \to \tilde{y} \in Y_\alpha - V$. The point $\tilde{y}$ is a member of $Y_\alpha - Y$, because the opposite assertion contradicts the fact that $f \in U(X, Y)$.

**Theorem 4.** Let $X$, $Y$, $Z$ be topological spaces and $Y_\alpha \supset Y$ a compactification of the topological space $Y$. Then for every function $f \in M(Y_\alpha, Z)$ and every $g \in U(X, Y)$ it holds that $f(g) \in U(X, Z)$, i.e. $M(Y_\alpha, Z) \subset M^{out}(X, Y, Z)$.

**Proof.** Let $g$ be any function of the family $U(X, Y)$. If $f \in U(Y, Z)$ is constant, then $f(g) \in U(X, Z)$, since $\{f(y)\}$ is a closed set in $Z$ for every $y \in Y$ (Theorem 1 [5]).

If $f \in M(Y_\alpha, Z)$ is not constant, then if a net $\{(x_i, f(g(x_i))), i \in I\}$ converges to a point $(x_o, z) \in X \times Z$, the net $\{(x_i, g(x_i)), i \in I\}$ converges to the point $(x_o, g(x_o))$. If that is not the case, there is a subnet $\{x_{i_j}, j \in J\}$ (Lemma 1) such that $g(x_{i_j}) \to \tilde{y} \in Y_\alpha - Y$. Since $\{f(g(x_{i_j})), j \in J\}$ converges to $z \in Z$, we have a contradiction with the assumption $f \in M(Y_a, Z)$. Because the net $\{(g(x_i), f(g(x_i))), i \in I\}$ converges to the point $(g(x_o), z) \in Y \times Z$ and $f \in U(Y, Z)$, it holds that $z = f(g(x_o))$, i.e. $f(g) \in U(X, Z)$.

Let $M(Y, Z)$ be a union of all families $M(Y_a, Z)$. If a topological space $Y$ is Tychonoff, then for Stone-Cech compactification $(\beta, Y_\beta)$ is holds, that $M(Y, Z) = M(Y_\beta, Z)$.

**Corollary.** For any topological space $X$, $Y$, $Z$ it holds that $M(Y, Z) \subset M^{out}(X, Y, Z)$.

In the present section we construct topological spaces $X$, $Y$, $Z$ for which $C(X, Y) \cap U(X, Y) \subsetneqq M^{in}(X, Y, Z) \subsetneqq U(X, Y)$, or $M(Y, Z) \subsetneqq M^{out}(X, Y, Z) \subsetneqq U(Y, Z)$ respectively.

**Example 1.** Let us give topological spaces $(X, T_x)$, where $X = N$ and $T_x = \{A \in P(N),\ N - A \text{ is finite}\} \cup \{\emptyset\}$, $(Z = \{z_1, z_2\}, P(Z))$ and $(Y = \{y_o, y'_o, y_1, y'_1, \dots\}, T_y)$, where the subbase for the topology $T_y$ is the family of all subsets of the forms $\{y_o, y_i\}$ or $\{y'_o, y'_i\}$, $i = 1, 2, \dots$ .
It is easy to verify that the following propositions hold.

Proposition $1^1$: A function $f \in U(X, Y)$ if and only if $f(X) \cap \{y_o, y'_o\} = \emptyset$ and $f$ satisfies any from the following conditions:

**i)** $f$ is constant, **ii)** $\operatorname{card} f^{-1}(y) < +\infty$ for every $y \in Y$.

Proposition $2^1$. $U(Y, Z) = C(Y, Z)$ and $f \in C(Y, Z)$ if and only if $f(y_i) = f(y_o)$, $f(y'_i) = f(y'_o)$ for every $i = 1, 2, \dots$ .
We are going to show that:

**a)** $U(X, Y) \supsetneqq M^{in}(X, Y, Z)$.

We define the function $f\colon X \to Y$ by

$$f(x) = \begin{cases} y_n, & \text{if } x = 2n - 1 \\ y'_n, & \text{if } x = 2n \end{cases}$$

and the function $g\colon Y \to Z$ by

$$g(y) = \begin{cases} z_1, & \text{if } y \in \{y_o, y_1, \dots\} \\ z_2, & \text{if } y \in \{y'_o, y'_1, \dots\}. \end{cases}$$

Evidently $f \in U(X, Y)$ (Proposition $1^1$) and $g \in U(Y, Z)$ (Proposition $2^1$), but $g(f) \notin U(X, Z)$ because the net $\{(2n, g(f(2n))), n = 1, 2, \dots\}$ converges to the point $(1, z_2)$ but $g(f(1)) = z_1 \neq z_2$. Thus the function $f \in U(X, Y) - M^{in}(X, Y, Z)$.

