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OSCILLATORY PROPERTIES OF THE SOLUTIONS OF THREE-DIMENSIONAL NONLINEAR DIFFERENTIAL SYSTEMS WITH DEVIATING ARGUMENTS

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Introduction

In this paper we consider a nonlinear differential system with deviating arguments:

$$\begin{aligned}(S)\quad & y_1'(t) = (-1)^{\nu_1} p_1(t) f_1(y_2(h_2(t)))\\ & y_2'(t) = (-1)^{\nu_2} p_2(t) f_2(y_3(h_3(t)))\\ & y_3'(t) = (-1)^{\nu_3} p_3(t) f_3(y_1(h_1(t))), \quad t \ge 0, \quad \nu_i \in \{0, 1\}, \quad i = 1, 2, 3.\end{aligned}$$

The following conditions are always assumed to be fulfilled:

(a) p_i: $[0, \infty) \to [0, \infty)$, $i = 1, 2, 3$, are continuous and not identically zero on any subinterval of $[T, \infty) \subset [0, \infty)$; $\int^{\infty} p_i(t)\,\mathrm{d}t = \infty$, $i = 1, 2$;

(b) h_i: $[0, \infty) \to R$, $i = 1, 2, 3$, are continuous and $\lim_{t \to \infty} h_i(t) = \infty$;

(c) f_i: $R \to R$, $i = 1, 2, 3$, are continuous and nondecreasing, u $f_i(u) > 0$ for $u \ne 0$.

Denote by W the set of all solutions $y(t) = \{y_1(t), y_2(t), y_3(t)\}$ of (S) which exist on some ray $[T_y, \infty) \subset [0, \infty)$ and satisfy $\sup\left\{\sum_{i=1}^{3} |y_i(t)|:\ t \ge T\right\} > 0$ for any $T \ge T_y$.

Definition 1. A solution $y \in W$ is called *oscillatory* (resp. weakly oscillatory) if each of its components (resp. at least one component) has arbitrarily large zeros. A solution $y \in W$ is called *nonoscillatory* (resp. weakly nonoscillatory) if each of its components (resp. at least one component) is eventually of a constant sing.

The oscillation theory of nonlinear differential systems with deviating arguments has been developed by many authors; see, for example, Kitamura and Kusano [1], Marušiak [2], Šeda [3] and Shevelo, Varech and Gritsai [4]. Our paper extends some of the results stablished in [4] to the system (S).

Oscillation theorems

We introduce the following notations:

$$h_i^*(t) = \min\{h_i(t), t\}, \quad i = 1, 2, 3,$$

$$\gamma_i(t) = \sup\{s \geq 0, h_i^*(s) < t\} \quad \text{for } t \geq 0, \quad i = 1, 2, 3,$$

$$\gamma(t) = \max\{\gamma_1(t), \gamma_2(t), \gamma_3(t)\}.$$

Lemma 1. Let $y(t) = \{y_1(t), y_2(t), y_3(t)\} \in W$ be a weakly nonoscillatory solution of (S). Then there exists a $t_1 \geq 0$ such that each of its components is monotone and of a constant sign on $[t_1, \infty)$.

Proof. Suppose that $y_1(t)$ is a nonoscillatory component of a solution $y(t) = \{y_1(t), y_2(t), y_3(t)\} \in W$ and $y_1(t) \neq 0$ for $t \geq T_0 \geq 0$. (In the case that $y_2(t)$ or $y_3(t)$ are nonoscillatory components of a solution $y(t)$, we can proceed analogously.) In view of (a), (b), (c), the third equation of (S) implies that either $y_3'(t) \geq 0$ or $y_3'(t) \leq 0$ for $t \geq \gamma(T_0) = T_1$ and $y_3(t) \not\equiv 0$ for $t \geq T_2 \geq T_1$. (If $y_3(t) \equiv 0$ for $t \geq T_2$, then $y_3'(t) \equiv 0$ for $t \geq T_2$ and the third equation of (S) gives that $p_3(t) \equiv 0$ for all $t \geq T_2$, which contradicts assumption (a)). We get that $y_3(t)$ is a monotone function and there exists a $T_3 \geq T_1$ such that either $y_3(t) > 0$ or $y_3(t) < 0$ for all $t \geq T_3$. Analogously we can prove that $y_1(t)$ and $y_2(t)$ are monotone functions of a constant sign on $[t_1, \infty)$, where $t_1 \geq T_3$.

