UNIVERSITAS COMENIANA
ACTA MATHEMATICA UNIVERSITATIS COMENIANAE
LIV—LV—1988

GRAPH CONVERGENCE, UNIFORM, QUASI-UNIFORM AND CONTINUOUS CONVERGENCE AND SOME CHARACTERIZATIONS OF COMPACTNESS

ĽUBICA HOLÁ — TIBOR ŠALÁT, Bratislava

Introduction

In this paper we deal with the notion of graph convergence of functions, which was introduced in the paper [9]. Graph convergence of functions was studied also by Beer in [2], [3], [4].

The paper has three parts. In the first part we complete and generalize some results of [9]. In the second part we give a characterization of compact metric spaces by using the notion of graph convergence of functions and by using Dini's theorem about convergence of sequences of functions. In the third part we give a characterization of compact metric spaces by using the notion of quasi-uniform and continuous convergence.

1 Graph convergence

Let X be a topological space and $\{S_n\}$ be a sequence of subsets of X. In accordance with [9] denote by $\operatorname{Lim\,inf} S_n$ the set of all points $x \in X$ each neighbourhood of which meets all but finitely many sets S_n, and $\operatorname{Lim\,sup} S_n$ the set of all points $x \in X$ each neighbourhood of which meets infinitely many sets S_n.

It is obvious that

$$\operatorname{Lim\,inf} S_n \subset \operatorname{Lim\,sup} S_n. \tag{1}$$

If these two sets coincide we say that the sequence $\{S_n\}$ converges and the set $\operatorname{Lim} S_n = \operatorname{Lim\,inf} S_n = \operatorname{Lim\,sup} S_n$ is said to be the limit of the sequence $\{S_n\}$.

Let X, Y be topological spaces and $f\colon X \to Y$. The graph of f is the set $G(f) = \{(x, y) \in X \times Y\colon y = f(x)\}$.

Consider $X \times Y$ with the product topology. Let $f, f_n \colon X \to Y$ $(n = 1, 2, \ldots)$. We say that the sequence $\{f_n\}$ graph converges to f if there exist $\operatorname{Lim} G(f_n)$ and $\operatorname{Lim} G(f_n) = G(f)$.

We shall write $f_n \xrightarrow[X]{} f$ if the sequence $\{f_n\}$ pointwise converges to f and $f_n \underset{X}{\not\longrightarrow} f$ if the sequence $\{f_n\}$ does not pointwise converge to f. If Y is a metric space, we shall similarly write $f_n \underset{X}{\rightrightarrows} f$ if $\{f_n\}$ uniformly converges to f, and $f_n \underset{X}{\not\rightrightarrows} f$ if $\{f_n\}$ fails to uniformly converge to f.

We shall often use the following theorem proved in the paper [9]. This theorem is a basic result about graph convergence of functions.

Theorem A. Let X and Y be compact metric spaces and $f_n \colon X \to Y$ $(n = 1, 2, \ldots)$. Then $\operatorname{Lim} G(f_n)$ exists and is the graph of a function f if and only if $\{f_n\}$ converges uniformly to f and f is continuous.

The following theorem is a theorem of "Dini's type".

Theorem 1.1. Let X be a topological space and $f_n \colon X \to R$ $(n = 1, 2, \ldots)$. Suppose that the sequence $\{f_n\}$ pointwise converges to a continuous function $f \colon X \to R$ and for any $x \in X$ the sequence $\{f_n(x)\}$ monotonically converges to $f(x)$. Assume that for any $x \in X$ infinitely many functions f_n are upper semicontinuous at x and infinitely many functions are lower semicontinuous at x. Then the sequence $\{f_n\}$ graph converges to f.

Proof. The assumption $f_n \xrightarrow[X]{} f$ implies that $G(f) \subset \operatorname{Lim\,inf} G(f_n)$. It is sufficient to prove (see (1)) that

$$\operatorname{Lim\,sup} G(f_n) \subset G(f). \tag{2}$$

Choose $(x, y) \in \operatorname{Lim\,sup} G(f_n)$. We prove that $(x, y) \in G(f)$.

Suppose $(x, y) \notin G(f)$. Then $f(x) \neq y$. Thus one of the two following possibilities holds:

a) $y > f(x)$ b) $y < f(x)$.