**b)** There is a function $f \in U(X, Y) - C(X, Y)$ such that $g(f) \in U(X, Z)$ for every $g \in U(Y, Z)$.

We define a function $f\colon X \to Y$ by $f(n) = y_n$, $n = 1, 2, \dots$ . The function $f \in U(X, Y)$ (Proposition $1^1$) and $f$ is discontinuous in every point $n \in X$. For every function $g \in U(Y, Z)$ $g(f)$ is a constant (Proposition $2^1$) and so $g(f) \in U(X, Z)$, that is $f \in M^{in}(X, Y, Z) - C(X, Y)$.

In Example 2 we shall use the Sierpinski theorem ([2], p. 440).

**The Sierpinski theorem.** If a continuum $X$ (a connected and compact topological space) has a countable cover $\{F_i\}_{i=1}^{\infty}$ by pairwise disjoint closed subsets, then at most one of the sets $F_i$ is non-empty.

**Remark.** The assertion of the Sierpinski theorem holds for a set $\emptyset \neq A = \bigcup_{i=1}^{\infty} A_i$ too, where $A_1 \subset A_2 \subset \dots$ is a sequence of continua.

**Example 2.** Let $X = [0, 1]$, $Z = [0, 1)$ and $Y = \left\{\left(x, \sin\frac{1}{x}\right), x \in \left(0, \frac{1}{\pi}\right]\right\} \cup \cup \left\{(x, 0), x \in \left[0, \frac{1}{\pi}\right]\right\}$ with the relative topology of $[0, 1]^2$.

For given topological spaces it holds:

Proposition 1[2]. If $f \in U(X, Y)$ is discontinuous at a point $x_o$, then $f(x_o) = (0, 0)$.

**Proof.** Assume that $f(x_o) \neq (0, 0)$. Then for every sequence $x_i \to x_o$ the sequence $\{f(x_i)\}_{i=1}^{\infty}$ does not converge to the point $(0, 0)$. Thus there is a neighbourhood $U = (x_o - \delta, x_o + \delta)$ of the point $x_o$ and a neighbourhood $V = (-\varepsilon, \varepsilon)^2 \cap Y$ of the point $(0, 0)$ such that $f(U) \cap V = \emptyset$. The set $Y - V$ may be expressed as a countable union of closed, pairwise disjoint sets $Y - V = \bigcup_{i=1}^{\infty} A_i$. Because $A_i$, $i = 1, 2, \dots$ is compact, the set $f^{-1}(A_i)$ is closed (Theorem 3.6 [3]). Evidently $\bigcup_{i=1}^{\infty} f^{-1}(A_i) \supset U$ and according to the remark above exactly one set $f^{-1}(A_i)$ is not empty. From this and from Theorem 4 [5] it follows that $f$ is continuous in the point $x_o$, which contradicts the assumption.

We shall show that:

**a)** $U(Y, Z) \supsetneqq M^{out}(X, Y, Z)$, that is we choose a function $f \in U(Y, Z)$ and $g \in U(X, Y)$ for which $f(g) \notin U(X, Z)$.

Define the function $f\colon Y \to Z$ by $f(x, y) = \frac{1}{2}|y|$ and the function

$$g\colon X \to Y \text{ by } g(x) = \begin{cases} \left(x, \sin\frac{1}{x}\right) & \text{if } x \neq 0 \\ (0, 0) & \text{if } x = 0. \end{cases}$$

The function $f \in U(Y, Z)$, $g \in U(X, Y)$ but $f(g) \notin U(X, Z)$, because the sequence $\left\{\left(\frac{2}{(4n+1)\pi}, f\left(g\left(\frac{2}{(4n+1)\pi}\right)\right)\right)\right\}_{n=1}^{\infty}$ converges to the point $\left(0, \frac{1}{2}\right)$ but $f(g(0)) = 0 \neq \frac{1}{2}$.