Theorem 1. Let the following conditions be satisfied:

$$h_3(h_2(h_1(t))) \leq t, \; h_i(t) \text{ are nondecreasing functions } i = 2, 3 \tag{1}$$

$$xyf_i(xy) \geq Kxyf_i(x)f_i(y) \quad (0 < K = \text{const.}) \quad i = 1, 2, 3 \tag{2}$$

$$\int_0^\infty p_3(t) f_3\left[\int_0^{h_1(t)} p_1(s) f_1\left(\int_0^{h_2(s)} p_2(x)\,\mathrm{d}x\right)\mathrm{d}s\right]\mathrm{d}t = \infty \tag{3}$$

$$\int_0^\infty p_2(t) f_2\left(\int_{h_3(t)}^\infty p_3(s)\,\mathrm{d}s\right)\mathrm{d}t = \infty \tag{4}$$

$$\int_0^\alpha \frac{\mathrm{d}u}{f_3(f_1(f_2(u)))} < \infty, \quad \int_0^{-\alpha} \frac{\mathrm{d}u}{f_3(f_1(f_2(u)))} < \infty \tag{5}$$

for every constant $\alpha > 0$.

If $\nu_1 + \nu_2 + \nu_3 \equiv 1 \pmod 2$, then every solution $y \in W$ is either oscillatory or $y_i(t)$ $(i = 1, 2, 3)$ tend monotonically to zero as $t \to \infty$.

Proof. Suppose that (S) has a weakly nonoscillatory solution $y(t) = \{y_1(t), y_2(t), y_3(t)\} \in W$. Then with regard to Lemma 1, each of its components is a monotone function of a constant sign on $[t_1, \infty)$. Without loss of generality we may suppose that $y_1(t) > 0$ for $t \geq t_1$. Then the third equation of (S) implies that $(-1)^{\nu_3} y_3(t)$ is nondecreasing for $t \geq t_1$ and either $(-1)^{\nu_3} y_3(t) > 0$ for $t \geq t_1$ or $(-1)^{\nu_3} y_3(t) < 0$ for $t \geq t_1$.

I) Let $(-1)^{\nu_3} y_3(t) > 0$ for $t \geq t_1$. Then $(-1)^{\nu_3} y_3(t) \geq (-1)^{\nu_3} y_3(t_1) = C_1 > 0$ for $t \geq t_1$. In view of (b), there exists a $t_2 = \gamma(t_1)$ such that $(-1)^{\nu_3} y_3(h_3(t)) \geq C_1$, $t \geq t_2$. According to (c) the last inequality implies

$$(-1)^{\nu_3} f_2(y_3(h_3(t))) \geq C_2 > 0, \quad t \geq t_2 \tag{6}$$

where $C_2 = \min\{f_2(C_1), -f_2(-C_1)\}$.
Multiplying the second equation of (S) by $(-1)^{\nu_2 + \nu_3}$, integrating from t_2 to t and then using (6), we get

$$(-1)^{\nu_2 + \nu_3}[y_2(t) - y_2(t_2)] \geq C_2 \int_{t_2}^{t} p_2(s)\,ds, \quad t \geq t_2. \tag{7}$$

With regard to (a) and the monotonicity of $y_2(t)$, (7) implies: $\lim_{t \to \infty} (-1)^{\nu_2 + \nu_3} y_2(t) = \infty$ and

$$(-1)^{\nu_2 + \nu_3} y_2(h_2(t)) \geq (-1)^{\nu_2 + \nu_3} y_2(t_1) = C_3 > 0, \quad t \geq t_2 \tag{8}$$

holds. From (8), in view of (c), we get

$$(-1)^{\nu_2 + \nu_3} f_1(y_2(h_2(t))) \geq C_4 > 0, \quad t \geq t_2, \tag{9}$$

where $C_4 = \min\{f_1(C_3), -f_1(-C_3)\}$.

