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THE CONSTRUCTION OF A VECTOR MEASURE FROM A CONTENT

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Introduction

The construction of a measure from a content is well-known in the case of a real-valued measure. In [2] we can find the case when X is a locally compact Hausdorff space. The papers by Neubrunn [4] and Kalas [3] are generalizations of [2]. If X is a compact Hausdorff topological space and the content is vector valued, then a construction of a measure from this content can be found in [6]. In that paper the extended vector measure is finite and it is supposed that the valued vector lattice is weakly (σ, ∞)-distributive. The assertions are proved by the Stone representation theorem. In this paper we suppose that X is an arbitrary set and the vector lattice is weakly σ-distributive. We use Theorem 1, which was proved in [1] and [5].

Some notions and notes.

Troughout the paper W will denote a boundedly complete vector lattice and X and arbitrary non-empty set. Supremum of any unbounded set of positive elements in W is taken to be infinity and the infinity is denoted by "∞". Let $\mathscr{U}$, $\mathscr{B}$ be systems of subsets of X with the following properties:

(A) $\emptyset \in \mathscr{U} \cap \mathscr{B}$;

(B) If $A_1, A_2 \in \mathscr{U}$, then $A_1 \cup A_2 \in \mathscr{U}$;

(C) if $(B_n)_n$ is a sequence from $\mathscr{B}$, then $\bigcup_{n=1}^{\infty} B_n$ belongs to $\mathscr{B}$;

(D) if $A \in \mathscr{U}$, $B_1, B_2 \in \mathscr{B}$ and $A \subset B_1 \cup B_2$, then there exist $A_1, A_2 \in \mathscr{U}$ such that $A_1 \subset B_1$, $A_2 \subset B_2$ and $A = A_1 \cup A_2$;

(E) if $A \in \mathscr{U}$, $A \subset \bigcup_{n=1}^{\infty} B_n$, $B_n \in \mathscr{B}$, then there exists $k \in N$ such that $A \subset \bigcup_{n=1}^{k} B_n$;

(F) if $B \in \mathscr{B}$, $C \in \mathscr{U}$, then $B - C \in \mathscr{B}$, $C - B \in \mathscr{U}$;

(G) if $A \in \mathscr{U}$, then there exist $B \in \mathscr{B}$, $C \in \mathscr{U}$ such that $A \subset B \subset C$.

Definition 1. A triple sequence $(a_{n,i,j})$ has the P-property iff:

(i) $a_{n,i,j} \geqq 0$ for each $n, i, j \in N$;

(ii) $(a_{n,i,j})$ is bounded from upper;

(iii) $a_{n,i,j} \searrow 0$ for $j \to \infty$, $n, i = 1, 2, 3, \ldots$ $\left(\text{i.e. } a_{n,i,j} \geqq a_{n,i,j+1} \text{ and } \bigwedge_{j=1}^{\infty} a_{n,i,j} = 0\right)$.

For each f: $N \to N$ we denote $\bigvee_{i=1}^{\infty} a_{n,i,f(n+i)}$ by $a_{f,n}$. Analogously we define the P-property for double sequences $(a_{i,j})$ and we denote $\bigvee_{i=1}^{\infty} a_{i,f(i)}$ by a_f.

Definition 2. A boundedly σ-complete vector lattice W is weakly σ-distributive if for an arbitrary sequence $(a_{i,j})$ with the P-property the equality

$$\bigwedge_{f \in N^N} \bigvee_{i=1}^{\infty} a_{i,f(i)} = 0 \tag{1}$$

is satisfied.

Theorem 1. Let $(a_{n,i,j})$ be a sequence in W with the P-property. Then for each $a \in W$, $a \geqq 0$, there is a sequence $(a_{i,j})$ with the P-property such that

$$a \bigwedge \sum_{n=1}^{\infty} a_{f,n} \leqq a_f.$$

The proof of Theorem 1 can be found in [1] and [5].

A content and a measure

Let c: $\mathscr{U} \to W$ be a content on $\mathscr{U}$, i.e. it holds

(I) $c(A) \geqq 0$ for each $A \in \mathscr{U}$;

(II) $c(A) \leqq c(B)$ if $A, B \in \mathscr{U}$ and $A \subset B$;

(III) if $A, B \in \mathscr{U}$, then $c(A \cup B) \leqq c(A) + c(B)$;

(IV) if $A, B \in \mathscr{U}$, $A \cap B = \emptyset$, then $c(A \cup B) = c(A) + c(B)$;

(V) for $A \in \mathscr{U}$ there exists $(a_{i,j})$ with the P-property such that for each $\mathfrak{f} \in N^N$ there exist $B_{\mathfrak{f}} \in \mathscr{B}$, $C_{\mathfrak{f}} \in \mathscr{U}$ such that

$$A \subset B_{\mathfrak{f}} \subset C_{\mathfrak{f}},\ c(A) \geqq c(C_{\mathfrak{f}}) - a_{\mathfrak{f}}.$$

Evidently, $c(\emptyset) = 0$.