In case of a) choose $\varepsilon > 0$ such that

$$y - \varepsilon > f(x) + \varepsilon. \tag{3}$$

Since $f_n \xrightarrow[X]{} f$, there exists $m \in N$ such that f_m is upper semicontinuous at x and

$$f_m(x) < f(x) + \varepsilon/2. \tag{4}$$

The continuity of the function f at x and the upper semicontinuity of the function f_m at x imply that there exists a neighbourhood $V = V(x)$ of the point x such that

$$\text{for any } z \in V(x) \text{ we have } f(z) \in (f(x) - \varepsilon, f(x) + \varepsilon) \tag{5}$$

and

$$\text{for any } z \in V(x) \text{ we have } f_m(z) < f_m(x) + \varepsilon/2. \tag{5'}$$

From (3), (4) and (5′) we get $f_m(z) < f(x) + \varepsilon < y - \varepsilon$ for any $z \in V(x)$. Thus

$$\text{for any } z \in V(x) \text{ we have } f_m(z) < y - \varepsilon. \tag{6}$$

Since $(x, y) \in \operatorname{Lim\,sup} G(f_n)$, there exists $p \geq m$ such that $G(f_p) \cap V(x) \times \times (y - \varepsilon, \infty) \neq \emptyset$. Choose $(s, f_p(s)) \in V(x) \times (y - \varepsilon, \infty)$. From (3) and (5) we have $f_p(s) > y - \varepsilon > f(x) + \varepsilon > f(s)$ and so

$$f_p(s) > f(s). \tag{7}$$

Since the sequence $\{f_n(x)\}$ monotonically converges to $f(x)$, the inequality (7) implies that $\{f_n(s)\}$ is a non-increasing sequence. Thus from (6) we have $f_n(s) \leq f_m(s) < y - \varepsilon$ for any $n \geq m$. In particular, $f_p(s) < y - \varepsilon$ and that contradicts $(s, f_p(s)) \in V(x) \times (y - \varepsilon, \infty)$.

b) The proof follows from a) if we consider functions $-f_n, f$ instead of f_n, f.

The proof of Theorem is finished.

By using Theorem A and Theorem 1.1 we obtain the following result.

Theorem 1.2. Let X be a compact metric space and $f_n\colon X \to R$ $(n = 1, 2, \ldots)$, $f\colon X \to R$. Suppose that the function f and the sequence $\{f_n\}$ satisfy the assumptions of Theorem 1.1. Then $f_n \underset{X}{\rightrightarrows} f$.

Proof. We show that all but finitely many functions f_n are bounded. Suppose that infinitely many functions are unbounded.

There exists $L > 0$ such that $|f(x)| < L$ for every $x \in X$. There exists an incresing sequence $\{n_k\}$ of positive numbers and a sequence $\{x_k\}$ of points of X such that $|f_{n_k}(x_k)| > L$ for every k. We have two possibilities:

a) for infinitely many k $\quad f_{n_k}(x_k) > L$

b) for infinitely many k $\quad f_{n_k}(x_k) < -L$

a) Without loss of generality we can suppose that $f_{n_k}(x_k) > L$ for any k and that the sequence $\{x_k\}$ converges to a point $x \in X$.

Since the sequence $\{f_n(z)\}$ monotonically converges to $f(z)$ for every $z \in X$, $\{f_n(x_k)\}$ is a non-increasing sequence for every k.

There exists $j \in N$ such that f_j is upper semicontinuous at x and $f_j(x) < L$. The upper semicontinuity of f_j at x implies, that there exists a neighbourhood $V = V(x)$ of the point x such that $f_j(z) < L$ for every $z \in V$.

Let $l \in N$ be such that $n_l > j$ and $x_l \in V$. Then $f_{n_l}(x_l) \leq f_j(x_l) < L$ and that is a contradiction.

The proof of b) is similar.

Let $m \in N$ be such that f_n is a bounded function for any $n \geq m$.

Put $K = \langle \min_{x \in X} f(x), \max_{x \in X} f(x) \rangle \cup \langle \inf_{x \in X} f_m(x), \sup_{x \in X} f_m(x) \rangle$. By theorem 1.1 we have $\operatorname{Lim} G(f) = G(f)$ and by Theorem A we get $f_n \underset{X}{\rightrightarrows} f$.