**b)** $M^{out}(X, Y, Z) \supsetneq M(Y, Z)$, that is there is $f \in U(Y, Z) - M(Y, Z)$ such that $f(g) \in (X, Z)$ for every $g \in U(X, Y)$.

Define the function $f\colon\ Y \to Z,\ f(x,\ y) = -x\left(x - \frac{1}{\pi}\right)$. The function $f \in U(Y, Z) - M(Y, Z)$, since the sequence $\left\{\left(\frac{2}{(4n+1)\pi}, 1\right)\right\}_{n=1}^{\infty}$ converges to the point $(0, 1) \in Y_\beta - Y$ and the sequence $f\left(\frac{2}{(4n+1)\pi}, 1\right) \to 0 \in Z$. Let $g \in U(X, Y)$ be any function and let a sequence $\{(x_i, f(g(x_i))), i = 1, 2, \ldots\}$ converge to a point $(x_o, z) \in X \times Z$. We shall show that $z = f(g(x_o))$.

If $x_o$ is a point of continuity of the function $g$, then $f(g)$ is continuous at $x_o$ and thus $z = f(g(x_o))$.

If $x_o$ is a point of discontinuity of the function $g$, then the set of accumulation points of the sequence $\{g(x_i)\}_{i=1}^{\infty}$ is a subset of $\{(x, y), x = 0 \text{ and } y \in [-1, 1]\}$. Because $f \in C([0, 1]^2, Z)$, the sequence $\{f(g(x_i))\}_{i=1}^{\infty}$ converges to $z = 0$. From Proposition $1^2$ it follows that $g(x_o) = 0$ and so $f(g(x_o)) = 0$, i.e. $f(g(x_o)) = z$.

## REFERENCES


1. Bruckner, A. M.: Differentation of real functions, Berlin Heidelberg New York 1978.
2. Engelking, R.: General topology, Warszawa 1977 PWN.
3. Fuller, R. V.: Relations among continuous and various non-continuous functions, Pacific Math. J. 25 (3), (1968), 495—509.
4. Kelley, J. L.: General topology, New York 1955.
5. Kostyrko, P.: A note on the functions with closed graphs, Časopis pro pěstování matematiky, roč. 94 (1969), 202—205.
6. Kostyrko, P.—Neubrunn, T.—Šalát, T.: О функциях, графы которых являются замкнутыми множествами II, Acta fac. rer. nat. Univ. Com., (1965), 51—61.
7. Menkyna, R.: The maximal additive and multiplicative families for functions with closed graph, Acta fac. rer. nat. Univ. Com., (1987).



*Author's address:*

Robert Menkyna
Cédrová 490/5.
031 01 Liptovský Mikuláš



Received: 14. 5. 1987

## SÚHRN

## MAXIMÁLNA TRIEDA FUNKCIÍ VZHĽADOM NA SKLADANIE FUNKCIÍ S UZAVRETÝM GRAFOM

Robert Menkyna, Liptovský Mikuláš

Nech $X$, $Y$, $Z$ sú topologické priestory. Označme $U(X, Y)$ triedu všetkých funkcií $f$: $X \to Y$ s uzavretým grafom v $X \times Y$. V článku sa zaoberám triedou všetkých funkcií $f \in U(x, Y)/f \in U(Y, Z)$ / takých, že $g(f) \in U(X, Z)$ $/f(g) \in U(X, Z)/$ pre každú funkciu $g \in U(Y, Z)$ $/g \in U(X, Y)/$.



## РЕЗЮМЕ

## МАКСИМАЛЬНАЯ СИСТЕМА ОТОБРАЖЕНИЙ ПО ОТНОЩЕНИЮ К КОМПОЗИЦИИ ОТОБРАЖЕНИЙ С ЗАМКНУТЫМ ГРАФИКОМ

Роберт Менкина, Липтовски Микулаш

Пусть $X$, $Y$, $Z$ являются топологическими пространствами. Обозначим $U(X, Y)$ систему всех тех отображений пространства $X$ в пространство $Y$, графики которых являются замкнутыми подмножествами в $X \times Y$. В этой работе автор рассматривает систему всех отображений $f \in U(X, Y)/f \in U(Y, Z)/$, для которых имеет место $g(f) \in U(X, Z)$ $/f(g) \in U(X, Z)$ для любого $g \in U(Y, Z)$ $/g \in U(X, Y)$.