Lemma 1. If $A \in \mathscr{U}$ and W is weakly σ-distributive, then

$$c(A) = \bigwedge \{c(C) \colon C \in \mathscr{U}(A)\},$$

where $\mathscr{U}(A) = \{C \in \mathscr{U}$: there exists $B \in \mathscr{B}$ such that $A \subset B \subset C\}$.

Proof. From (II) we have that $c(A)$ is a lower bound of the set $\{c(C) \colon C \in \mathscr{U}(A)\}$. Let w be another lower bound of the set $\{c(C) \colon C \in \mathscr{U}(A)\}$.

From (V) we have $C_f \in \mathscr{U}(A)$ and

$$c(A) \geqq c(C_f) - a_f.$$

We consider

$$w - c(A) \leqq c(C_f) - c(A) \leqq a_f.$$

With respect to the weak σ-distributivity we obtain

$$w \leqq c(A).$$

Q.E.D.

We define the mapping $\hat{c}$: $\mathscr{U} \cup \mathscr{B} \to W \cup \{\infty\}$ in the following way: $\hat{c}(B) = \bigvee \{c(C)\colon C \subset B,\ C \in \mathscr{U}\}$.
Evidently, $\hat{c}$ is non-negative.

Lemma 2. $\hat{c}$ has the following properties:
(i) if $A \in \mathscr{U}$, then $\hat{c}(A) = c(A)$;
(ii) if $A \subset B$, A, $B \in \mathscr{U} \cup \mathscr{B}$, then $\hat{c}(A) \leqq \hat{c}(B)$;
(iii) $\hat{c}$ is σ-subadditive on $\mathscr{B}$.
(iv) $\hat{c}$ is σ-additive on $\mathscr{B}$.

Proof. (i) and (ii) are evident.
(iii) Let B_1, $B_2 \in \mathscr{B}$, $C \in \mathscr{U}$, $C \subset B_1 \cup B_2$. Then there are C_1, $C_2 \in \mathscr{U}$ such that $C_1 \subset B_1$, $C_2 \subset B_2$ and $C = C_1 \cup C_2$. Thus

$$c(C) \leqq \hat{c}(B_1) + \hat{c}(B_2) \quad .$$

and therefore $\hat{c}$ is subadditive on $\mathscr{B}$. Let now $B_n \in \mathscr{B}$, $C \in \mathscr{U}$ and $C \subset \bigcup_{n=1}^{\infty} B_n$. Then there exists $k \in N$ such that $C \subset \bigcup_{n=1}^{k} B_n$. We obtain

$$c(C) \leqq \hat{c}\left(\bigcup_{n=1}^{k} B_n\right) \leqq \sum_{n=1}^{k} \hat{c}(B_n) \leqq \sum_{n=1}^{\infty} \hat{c}(B_n)$$

and thus $\hat{c}$ is σ-subadditive.
(iv) Let B_1, $B_2 \in \mathscr{B}$, $B_1 \cap B_2 = \emptyset$, C_1, $C_2 \in \mathscr{U}$ and $C_1 \subset B_1$, $C_2 \subset B_2$. Then $C_1 \cap \cap C_2 = \emptyset$,

$$c(C_1) + c(C_2) \leqq \hat{c}(B_1 \cup B_2)$$

and with respect to (iii) we obtain that $\hat{c}$ is additive. Let $B_n \in \mathscr{B}$ $(n = 1, 2, \ldots)$ and $B_i \cap B_j = \emptyset$ $(i \neq j)$. Then for each $n \in N$

$$\sum_{i=1}^{n} \hat{c}(B_i) = \hat{c}\left(\bigcup_{i=1}^{n} B_i\right) \leqq \hat{c}\left(\bigcup_{n=1}^{\infty} B_n\right)$$

and therefore $\hat{c}$ is σ-additive.

Q.E.D.