Multiplying the first equation of (S) by $(-1)^{\nu_1 + \nu_2 + \nu_3}$, integrating from t_2 to t and then using (9), we have

$$-[y_1(t) - y_1(t_2)] \geq C_4 \int_{t_2}^{t} p_1(s)\,ds, \quad t \geq t_2. \tag{10}$$

In view of (a) and the monotonicity of y_1, (10) implies:

$\lim_{t \to \infty} y_1(t) = -\infty$, which contradicts the assumption $y_1(t) > 0$ for $t \geq t_1$. The case I) cannot occur.

II) Let $(-1)^{\nu_3} y_3(t) < 0$ for $t \geq t_1$. Then from the second equation of (S) we obtain that $(-1)^{\nu_2+\nu_3}y_2(t)$ is nonincreasing on $[t_1, \infty)$ and either $(-1)^{\nu_2+\nu_3}y_2(t) < 0$ or $(-1)^{\nu_2+\nu_3}y_2(t) > 0$ for $t \geq t_1$.

IIa) Let $(-1)^{\nu_2+\nu_3}y_2(t) < 0$ for $t \geq t_1$. Then the system (S) implies:

$y_1(t) > 0$ nondecreasing	$(-1)^{\nu_2+\nu_3}y_2(t) < 0$ nonincreasing	$(-1)^{\nu_3}y_3(t) < 0$ nondecreasing	for $t \geq t_1$.

In view of the assumption (1), the function $(-1)^{\nu_3}y_3(h_3(t))$ is negative and nondecreasing on $[t_2, \infty)$. Multiplying the second equation of (S) by $(-1)^{\nu_2+\nu_3}$ and then integrating from t_2 to t, we have

$$(-1)^{\nu_2+\nu_3}[y_2(t) - y_2(t_2)] = \int_{t_2}^{t} p_2(s)(-1)^{\nu_3}f_2(y_3(h_3(s)))\,\mathrm{d}s, \quad t \geq t_2. \tag{11}$$

From (11), with regard to $(-1)^{\nu_2+\nu_3+1}y_2(t_2) > 0$, the monotonicity of $(-1)^{\nu_3}f_2(y_3(h_3(s)))$, we get

$$(-1)^{\nu_2+\nu_3}y_2(h_2(t)) \leq (-1)^{\nu_3}f_2(y_3(h_3(h_2(t)))) \cdot \int_{t_2}^{h_2(t)} p_2(s)\,\mathrm{d}s, \tag{12}$$

for $t \geq t_3 = \gamma(t_2)$.

Using the assumptions (c) and (2) we shall prove that (12) implies

$$(-1)^{\nu_2+\nu_3}f_1(y_2(h_2(t))) \leq C_5(-1)^{\nu_3}f_1(f_2(y_3(h_3(h_2(t))))) \cdot f_1\left(\int_{t_2}^{h_2(t)} p_2(s)\,\mathrm{d}s\right), \tag{13}$$

for $t \geq t_3$, where $0 < C_5 = \min\{K, -K^2 f_1(-1)\}$.

Multiplying the first equation of (S) by $(-1)^{\nu_1+\nu_2+\nu_3}$, integrating from t_3 to t and then using (13), $y_1(t_3) > 0$, we obtain

$$-y_1(t) \leq C_5(-1)^{\nu_3}\int_{t_3}^{t} p_1(s)f_1(f_2(y_3(h_3(h_2(s))))) \times$$
$$\times f_1\left(\int_{t_2}^{h_2(s)} p_2(x)\,\mathrm{d}x\right)\mathrm{d}s, \quad t \geq t_3. \tag{14}$$