2 Characterization of compactness by using graph convergence

In this part of the paper we give a characterization of compact metric spaces in the class of all metric spaces by using Theorem A and Theorem 1.2.

We shall use these usual notations:

If (X, d) is a metric space, $x \in X$ and $\varepsilon > 0$, then $K(x, \varepsilon) = \{y \in X: d(x, y) < \varepsilon\}$ denotes the open ε-ball with the centre x. If $A \subset X$, then $\bar{A}$ denotes the closure of A in X.

If (X, d_1), (Y, d_2) are metric spaces, we shall consider $X \times Y$ as a metric space with the metric $d = d_1 + d_2$ $(d((x_1, y_1), (x_2, y_2)) = d_1(x_1, x_2) + d_2(y_1, y_2))$.

Theorem 2.1. A metric space (Y, d) is compact if and only if for any compact metric space X with infinitely many points and for any sequence $\{f_n\}$ of functions f_n: $X \to Y$ $(n = 1, 2, \ldots)$ the following statement holds:

The sequence $\{f_n\}$ graph converges to a function f if and only if $f_n \underset{X}{\rightrightarrows} f$ and f is continuous on X.

Proof. 1. If (Y, d) is a compact metric space, the assertion follows from Theorem A.

2. Suppose that (Y, d) is not a compact space. Then there exists a one-to-one sequence $\{y_n\}$ of points from Y which has no cluster point in Y. Choose $\varepsilon_0 > 0$ such that $K(y_1, \varepsilon_0) \cap \{y_n: n = 2, 3, \ldots\} = \emptyset$.

Since X has infinitely many points, there exists a one-to-one sequence $\{x_n\}$ of points from X.

Define the functions f_n: $X \to Y$ $(n = 1, 2, \ldots)$ as follows:

$$f_n(x) = y_1 \text{ for } x \neq x_n$$

and $f_n(x_n) = y_n$ $(n = 1, 2, \ldots)$. The function f_1 is identically equal to y_1. It is easy to verify that $f_n \underset{X}{\longrightarrow} f_1$. This fact implies

$$G(f_1) \subset \operatorname{Lim\,inf} G(f_n). \tag{13}$$

We show that the following inclusion holds

$$\operatorname{Lim\,sup} G(f_n) \subset G(f_1). \tag{13'}$$

Let $(x, y) \in \operatorname{Lim\,sup} G(f_n)$. The definition of $\operatorname{Lim\,sup} G(f_n)$ implies that there exists a sequence $n_1 < n_2 < \ldots n_j < \ldots$ of the positive integers and a sequence $\{z_j\}$ of points of X such that $(z_j, f_{n_j}(z_j)) \to (x, y)$ $(j \to \infty)$.

That means $z_j \to x$ $(j \to \infty)$ and

$$f_{n_j}(z_j) \to y \ (j \to \infty). \tag{14}$$

The function f_{n_j} takes only on the values y_1 and y_{n_j}. If $f_{n_j}(z_j) = y_{n_j}$ for infinitely many j, the point y will be a cluster point of the sequence $\{y_n\}$. That means, there exists $j_0 \in N$ such that $f_{n_j}(z_j) = y_1$ for $j \geq j_0$. From (14) we get $y = y_1$.

The validity of (13′) is proved.

The relations (13) and (13′) imply that $\operatorname{Lim} G(f_n) = G(f_1)$. Thus the sequence $\{f_n\}$ graph converges to the continuous function $f_1 \equiv y_1$. But $f_n \underset{X}{\not\rightrightarrows} f_1$, since for any $n \geq 2$ we have $d(f_n(x_n), f(x_n)) = d(y_n, y_1) \geq \varepsilon_0$.

The proof of Theorem is over.

Theorem 2.2. A metric space (X, d_1) is compact if and only if for any metric space (Y, d_2) which contains at least two points and for any sequence $\{f_n\}$ of functions f_n: $X \to Y$ $(n = 1, 2, \ldots)$ the following statement holds.
$\operatorname{Lim} G(f_n) = G(f)$ if and only if $f_n \underset{X}{\rightrightarrows} f$ anf f: $X \to Y$ is a continuous function on X.

