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A GENERALIZATION OF THE KOLMOGOROV CONSISTENCY THEOREM FOR VECTOR MEASURES

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Introduction

In the present paper we generalize the well-known Kolmogorov consistency theorem. The proof of this theorem can be found in the case of weakly regular lattice group-valued measures in [3]. In this paper we shall prove the Kolmogorov theorem for vector lattice measures. The inner regularity in this paper is generalized of the inner regularity in [3]. We also deal with measures defined on rings.

Throughout the paper V will denote a boundedly σ-complete vector lattice, N the set of all positive integers. We shall say that V is weakly σ-distributive, if whenever $(a_{i,j})_{i,j\in N}$ is a bounded family of V such that $a_{i,j} \searrow 0$ $(j \to \infty;\ i = 1, 2, \ldots)$, then

$$\bigwedge_{\mathrm{f}\in N^N} \bigvee_{i} a_{i,\mathrm{f}(i)} = 0.$$

The following lemma is proved in [4].

Lemma 1. Let $(a_{n,i,j})_{n,i,j\in N}$ be a triple bounded sequence such that $a_{n,i,j} \searrow 0$ $(j \to \infty;\ n = 1, 2, \ldots;\ i = 1, 2, \ldots)$. Then for each $a \in V$, $a \geqq 0$ there is a double bounded sequence $(a_{i,j})_{i,j\in N}$ such that

$$a_{i,j} \searrow 0 \quad (j \to \infty;\ i = 1, 2, \ldots)$$

and

$$a \wedge \sum_{n=1}^{\infty} \bigvee_{i=1}^{\infty} a_{n,i,\mathrm{f}(n+i)} \leqq \bigvee_{i=1}^{\infty} a_{i,\mathrm{f}(i)}$$

for each $\mathrm{f} \in N^N$.

Let $(X, \mathscr{S})$ be a measurable space, i. e. X be an arbitrary non-empty set and $\mathscr{S}$ be a ring of subsets of X. A system $\mathscr{C}$ of subsets of X is called a compact system, if to any sequence $(C_n)_n$ of sets from $\mathscr{C}$ such that

$$\bigcap_{n=1}^{\infty} C_n = \emptyset$$

there exists $k \in N$ such that

$$\bigcap_{n=1}^{k} C_n = \emptyset.$$

Definition 1. An additive non-negative set mapping $m\colon \mathscr{S} \to V$ is called compact if there is a compact system $\mathscr{C}$ and if for each $E \in \mathscr{S}$ there is a bounded sequence $(a_{i,j})$ of points from V such that $a_{i,j} \searrow 0$ $(j \to \infty;\ i = 1, 2, \ldots)$ and to any $\mathrm{f} \in N^N$ there are $K \in \mathscr{C}$, $F \in \mathscr{S}$ such that it holds:

$$F \subset K \subset E,\ m(E - F) \leqq \bigvee_{i=1}^{\infty} a_{i,\mathrm{f}(i)}.$$

Lemma 2. Let $m\colon \mathscr{S} \to V$ be a compact additive non-negative set mapping and V be weakly σ-distributive. Then m is σ-additive.

Proof. By [4] it is sufficient to prove that m is continuous from above in $\emptyset$. Let $E_n \searrow \emptyset$, $E_n \in \mathscr{S}$ and $\mathscr{C}$ be a compact system. Then there exists a bounded triple sequence $(a_{n,i,j})$, $a_{n,i,j} \searrow 0$ $(j \to \infty;\ n, i = 1, 2, \ldots)$ such that for each $\mathrm{f} \in N^N$ there exist $K_n \in \mathscr{C}$, $F_n \in \mathscr{S}$ that

$$F_n \subset K_n \subset E_n,\ m(E_n - F_n) \leqq \bigvee_{i=1}^{\infty} a_{n,i,\mathrm{f}(n+i)} \leqq m(E_1).$$