From (14), in view of the fact that $(-1)^{\nu_3}f_1(f_2(y_3(h_3(h_2(s)))))$ is the nondecreasing function on $[t_3, \infty)$, we get

$$y_1(h_1(t)) \geq C_5(-1)^{\nu_3+1}f_1(f_2(y_3(h_3(h_2(h_1(t)))))) \times$$
$$\times \int_{t_3}^{h_1(t)} p_1(s)f_1\left(\int_{t_2}^{h_2(s)} p_2(x)\,\mathrm{d}x\right)\mathrm{d}s, \quad \text{for } t \geq t_4 = \gamma(t_3). \tag{15}$$

By virtue of (1) and the monotonicity of $(-1)^{\nu_3+1} f_1(f_2(y_3(s)))$, from (15) we have

$$y_1(h_1(t)) \geq C_5(-1)^{\nu_3+1} f_1(f_2(y_3(t))) \times$$
$$\times \int_{t_3}^{h_1(t)} p_1(s) f_1\left(\int_{t_2}^{h_2(s)} p_2(x)\,dx\right) ds, \quad t \geq t_4. \tag{16}$$

Using (2) and (c), (16) implies

$$f_3(y_1(h_1(t))) \geq C_6 f_3(f_1(f_2(y_3(t)))) \times$$
$$\times f_3\left(\int_{t_3}^{h_1(t)} p_1(s) f_1\left(\int_{t_2}^{h_2(s)} p_2(x)\,dx\right) ds\right), \quad t \geq t_4, \tag{17}$$

where $C_6 = K^2 f_3(C_5(-1)^{\nu_3+1})$.

Multiplying (17) by $(-1)^{\nu_3} p_3(t)[f_3(f_1(f_2(y_3(t))))]^{-1}$ and using the third equation of (S), we get

$$\frac{y_3'(t)}{f_3(f_1(f_2(y_3(t))))} \leq -C_7 p_3(t) f_3\left(\int_{t_3}^{h_1(t)} p_1(s) f_1\left(\int_{t_2}^{h_2(s)} p_2(x)\,dx\right) ds\right), \quad t \geq t_4, \tag{18}$$

where $C_7 = (-1)^{\nu_3+1} . C_6 > 0$.

Integrating (18) from t_4 to t and then using (5) we get a contradiction to (3).

IIb) Let $(-1)^{\nu_2+\nu_3} y_2(t) > 0$ for $t \geq t_1$. The system (S) implies

$y_1(t) > 0$ nonincreasing	$(-1)^{\nu_2+\nu_3} y_2(t) > 0$ nonincreasing	$(-1)^{\nu_3} y_3(t) < 0$ nondecreasing	for $t \geq t_1$.

In this case $\lim_{t\to\infty} y_i(t)$ $(i = 1, 2, 3)$ is finite. We shall prove that $\lim_{t\to\infty} y_i(t) = 0$ $(i = 1, 2, 3)$. Let $\lim_{t\to\infty} [(-1)^{\nu_3} y_3(t)] = -k_3$, $k_3 > 0$. Then

$$(-1)^{\nu_3} y_3(h_3(t)) \leq -k_3, \quad \text{for } t \geq t_2 = \gamma(t_1) \quad \text{holds} \tag{19}$$

and (19) in view of (c) implies

$$(-1)^{\nu_3} f_2(y_3(t))) \leq (-1)^{\nu_3} f_2(k_3(-1)^{\nu_3+1}), \quad t \geq t_2. \tag{20}$$

Multiplying the second equation of (S) by $(-1)^{\nu_2+\nu_3}$, integrating from t_2 to t and then using (20), we get

$$(-1)^{\nu_2+\nu_3}[y_2(t) - y_2(t_2)] \leq (-1)^{\nu_3} f_2(k_3(-1)^{\nu_3+1}) \cdot \int_{t_2}^{t} p_2(s)\,ds, \quad t \geq t_2. \tag{21}$$

Using (a) and $(-1)^{\nu_3} f_2(k_3(-1)^{\nu_3+1}) < 0$, from (21) we get

$$\lim_{t\to\infty} (-1)^{\nu_2+\nu_3} y_2(t) = -\infty,$$

which contradicts the assumption $(-1)^{\nu_2+\nu_3} y_2(t) > 0$ on $[t_1, \infty)$. Therefore $\lim_{t\to\infty} y_3(t) = 0$. Similarly, we shall prove that $\lim_{t\to\infty} y_2(t) = 0$.