Proof. 1. If X is a compact metric space, then the assertion follows from Theorem A.

2. Suppose that X is not a compact space. There exists a sequence $\{x_n\}$ of points of X which has no cluster point in X. Choose $a, b \in Y$ such that $a \neq b$. Define functions f_n: $X \to Y$ $(n = 1, 2, \ldots)$ by

$$f_n(x) = a \text{ for } x \neq x_n \text{ and } f_n(x_n) = b \ (n = 1, 2, \ldots).$$

It is easy to see that the sequence $\{f_n\}$ pointwise converges to the function f, which is identically equal to a. From this fact we have

$$G(f) \subset \operatorname{Lim\,inf} G(f_n). \tag{15}$$

We show that the inclusion

$$\operatorname{Lim\,sup} G(f_n) \subset G(f) \text{ holds.} \tag{15′}$$

Let $(x, y) \in \operatorname{Lim\,sup} G(f_n)$. The definition of $\operatorname{Lim\,sup} G(f_n)$ implies that there exists a sequence of positive integers $n_1 < n_2 < \ldots n_j < \ldots$ and a sequence of points of X $\{z_j\}$ such that $\{(z_j, f_{n_j}(z_j))\}$ converges to (x, y).

Thus $\{z_j\}$ converges to x and $\{f_{n_j}(z_j)\}$ converges to y. It is easy to verify (similarly as in the proof of Theorem 2.1.) that there exists j_0 such that $f_{n_j}(z_j) = a, j \geq j_0$ and thus $y = a$. Then $(x, y) = (x, a) \in G(f)$. The inclusion (15′) is proved. The relations (15) and (15′) imply that $\operatorname{Lim} G(f_n) = G(f)$. Thus the sequence $\{f_n\}$ graph converges to the function f. Since for any $n \geq 1$ we have $d_2(f_n(x_n), f(x_n)) = d_2(a, b) > 0, f_n \underset{X}{\not\rightrightarrows} f$. The proof of Theorem is finished.

Theorem 2.3. A metric space (X, d) is compact if and only if the following implication holds: If g, g_n: $X \to R$ $(n = 1, 2, \ldots)$ are continuous functions and $\{g_n(x)\}$ monotonically converges to $g(x)$ for any $x \in X$, then $g_n \underset{X}{\rightrightarrows} g$.

Proof. 1. If X is a compact metric space, the assertion follows from Theorem 1.2.

2. Suppose that X is not a compact space. Choose a one-to-one sequence $\{x_n\}$, $x_n \in X$, which has no cluster point in X and choose $\varepsilon_n > 0$ $(n = 1, 2, \ldots)$ such that the family $\{K(x_n, \varepsilon_n)\colon n \in N\}$ is pairwise disjoint. Define the functions g_j: $X \to R$ $(j = 1, 2, \ldots)$ by

$$g_j(x) = \begin{cases} 1 - (d(x_n, x)/\varepsilon_n) & \text{for } x \in K(x_n, \varepsilon_n),\ n \geq j \\ 0 & \text{for other } x. \end{cases}$$

The functions g_j $(j = 1, 2, \ldots)$ are continuous and $g_j \underset{X}{\longrightarrow} g \equiv 0$. For any $x \in X$ the sequence $\{g_j(x)\}$ is decreasing. Since $|g_j(x_j) - g(x_j)| = |g_j(x_j)| = 1$ $(j = 1, 2, \ldots)$, we have $g_j \underset{X}{\not\rightrightarrows} g$.

The proof of Theorem is over.

The following theorem shows that Theorem 2.3 cannot be extended for topological spaces.

Theorem 2.4. Let X be a topological space such that any sequence of points of X contains a convergent subsequence. Let f_n: $X \to R$ $(n = 1, 2, \ldots)$, f: $X \to R$ be continuous functions and $f_n \underset{X}{\longrightarrow} f$. Suppose that for any $x \in X$ the sequence $\{f_n(x)\}$ is monotone. Then $f_n \underset{X}{\rightrightarrows} f$.

Remark 2.1. There exists a non-compact topological space for which any sequence of its points contains a convergent subsequence (see [5], 5 D p 220).