Lemma 3. If $B \in \mathscr{B}$, $C \in \mathscr{U}$, $B \supset C$ and W is weakly σ-distributive, then

$$\hat{c}(B) = \hat{c}(B - C) + c(C).$$

Proof. Let $C_1 \subset B - C$, $C_1 \in \mathscr{U}$, then $C \cup C_1 \subset B$ and $C_1 \cap C = \emptyset$. Therefore

$$c(C) + c(C_1) \leqq \hat{c}(B)$$

and hence

$$c(C) + \hat{c}(B - C) \leqq \hat{c}(B).$$

With respect to (V) we have $C \subset B_f \subset A_f$ and

$$c(C) \geqq c(A_f) - a_f.$$

Also

$$\hat{c}(B_f) \leqq c(A_f) \leqq c(C) + a_f.$$

Then

$$\hat{c}(B) \leqq \hat{c}(B - C) + \hat{c}(B_f) \leqq \hat{c}(B - C) + c(C) + a_f.$$

Since W is weakly σ-distributive we obtain

$$\hat{c}(B) = \hat{c}(B - C) + c(C).$$

Q.E.D.

Definition 3. A set L belongs to $\mathscr{P}$ if there is a sequence $(a_{i,j})$ with the P-property and for each $f \in N^N$ there is $B_f \in \mathscr{B}$, $C_f \in \mathscr{U}$ such that

$$C_f \subset L \subset B_f,\ \hat{c}(B_f - C_f) \leqq a_f.$$

Further, let W be weakly σ-distributive. Evidently, $\hat{c}(B_f) \in W$ for the set B_f in Definition 3, since

$$0 \leqq \hat{c}(B_f) \leqq \hat{c}(B_f - C_f) + c(C_f) \leqq a_f + c(C_f) \in W.$$

Let $L \in \mathscr{P}$ and put

$$w = \bigwedge \{\hat{c}(B) : B \supset L,\ B \in \mathscr{B}\}.$$

Then $w \in W$.

The proofs of the following theorems are easy.

Theorem 2. $\mathscr{U} \subset \mathscr{P}$ if W is weakly σ-distributive.

Theorem 3. $\mathscr{P}$ is a ring.

Lemma 4. If $L \in \mathscr{P}$, then $w = \bigvee \{c(C) : C \subset L,\ C \in \mathscr{U}\}$.

Proof. Evidently, w is an upper bound of the set

$$\{c(C) : C \subset L,\ C \in \mathscr{U}\}.$$

Let z be another upper bound. Then $z \geqq c(C_{\mathfrak{f}})$. We consider

$$w - z \leqq w - c(C_{\mathfrak{f}}) \leqq \hat{c}(B_{\mathfrak{f}} - C_{\mathfrak{f}}) \leqq a_{\mathfrak{f}}.$$

From the weak σ-distributivity we obtain $w \leqq z$.

Q.E.D.

Definition 4. We define on $\mathscr{P}$ a mapping m in the following way

$$m(A) = \bigvee \{c(C): C \subset A, C \in \mathscr{U}\}$$

for $A \in \mathscr{P}$.

Furthermore, the following lemmas can be easily proved.

Lemma 5. m is finite.

Lemma 6. $L \in \mathscr{P}$ if and only if there is $(a_{i,j})$ with the P-property and for each $\mathfrak{f} \in N^N$ there are $B_{\mathfrak{f}} \in \mathscr{B}$, $C_{\mathfrak{f}} \in \mathscr{U}$ such that $\hat{c}(B_{\mathfrak{f}}) < \infty$, $C_{\mathfrak{f}} \subset L \subset B_{\mathfrak{f}}$ and $\hat{c}(B_{\mathfrak{f}}) - a_{\mathfrak{f}} \leqq m(L) \leqq c(C_{\mathfrak{f}}) + a_{\mathfrak{f}}$.

Theorem 4. The mapping m is non-negative, monotone and σ-subadditive.

Proof. It can be easily proved that m is non-negative and monotone. We prove the σ-subadditivity. Let $A_n \in \mathscr{P}$, $A = \bigcup_{n=1}^{\infty} A_n \in \mathscr{P}$. If

$$\sum_{n=1}^{\infty} m(A_n) = \infty,$$

then the corresponding inequality holds. Let

$$\sum_{n=1}^{\infty} m(A_n) = z \in W.$$

By Lemmas 4, 5 and 6 there exists $(a_{n,i,j})$ with the P-property, $B \in \mathscr{B}$ with $\hat{c}(B) < \infty$ and $A \subset B$ and for each $\mathfrak{f} \in N^N$ there are $B_{n,\mathfrak{f}} \in \mathscr{B}$ such that $\hat{c}(B_{n,\mathfrak{f}}) < \infty$, $A_n \subset B_{n,\mathfrak{f}} \subset B$ and

$$m(A_n) \geqq \hat{c}(B_{n,\mathfrak{f}}) - a_{\mathfrak{f},n}.$$

Put $a = z \vee \hat{c}(B)$. Evidently, $m(A) \leqq a$ and

$$m(A) \leqq \hat{c}\left(\bigcup_{n=1}^{\infty} B_{n,\mathfrak{f}}\right) \leqq \bigvee_{n=1}^{\infty} \sum_{i=1}^{n} \hat{c}(B_{i,\mathfrak{f}}).$$