Let $\lim_{t\to\infty} y_1(t) = k_1$. Then

$$y_1(h_1(t)) \geq k_1, \quad \text{for } t \geq t_2 \text{ holds.} \tag{22}$$

In view of (c), (22) implies

$$f_3(y_1(h_1(t))) \geq f_3(k_1), \quad t \geq t_2. \tag{23}$$

Multiplying the third equation of (S) by $(-1)^{\nu_3}$, integrating from t_2 to t and then using (23), we have

$$(-1)^{\nu_3+1} y_3(t_2) \geq f_3(k_1) \cdot \int_{t_2}^{\infty} p_3(s)\,\mathrm{d}s. \tag{24}$$

Because (24) is valid for arbitrary $t \geq t_2$, we get

$$(-1)^{\nu_3+1} y_3(h_3(t)) \geq f_3(k_1) \int_{h_3(t)}^{\infty} p_3(s)\,\mathrm{d}s, \quad t \geq t_3 = \gamma(t_2). \tag{25}$$

It is easy to prove that from (25), in view of (2) and (c), we obtain

$$(-1)^{\nu_3} f_2(y_3(h_3(t))) \leq -C_8 f_2\left(\int_{h_3(t)}^{\infty} p_3(s)\,\mathrm{d}s\right), \quad t \geq t_3, \tag{26}$$

where $0 < C_8 = \min\{-Kf_2(-f_3(k_1)), Kf_2(f_3(k_1))\}$.

Multiplying the second equation of (S) by $(-1)^{\nu_2+\nu_3}$, integrating from t_3 to t and then using (26) we obtain

$$(-1)^{\nu_2+\nu_3}[y_2(t) - y_2(t_3)] \leq -C_8 \int_{t_3}^{t} p_2(s) f_2\left(\int_{h_3(s)}^{\infty} p_3(x)\,\mathrm{d}x\right)\mathrm{d}s, \quad t \geq t_3.$$

By virtue of (4), the last inequality implies for $t \to \infty$ that $\lim_{t\to\infty} (-1)^{\nu_2+\nu_3} y_2(t) = -\infty$, which contradicts the assumption $(-1)^{\nu_2+\nu_3} y_2(t) > 0$ for $t \geq t_1$. Therefore $\lim_{t\to\infty} y_1(t) = 0$. The proof of Theorem 1 is complete. Theorem 1 generalizes Theorem 2.9 in the paper [4].

Theorem 2. Suppose that (1), (2), (4) hold and in addition

$$f_3(f_1(f_2(x))) = x \tag{27}$$

$$\int_0^\infty p_3(t)\left\{f_3\left[\int_0^{h_1(t)} p_1(s) f_1\left(\int_0^{h_2(s)} p_2(x)\,\mathrm{d}x\right)\mathrm{d}s\right]\right\}^{1-\varepsilon}\mathrm{d}t = \infty \ (0 < \varepsilon < 1). \tag{28}$$

If $\nu_1 + \nu_2 + \nu_3 \equiv 1 \pmod 2$, then the conclusion of Theorem 1 holds.

Proof. Let $y \in W$ be a weakly nonoscillatory solution of (S). Then with regard to Lemma 1, each of its components is a monotone function of a constant sign on $[t_1, \infty)$. Suppose that $y_1(t) > 0$ for $t \geq t_1$. As in the proof of Theorem 1, we get three cases: I) IIa) and IIb). In the cases I) and IIb) we proceed in the same way as in the proof of Theorem 1. Consider now the case IIa). In this case the inequality (17) holds. Using (27), (17) implies

$$f_3(y_1(h_1(t))) \geq C_6 y_3((t) f_3\left(\int_{t_3}^{h_1(t)} p_1(s) f_1\left(\int_{t_2}^{h_2(s)} p_2(x)\,\mathrm{d}x\right)\mathrm{d}s\right) > 0, \quad t \geq t_4. \tag{29}$$