Proof of Theorem. Suppose that $f_n \underset{X}{\not\rightrightarrows} f$. Then the following statement holds: There exists $\varepsilon > 0$ such that for any $n_0 \in N$ there exist $n \geq n_0$ and a point x such that $|f_n(x) - f(x)| \geq \varepsilon$. This fact implies that there exist a sequence $n_1 < n_2 < \ldots$ $\ldots$ of positive integers and a sequence $\{x_k\}$ of points of X such that the following inequalities hold:

$$|f_{n_k}(x_k) - f(x_k)| \geq \varepsilon \quad (k = 1, 2, \ldots). \tag{16}$$

We can already assume (owing to the assumption of Theorem) that there exists $x \in X$ such that $\{x_k\}$ converges to x.

There are two possibilities:

a) $x = x_k$ for infinitely many $k \in N$,

b) $x \neq x_k$ for infinitely many $k \in N$.

In the case of a) the assumption of Theorem implies that $f_{n_j}(x) \to f(x)$. Let $m_1 < m_2 < \ldots$ be such that $x = x_{m_i}$ $(i = 1, 2, \ldots)$. Then $f_{m_j}(x_{m_j}) \to f(x_{m_j})$ and that is a contradiction to (16).

In the case of b) let $v_1 < v_2 < \ldots$ be such that $x \neq x_{v_j}$ $(j = 1, 2, \ldots)$. Put $A = \{x\} \cup \{x_{v_1}, x_{v_2}, \ldots x_{v_j}, \ldots\}$. It is easy to verify that the set A with the induced topology from X is a metrizable compact space. The functions $f_{n_j}|A$ $(j = 1, 2, \ldots)$ and $f|A$ satisfy the assumptions of Theorem 1. 2. By this Theorem we obtain $f_{n_j}|A \rightrightarrows f|A$ and that is a contradiction to (16).

The proof of Theorem is finished.

3 Characterization of compactness by using quasi-uniform and continuous convergence

Let X be a topological space and f, f_n: $X \to R$ $(n = 1, 2, \ldots)$. We say that the sequence $\{f_n\}$ of functions converges quasi-uniformly to a function f (we write $f_n \underset{q}{\rightrightarrows} f$) on X if $f_n \xrightarrow[X]{} f$ and the following statement is true:

For each $\varepsilon > 0$ and each $m \in N$, $m \geq 0$ there exists a $p \in N$ such that $\min \{|f_{m+1}(x) - f(x)|, \ldots, |f_{m+p}(x) - f(x)|\} < \varepsilon$ for each $x \in X$ (see [1], p. 334—335, [6] p. 143).

If X is a compact metric space and $\{f_n\}$ is a sequence of continuous functions f_n: $X \to R$ $(n = 1, 2, \ldots)$ pointwise convergent to a function f: $X \to R$, then f is a continuous function if and only if $f_n \underset{q}{\rightrightarrows} f$ (see [6], p. 143).

This result can be extended for functions f_n: $X \to Y$ $(n = 1, 2, \ldots)$, where Y is a metric space and X is a compact metric space.

The foregoing result suggests the following characterization of compact metric spaces.

Theorem 3.1. A metric space (X, d) is compact iff and only if the following assertion holds:

(V) If f, f_n: $X \to R$ $(n = 1, 2, \ldots)$ are continuous functions on X and $f_n \xrightarrow[X]{} f$, then $f_n \underset{q}{\rightrightarrows} f$.

Proof. 1. If (X, d) is a compact metric space, then (V) holds (see [6] p. 143, [1] p. 334—335).

2. Suppose that (X, d) is not a compact space. Then there exists a one-to-one sequence $\{x_k\}$ of points of X such that there exists no convergent subsequence of $\{x_k\}$. It is obvious that any point of the set $\{x_1, x_2, \ldots x_n, \ldots\}$ is an isolated point of the set $\{x_1, x_2, \ldots, x_n, \ldots\}$. That means there exist $\delta_k > 0$ $(k = 1, 2, \ldots)$ such that

$$\delta_k \to 0 \quad (k \to \infty) \tag{17}$$

and the closed balls $\bar{K}(x_k, \delta_k) = \{x \in X: d(x, x_k) \leq \delta_k\}$ $(k = 1, 2, \ldots)$ are pairwise disjoint. Put $H = \bigcup_{k=1}^{\infty} \bar{K}(x_k, \delta_k)$. We show that H is a closed set.