Since

$$m\left(\bigcap_{i=1}^{n} E_i - \bigcap_{i=1}^{n} F_i\right) \leqq \sum_{i=1}^{n} m(E_i - F_i),$$

then

$$m\left(\bigcap_{i=1}^{n} E_i - \bigcap_{i=1}^{n} F_i\right) \leqq m(E_1) \wedge \sum_{n=1}^{\infty} \bigvee_{i=1}^{\infty} a_{n,i,\mathrm{f}(n+i)} \leqq \bigvee_{i=1}^{\infty} a_{i,\mathrm{f}(i)},$$

where $(a_{i,j})$ is a sequence from Lemma 1 depending on $m(E_1)$.
Because $E_n \searrow \emptyset$, then

$$\bigcap_{n=1}^{\infty} K_n = \emptyset$$

and therefore there is $k \in N$ such that

$$\bigcap_{n=1}^{k} K_n = \emptyset.$$

And so

$$\bigcap_{n=1}^{r} F_n = \emptyset$$

for $r \geqq k$. Therefore

$$m\left(\bigcap_{n=1}^{r} E_n\right) = m(E_r) \leqq \bigvee_{i=1}^{\infty} a_{i,\mathrm{f}(i)}$$

for each $\mathrm{f} \in N^N$. With respect to the weak σ-distributivity we obtain

$$\bigwedge_{n=1}^{\infty} m(E_n) = 0. \qquad \text{Q.E.D.}$$

The Kolmogorov theorem

Let I be an index set, T be a set of all finite subsets of I and $(X_i)_{i \in I}$ be a system of topological spaces. Let us define

$$X^t = \mathop{\times}_{i \in t} X_i, \text{ for each } t \in T;$$

$$X = \mathop{\times}_{i \in I} X_i.$$

We denote the projection of X^t into X^s by p_{ts} for any $s, t \in T$, $s \subset t$ and the projection of X into X^t by p_t for any $t \in T$. Further, let $\mathscr{C}_t$ be a system of all compact sets of a topological space X^t and $(\mathscr{S}_t)_{t \in T}$ the system of rings with the following properties:

(i) $\mathscr{S}_t$ is a ring of subsets of X^t;

(ii) $\mathscr{C}_t \subset \mathscr{S}_t$;

(iii) for any $(s, t) \in T \times T$ there is $r \in T$ such that $s \cup t \subset r$ and

$$p_t^{-1}(\mathscr{S}_t) \cup p_s^{-1}(\mathscr{S}_s) \subset p_r^{-1}(\mathscr{S}_r),$$

where

$$p_k^{-1}(\mathscr{S}_k) = \{p_k^{-1}(E);\ E \in \mathscr{S}_k\}$$

for each $k \in T$. Put

$$\mathscr{B}_t = p_t^{-1}(\mathscr{S}_t), \text{ for each } t \in T,$$

$$\mathscr{V} = \bigcup_{t \in T} \mathscr{B}_t.$$

Then the proof of the following lemma is easy.

Lemma 3. $\mathscr{B}_t$ is a ring for each $t \in T$ and $\mathscr{V}$ is a ring, too.

Definition 2. Let $m_t \colon \mathscr{S}_t \to V$ be a measure for each $t \in T$. A system of measures $(m_t)_{t \in T}$ will be called consistent if for every $s, t \in T$, $s \subset t$ and $E \in \mathscr{S}_s$ it holds

$$m_t(p_{ts}^{-1}(E)) = m_s(E).$$

Lemma 4. Let $A \in \mathscr{B}_t \cap \mathscr{B}_s$ and $(m_t)_{t \in T}$ be consistent system of measures, then

$$m_t(p_t(A)) = m_s(p_s(A)).$$

Proof. Let $A \in \mathscr{B}_t$, i.e. $A = p_t^{-1}(E)$ for some $E \in \mathscr{S}_t$. Put $r = s \cup t$. Since

$$p_t(A) = E$$

$$p_r(p_t^{-1}(E)) = p_{rt}^{-1}(E)$$

and the system of measures is consistent, we have

$$m_t(p_t(A)) = m_r(p_{rt}^{-1}(E)) = m_r(p_r(A)).$$

We prove

$$m_s(p_s(A)) = m_r(p_r(A))$$

analogously. Q.E.D.