Raising (29) to the $(1-\varepsilon)$ power, we obtain

$$[C_6 y_3(t)]^{1-\varepsilon} \cdot \left\{f_3\left(\int_{t_3}^{h_1(t)} p_1(s) f_1\left(\int_{t_2}^{h_2(s)} p_2(x)\,\mathrm{d}x\right)\mathrm{d}s\right)\right\}^{1-\varepsilon} \leq$$

$$\leq \{f_3(y_1(h_1(t)))\}^{1-\varepsilon}, \quad t \geq t_4. \tag{30}$$

Using (c) and the fact that the function $y_1(t)$ is positive and nondecreasing on $[t_1, \infty)$, we have

$$f_3(y_1(h_1(t))) \geq f_3(y_1(t_1)) = C_9 > 0, \quad t \geq t_2. \tag{31}$$

Now (31) implies

$$\{f_3(y_1(h_1(t)))\}^{1-\varepsilon} \leq K_1 f_3(y_1(h_1(t))), \quad t \geq t_2, \tag{32}$$

where $K_1 = C_9^{-\varepsilon} > 0$.

Combining (30) with (32), we get

$$[C_6 y_3(t)]^{1-\varepsilon}\left\{f_3\left(\int_{t_3}^{h_1(t)} p_1(s) f_1\left(\int_{t_2}^{h_2(s)} p_2(x)\,\mathrm{d}x\right)\mathrm{d}s\right)\right\}^{1-\varepsilon} \leq$$

$$\leq K_1 f_3(y_1(h_1(t))), \quad t \geq t_2. \tag{33}$$

Multiplying (33) by $p_3(t)[C_6 y_3(t)]^{\varepsilon-1}$, using the third equation of (S) and then integrating from t_4 to t, we get

$$\int_{t_4}^{t} p_3(z)\left\{f_3\left[\int_{t_3}^{h_1(z)} p_1(s) f_1\left(\int_{t_2}^{h_2(s)} p_2(x)\,\mathrm{d}x\right)\mathrm{d}s\right]\right\}^{1-\varepsilon}\mathrm{d}z \leq$$

$$\leq (-1)^{\nu_3} K_1 (\varepsilon C_6)^{-1}[(C_6 y_3(t))^\varepsilon - (C_6 y_3(t_4))^\varepsilon] < \infty, \quad t \geq t_4,$$

which contradicts (28). Therefore the case IIa) cannot occur. The proof of Theorem 2 is complete. Theorem 2 generalizes Theorem 2.10 in the paper [4].

Theorem 3. Suppose that (2), (4), (5) hold and in addition

$$h_2(t) \geq t, \quad h_3(t) \leq t \tag{34}$$

$$\int_0^\infty p_3(t) f_3\left[\int_0^{g(t)} p_1(s) f_1\left(\int_0^s p_2(x)\,\mathrm{d}x\right)\mathrm{d}s\right]\mathrm{d}t = \infty, \tag{35}$$

where $g(t) = \min_{t \geq 0}\{t, h_1(t)\}$.

If $\nu_1 + \nu_2 + \nu_3 \equiv 1 \pmod 2$, then the conclusion of Theorem 1 holds.

Proof. Let $y \in W$ be a weakly nonoscillatory solution of (S). Further proceeding in the same way as in the proof of Theorem 2, we consider only the case IIa). Ich view of the assumption (34), we modify the system (S) to the form

(S*)
$$-y_1'(t) \leq (-1)^{\nu_2+\nu_3} p_1(t) f_1(y_2(t))$$
$$(-1)^{\nu_2+\nu_3} y_2'(t) \leq (-1)^{\nu_3} p_2(t) f_2(y_3(t))$$
$$(-1)^{\nu_3} y_3'(t) \geq p_3(t) f_3(y_1(g(t))), \quad t \geq t_2 = \gamma(t_1).$$