Let $y_n \in H$ $(n = 1, 2, \ldots)$ and $y_n \to y \in X$. There are two possibilities:

a) There exists a $m \in N$ such that $y_n \in \bar{K}(x_m, \delta_m)$ for infinitely many n.

b) Such a m does not exist.

In the case of a) we have $y \in \bar{K}(x_m, \delta_m) \subset H$ and so $y \in H$.

In the case of b) there exist two sequences $n_1 < n_2 < \ldots n_j, s_1 < s_2 < \ldots s_j < \ldots$ of positive integers such that $y_{n_j} \in \bar{K}(x_{s_j}, \delta_{s_j})$ $(j = 1, 2, \ldots)$. Then $d(x_{s_j}, y_{n_j}) \leq \delta_{s_j} \to 0$ $(j \to \infty)$ (see (17)). But $y_{n_j} \to y$ $(j \to \infty)$ and therefore also $x_{s_j} \to y$ $(j \to \infty)$. However this contradicts the fact that the sequence $\{x_n\}$ has no convergent subsequence. That means the statement a) is valid and so $y \in H$. The closedness of the set H is proved.

Define a sequence $\{f_p\}$ of real functions on the set $\{x_1, x_2, \ldots x_n, \ldots\}$ by

$$f_1(x_n) = 0 \ (n = 1, 2, \ldots) \text{ and for } p > 1$$

$$f_p(x_1) = (1 - 1/1)^{p-1}, f_p(x_2) = (1 - 1/2)^{p-1}, \ldots f_p(x_{p-1}) =$$
$$= (1 - 1/p - 1)^{p-1}, f_p(x_p) = (1 - 1/p)^{p-1}, f_p(x_{p+j}) = f_p(x_p) \quad (j = 1, 2, \ldots).$$

Put for $p \in N$: $f_p^*(x) = 0$ if $x \notin H$ and $f_p^*(x) = f_p(x_j) . (\delta_j - d(x, x_j))/\delta_j$ if $x \in \bar{K}(x_j, \delta_j)$ $(j = 1, 2, \ldots)$.

It is easy to see that the function f_p^* is a nonnegative continuous extension of f_p $(p = 1, 2, \ldots)$ from the set $\{x_1, x_2, \ldots x_n, \ldots\}$ on the whole X and

$$0 \leq f_p^*(x) \leq (1 - 1/p)^{p-1} \quad (p \geq 2).$$

The function $f_p^*(x)$ takes on its global maximum at the point x_p. We show that the sequence $\{f_p^*\}$ pointwise converges to $f_0 \equiv 0$.

Let $x \notin H$. Then $f_p^*(x) = 0$ $(p = 1, 2, \ldots)$ and hence $f_p^*(x) \to f_0(x) = 0$.

Let $x \in H$. Then there exists a $m \in N$ such that $x \in \bar{K}(x_m, \delta_m)$. Then for any $p > m$ we have $0 \leq f_p^*(x) \leq f_p^*(x_m) = f_p(x_m) = (1 - 1/m)^{p-1}$ and $(1 - 1/m)^{p-1} \to$ $\to 0$ $(p \to \infty)$. Thus again $f_p^*(x) \to f_0(x) = 0$ $(p \to \infty)$.

We prove that the sequence $\{f_p^*\}$ does not converge quasi-uniformly to f_0. Suppose the contrary, i.e. $f_p^* \underset{q}{\longrightarrow} f_0$. Put

$$a_p = (1 - 1/p)^{p-1} \quad \text{for } p > 1.$$

Then $a_p = ((p-1)/p)^{p-1} = 1/(1 + 1/p - 1)^{p-1}$.

It is well-known that $(1 + 1/p - 1)^{p-1} \uparrow \mathrm{e}$. Hence (18) $a_p \to \mathrm{e}^{-1}$ (i.e.