Definition 3. Let $A \in \mathscr{V}$, i.e. $A \in \mathscr{B}_t$ for some $t \in T$. Then we define a mapping $m: \mathscr{V} \to V$ as

$$m(A) = m_t(p_t(A)).$$

Evidently, m is a nonnegative and additive mapping. Put $\mathscr{C} = \{p_t^{-1}(C): C \in \mathscr{C} \in_t, t \in T\}$. Then $\mathscr{C}$ is a compact system. (see [3], p. 348).

Theorem (Kolmogorov). Let V be a boundedly σ-complete weakly σ-distributive vector lattice, $(m_t)_{t \in T}$ consistent system of measures and m_t be compact with respect to $\mathscr{C}_t$ for each $t \in T$. Then there exists a measure μ: $s(\mathscr{V}) \to V$, where $s(\mathscr{V})$ is a σ-ring generated by $\mathscr{V}$, such that

$$m_t(E) = \mu(p_t^{-1}(E))$$

for any $t \in T$, $E \in \mathscr{S}_t$.

Proof. We have the mapping m on $\mathscr{V}$, which is non-negative and additive. From the additivity we have $m(\emptyset) = 0$, too. Let now $A \in \mathscr{V}$, $A = p_t^{-1}(E)$. Since m_t is compact, there exists a bounded double sequence $a_{i,j} \searrow 0$ $(j \to \infty$, $i = 1, 2, \dots)$ such that for each $f \in N^N$ there is $C \in \mathscr{C}_t$ such that

$$C \subset E,\ m_t(E - C) \leqq \bigvee_{i=1}^{\infty} a_{i,f(i)}.$$

Put $D = p_t^{-1}(C)$. Then $D \subset A$ and

$$m(A - D) = m_t(E - C) \leqq \bigvee_{i=1}^{\infty} a_{i,f(i)}$$

and therefore m is compact. By Lemma 2 m is a measure on $\mathscr{V}$ and by [7] there exists a measure μ defined on $s(\mathscr{V})$ such that is an extension of m. Q.E.D.

REFERENCES

1. Halmos, P. R.: Measure Theory, Springer Berlin 1974.
2. Kelley, J. L.: General topology, New Jersy, 1957.
3. Riečan, B.: On the lattice group valued measures, Čas. pěst. mat. 101, 1976, 343—349.
4. Riečan, B.: Notes on lattice-valued measures, AMUC 42—43, 1983, 181—192.
5. Riečan, B.—Volauf, P.: On technical lemma in lattice ordered groups, AMUC 44—45, 1984, 31—36.
6. Riečan, J.: On projective limits of small systems, AMUC 44—45, 1984, 203—213.
7. Wright, J. D. M.: Extensions of infinite vector lattice measures, The Quarter J. of Math., 23, 91, 1972, 259—265.
8. Yeh, J.: Stochastic processes and the Wiener integral, New York 1973.

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SÚHRN

ZOVŠEOBECNENIE KOLMOGOROVOVEJ KONZISTENTNEJ VETY PRE VEKTOROVÉ MIERY

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V práci je zovšeobecnená Kolmogorovova konzistentná veta. Miery, s ktorými sa v práci pracuje, nadobúdajú hodnoty v slabo σ-distributívnom podmienene σ-úplnom vektorovom zväze.

РЕЗЮМЕ

ОБОБЩЕНИЕ КОНСИСТЕНТНОЙ ТЕОРЕМЫ КОЛМОГОРОВА ДЛЯ МЕР СО ЗНАЧЕНИЯМИ ВО ВЕКТОРНОЙ СТРУКТУРЕ

Эрвин Хraховина, Братислава

В статье доказывается консистентная теорема Колмогорова. Меры, с которыми здесь работается, принимают значения в условно σ-полной слабо σ-дистрибутивной векторной структуре.