Integrating the first inequality of (S*) from t_2 to t and then using $y_1(t_2) > 0$, we obtain

$$-y_1(t) \leq (-1)^{\nu_2+\nu_3}\int_{t_2}^t p_1(s) f_1(y_2(s))\,\mathrm{d}s, \quad t \geq t_2. \tag{36}$$

Integrating the second inequality of (S*) from t_2 to s and using $(-1)^{\nu_2+\nu_3+1} y_2(t_2) > 0$, we obtain

$$(-1)^{\nu_2+\nu_3} y_2(s) \leq (-1)^{\nu_3}\int_{t_2}^s p_2(x) f_2(y_3(x))\,\mathrm{d}x, \quad s \leq t_2. \tag{37}$$

Because the function $(-1)^{\nu_3} f_2(y_3(t))$ is nondecreasing on $[t_2, \infty)$, the inequality (37) implies

$$(-1)^{\nu_2+\nu_3} y_2(s) \leq (-1)^{\nu_3} f_2(y_3(s))\int_{t_2}^s p_2(x)\,\mathrm{d}x, \quad s \geq t_2. \tag{38}$$

In view of (c) and (2), (38) implies

$$(-1)^{\nu_2+\nu_3} f_1(y_2(s)) \leq (-1)^{\nu_3} C_5 f_1[f_2(y_3(s))] f_1\left(\int_{t_2}^s p_2(x)\,\mathrm{d}x\right), \quad s \geq t_2, \tag{39}$$

where $0 < C_5 = \min\{K, -K^2 f_1(-1)\}$.

Combining (36) with (39), we get

$$-y_1(t) \leq (-1)^{\nu_3} C_5 \int_{t_2}^{t} p_1(s) f_1(f_2(y_3(s))) f_1\left(\int_{t_2}^{s} p_2(x)\,\mathrm{d}x\right)\mathrm{d}s, \quad t \geq t_2. \tag{40}$$

Using $g(t) = \min_{t \leq 0} \{h_1(t), t\}$ and the fact that $(-1)^{\nu_3} f_1(f_2(y_3(t)))$ is nondecreasing on $[t_2, \infty)$, the inequality (40) implies

$$y_1(g(t)) \geq (-1)^{\nu_3+1} C_5 f_1(f_2(y_3(t))) \int_{t_2}^{g(t)} p_1(s) f_1\left(\int_{t_2}^{s} p_2(x)\,\mathrm{d}x\right)\mathrm{d}s, \tag{41}$$

for $t \geq t_3 = \gamma(t_2)$.

With regard to (2) and (c), from (41) we get

$$f_3(y_1(g(t))) \geq C_6 f_3(f_1(f_2(y_3(t)))) \times$$
$$\times f_3\left[\int_{t_2}^{g(t)} p_1(s) f_1\left(\int_{t_2}^{s} p_2(x)\,\mathrm{d}x\right)\mathrm{d}s\right], \quad t \geq t_3, \tag{42}$$

where $C_6 = K^2 . f_3(C_5(-1)^{\nu_3+1})$.

Multiplying (42) by $p_3(t)[C_6 f_3(f_1(f_2(y_3(t))))]^{-1}$, using the third inequality of (S*) and then integrating from t_3 to t we have

$$\int_{t_3}^{t} p_3(z) f_3\left[\int_{t_2}^{g(z)} p_1(s) f_1\left(\int_{t_2}^{s} p_2(x)\,\mathrm{d}x\right)\mathrm{d}s\right]\mathrm{d}z \leq$$
$$\leq \frac{(-1)^{\nu_3+1}}{C_6} \int_{y_3(t)}^{y_3(t_3)} \frac{\mathrm{d}u}{f_3(f_1(f_2(u)))} < \infty, \quad t \geq t_3,$$

which contradicts (35) and therefore the case IIa) cannot occur. The proof of Theorem 3 is complete. Theorem 3 generalizes Theorem 2.12 in the paper [4].