$\alpha_1 > \alpha_2 > \dots \alpha_p \dots \alpha_p \to e^{-1}$). Choose $\varepsilon = e^{-1}$, $m = 1$ in the definition of a quasi-uniform convergence. Then there exists $p > 1$ such that for each $x \in X$ the following inequality holds:

$$\min\{f_2^*(x), \dots f_{1+p}^*(x)\} < e^{-1}. \tag{19}$$

For $x = x_{p+1}$ we obtain from (19) that $\alpha_{p+1} < e^{-1}$ and that is a contradiction to (18).

The proof of Theorem is finished.

Another characterization of compact metric spaces we obtain by using the notion of a continuous convergence of functions.

Let X be a metric space and f, f_n: $X \to R$ $(n = 1, 2, \dots)$. The sequence $\{f_n\}$ is said to continuously converge to the function f if for any $x \in X$ and for any sequence $\{x_n\}$ of points of X convergent to x we have $f_n(x_n) \to f(x)$ (see [7], [8], p. 103).

It is known (see [8], p. 106) that if X is a compact metric space, then the continuous convergence of functions $\{f_n\}$ to f implies a uniform convergence of $\{f_n\}$ to f. By using this fact we can give the following characterization of compact metric spaces.

Theorem 3.2. A metric space (X, d) is compact if and only if the following statement holds for functions f, f_n: $X \to R$ $(n = 1, 2, \dots)$:

(W) If $\{f_n\}$ continuously converges to f, then $f_n \underset{X}{\rightrightarrows} f$.

Proof. 1. If X is a compact metric space, then (W) holds by the result [8], p. 206.

2. Suppose that X is not a compact metric space. Let f_p^* $(p = 1, 2, \dots)$ be the functions from the proof of Theorem 3.1. Then $f_p^* \underset{X}{\to} f_0 \equiv 0$, but $f_p^* \underset{q}{\not\rightrightarrows} f_0$ and thus $f_p^* \underset{X}{\not\rightrightarrows} f_0$. We prove that the sequence $\{f_p^*\}$ continuously converges to f_0.

Let $y_n \to y_0$. First, suppose that $y_0 \notin H$. Since the set H is closed, there exists $n_1 \in N$ such that $y_n \notin H$ for any $m > n_1$. Then $f_n^*(y_n) = 0$ $(n > n_1)$ and thus $f_n^*(y_n) \to 0 = f_0(y_0)$.

Second, suppose that $y_0 \in H$. There exists $m \in N$ such that $y_0 \in \bar{K}(x_m, \delta_m)$. There are two possibilities:

a) $y_0 \in K(x_m, \delta_m)$,

b) $y_0 \in \{x \in X$: $d(x, x_m) = \delta_m\}$.

In case of a) there exists $n_2 \in N$ such that $y_n \in K(x_m, \delta_m)$ for any $n > n_2$. Then for any $n > \max\{n_2, m\}$ we have

$$0 \le f_n^*(y_n) \le f_n^*(x_m) = (1 - 1/m)^{n-1} \to 0 \ (n \to \infty);$$

thus $f_n^*(y_n) \to 0 = f_0(y_0)$.

In case of b) there are two possibilities:

b_1) $y_n \in K(x_m, \delta_m)$ for all but finitely many n' s,

b_2) $y_n \notin K(x_m, \delta_m)$ for infinitely many n' s.

In case of b_1) using the same method as in case of a) we obtain that $f_n^*(y_n) \to 0 = f_0(y_0)$.

b_2) Notice that if $y_n \in \bar{K}(x_m, \delta_m) \setminus K(x_m, \delta_m)$, then by the definition of the function f_n^* we have $f_n^*(y_n) = 0$ (since $d(y_n, x_m) = \delta_m$).

Because of a) we can assume that $y_n \notin \bar{K}(x_m, \delta_m)$ $(n = 1, 2, \ldots)$. Since $y_n \to y_0$, $y_0 \in \bar{K}(x_m, \delta_m)$ the consideration analogous to that used in the proof of closedness of H (see the proof of Theorem 3.1.) we can check that the set $\{y_1, y_2, \ldots y_n, \ldots\}$ has a non-empty intersection only with a finite number of balls $\bar{K}(x_j, \delta_j)$ $(j \geq m)$ and each such intersection is a finite set. Hence there exists $n_0 \in N$ such that $y_n \notin H$ for any $n > n_0$. Then $f_n^*(y_n) = 0$ for any $n > n_0$ and so $f_n^*(y_n) \to 0 = \mathrm{f}_0(\mathrm{y}_0)$.