Then

$$m(A) \leqq \bigvee_{n=1}^{\infty} \left[a \wedge \left(\sum_{i=1}^{n} m(A_i) + \sum_{i=1}^{n} a_{\mathfrak{f},i}\right)\right] \leqq$$

$$\leqq \bigvee_{n=1}^{\infty} \sum_{i=1}^{n} m(A_i) + a \wedge \sum_{n=1}^{\infty} a_{\mathfrak{f},n} \leqq \sum_{n=1}^{\infty} m(A_n) + a_{\mathfrak{f}}.$$

Since

$$m(A) \leqq \sum_{n=1}^{\infty} m(A_n) + a_{\mathrm{f}}$$

holds for each $\mathrm{f} \in N^N$, thus by the weak σ-distributivity we obtain that m is σ-subadditive.

Q.E.D.

Theorem 5. m is additive.

Proof. Let $A, B \in \mathscr{P}$ and $A \cap B = \emptyset$. Then there is $a_{i,j}$ with the P-property such that for each $\mathrm{f} \in N^N$ there are $C, D \in \mathscr{U}$ such that $C \subset A$, $D \subset B$ and

$$c(C) + a_{\mathrm{f}} \geqq m(A),\ c(D) + b_{\mathrm{f}} \geqq m(B).$$

Evidently, $C \cap D = \emptyset$. Then

$$c(C \cup D) + a_{\mathrm{f}} \geqq m(A) + m(B).$$

Hence

$$m(A \cup B) + a_{\mathrm{f}} \geqq m(A) + m(B)$$

for each $\mathrm{f} \in N^N$. Therefore by the weak σ-distributivity and by Theorem 4 m is additive.

Q.E.D.

Theorem 6. m is σ-additive.

Proof. Let $A_n \in \mathscr{P}$, $A = \bigcup_{n=1}^{\infty} A_n \in \mathscr{P}$ and $A_i \cap A_j = \emptyset$ for $i \neq j$. Then for each n

$$\sum_{i=1}^{n} m(A_i) = m\left(\bigcup_{i=1}^{n} A_i\right) \leqq m(A)$$

holds. Therefore,

$$\sum_{n=1}^{\infty} m(A_n) = \bigvee_{n=1}^{\infty} \sum_{i=1}^{n} m(A_i) \leqq m(A).$$

Hence by Theorem 4 m is σ-additive.

Q.E.D.

Theorem 7. $\hat{c}(B) = m(B)$ for each $B \in (\mathscr{B} \cup \mathscr{U}) \cap \mathscr{P}$.

The proof of this theorem is easy.

We have constructed the measure m on the ring $\mathscr{P}$ and this measure is an extension of the content c. By [8] there is a measure μ defined on the σ-ring generated by $\mathscr{P}$, which is an extension of m. Thus the following theorem holds.

Theorem 8. Let X be an arbitrary set, W be a boundedly complete weakly σ-distributive vector lattice, $\mathscr{U}$, $\mathscr{B}$ be systems of subsets of X with the properties (A)—(G) and c be a content defined on $\mathscr{U}$ with the properties (I)—(V). Then there exists a measure μ defined on a σ-ring generated by $\mathscr{U}$ such that μ is an extension of c.

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SÚHRN

KONŠTRUKCIA VEKTOROVEJ MIERY Z VEKTOROVÉHO OBJEMU

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V práci konštruujeme mieru z objemu. Miera i objem nadobúdajú hodnoty v podmienene úplnom slabo σ-distributívnom zväze. Objem, s ktorým sa pracuje je naviac regulárny.

РЕЗЮМЕ

ПОСТРОЕНИЕ ВЕКТОРНОЙ МЕРЫ ИЗ ВЕКТОРНОГО ОБЪЕМА

Эрвин Храховина, Братислава

В статье мы занимаемся продолжением регулярного объема в меру. Эти изображения принимают значения в условно полной векторной структуре.