Theorem 4. Suppose that (2), (4), (27), (34) hold and in addition

$$\int_0^{\infty} p_3(t) \cdot \left\{f_3\left(\int_0^{g(t)} p_1(s) f_1\left(\int_0^{s} p_2(x)\,\mathrm{d}x\right)\mathrm{d}s\right)\right\}^{1-\varepsilon} \mathrm{d}t = \infty, \tag{43}$$

$0 < \varepsilon < 1$, where $g(t) = \min_{t \geq 0} \{t, h_1(t)\}$.

If $\nu_1 + \nu_2 + \nu_3 \equiv 1 \pmod 2$, then the conclusion of Theorem 1 holds.

Proof. Let $y \in W$ be a weakly nonoscillatory solution of (S). Further proceeding in the same way as in the proof of Theorem 2, we consider only the case IIa). Analogously as the inequalities (31) and (32) we prove the following ones

$$f_3(y_1(g(t))) \geq C_9, \quad t \geq t_2 = \gamma(t_1) \tag{44}$$

$$\{f_3(y_1(g(t)))\}^{1-\varepsilon} \leq K_1 f_3(y_1(g(t))), \quad t \geq t_3. \tag{45}$$

Proceeding the same way as in the proof of Theorem 3 we derive the system (S*) and (42). Combining (27) with (42), we get

$$f_3(y_1(g(t))) \geq C_6 y_3(t) f_3\left[\int_{t_2}^{g(t)} p_1(s) f_1\left(\int_{t_2}^{s} p_2(x)\,\mathrm{d}x\right)\mathrm{d}s\right] > 0, \quad t \geq t_3. \tag{46}$$

Raising the inequality (46) to the $1 - \varepsilon$ power and using (45), we obtain

$$[C_6 y_3(t)]^{1-\varepsilon}\left\{f_3\left[\int_{t_2}^{g(t)} p_1(s) f_1\left(\int_{t_2}^{s} p_2(x)\,\mathrm{d}x\right)\mathrm{d}s\right]\right\}^{1-\varepsilon} \leq$$
$$\leq K_1 f_3(y_1(g(t))), \quad t \geq t_3. \tag{47}$$

Multiplying (47) by $p_3(t)[C_6 y_3(t)]^{\varepsilon-1}$, using the third inequality of (S*) and then integrating from t_3 to t, we have

$$\int_{t_3}^{t} p_3(z)\left\{f_3\left[\int_{t_2}^{g(z)} p_1(s) f_1\left(\int_{t_2}^{s} p_2(x)\,\mathrm{d}x\right)\mathrm{d}s\right]\right\}^{1-\varepsilon}\mathrm{d}z \leq$$
$$\leq (-1)^{\nu_3} K_1 (\varepsilon C_6)^{-1}[(C_6 y_3(t))^{\varepsilon} - (C_6 y_3(t_3))^{\varepsilon}] < \infty, \quad t \geq t_3,$$

which contradicts (43). Therefore the case IIa) cannot occur. The proof of Theorem 4 is complete. Theorem 4 generalizes Theorem 2.13 in the paper [4].

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SÚHRN

OSCILATORICKÉ VLASTNOSTI RIEŠENÍ 3-DIMENZIONÁLNYCH NELINEÁRNYCH DIFERENCIÁLNYCH SYSTÉMOV S POSUNUTÝMI ARGUMENTAMI

Eva Špániková, Žilina

V práci sú uvedené postačujúce podmienky pre to, aby každé riešenie systému (S) bolo buď oscilatorické, alebo každá jeho komponenta monotónne konvergovala k nule pre $t \to \infty$.

РЕЗЮМЕ

ОСЦИЛЛЯТОРНЫЕ СВОЙСТВА РЕШЕНИЙ ТРЕХ-ДИМЕНЗИОНАЛЬНЫХ НЕЛИНЕЙНЫХ ДИФФЕРЕНЦИАЛЬНЫХ СИСТЕМ С ОТКЛОНЯЮЩИМСЯ АРГУМЕНТОМ

Ева Шпаникова, Жилина

В статье доказаны достаточные условия для то чтобы каждое решение системы (S) было осциллирующимся, или же каждая его компонента монотонно стремилась к нулю при $m \to \infty$.