The proof of Theorem is finished.

The question arises whether Theorem 3.1. can be extended for the topological spaces. We show that it is impossible.

It is known that there exists a non-compact, locally compact topological space in which any sequence of its points has a cluster point. Such a space is a space of all ordinal numbers less than the smallest uncountable ordinal numbers (see [5] p. 220 5 D).

Theorem 3.3. Let X be a locally compact topological space, in which any sequence of points of X has a cluster point. Let $f, f_n \colon X \to R$ $(n = 1, 2, \ldots)$ be continuous functions on X. If $f_n \underset{X}{\to} f$, then $f_n \underset{q}{\rightrightarrows} f$.

Proof. Suppose that $f_n \underset{X}{\to} f$, but $f_n \underset{q}{\not\rightrightarrows} f$. Then,

there exist $\varepsilon_0 > 0$ and $m_0 \geq 0$ such that for any $p \in N$ there exists x_p for which $\min\{|f_{m_0+1}(x_p) - f(x_p)|, \ldots |f_{m_0+p}(x_p) - f(x_p)|\} \geq \varepsilon_0$. (20)

Construct the sequence $\{x_p\}$. By assumption of Theorem there exists a cluster point $x_0 \in X$ of this sequence. The local compactness of X implies that there exists a compact neighbourhood $U(x_0)$ of x_0. The sequence $\{f_p|U(x_0)\}$ of continuous functions on the compact space $U(x_0)$ pointwise converges to f, $(U(x_0)$ and $f|U(x_0)$ is continuous on $U(x_0)$. By [1], p. 334—335, the sequence $\{f_p|U(x_0)\}$ converges quasi-uniformly to $f|U(x_0)$ on $U(x_0)$. The definition of the quasi--uniform convergence implies that

for any $\varepsilon > 0$ and for any $m \geq 0$ there exists $k \in N$ such that for any $x \in U(x_0)$ $\min\{|f_{m+1}(x) - f(x)|, \ldots |f_{m+k}(x) - f(x)|\} < \varepsilon$. (21)

Choose in (21) $\varepsilon = \varepsilon_0$ and $m = m_0$. There exists $k \in N$ such that for any $x \in U(x_0)$ we have

$$\min \{|f_{m_0+1}(x) - f(x)|, \dots |f_{m_0+k}(x) - f(x)|\} < \varepsilon_0. \tag{21'}$$

Since x_0 is a cluster point of the sequence $\{x_j\}$, there exists $i > k$ such that $x_i \in U(x_0)$. By using (20) we have

$$\min \{|f_{m_0+1}(x_i) - f(x_i)|, \dots |f_{m_0+i}(x_i) - f(x_i)|\} \geq \varepsilon_0. \tag{22}$$

Since for $i \geq k$ we have $\min \{|f_{m_0+1}(x_i) - f(x_i)|, \dots |f_{m_0+k}(x_i) - f(x_i)|, \dots$ $\dots |f_{m_0+i}(x_i) - f(x_i)|\} \leq \min \{|f_{m_0+1}(x_i) - f(x_i)|, \dots |f_{m_0+k}(x_i) - f(x_i)|\}$, (22) contradicts (21′).

REFERENCES

1. Alexandrov, P. S.: Úvod do obecné theorie množin a funkcí. N ČSAV, Prahá, 1954.
2. Beer, G.: The approximation of upper semicontinuous multifunctions by step multifunctions. Pacific. J. Math. 87 (1980).
3. Beer, G.: On uniform convergence of continuous functions and topological convergence of sets. Canad. Math. Bull 26 (1983), 418—424.
4. Beer, G.: More on convergence of continuous functions and topological convergence of sets. Canad. Math. Bull. 28 (1985), 52—59.
5. Kelley, J.: General topology, New York 1957.
6. Sikorski, R.: Funkcje rzeceywiste I. PWN, Warszawa, 1956.
7. Stoilov, S.: O nepreryvnoj schodimosti. Rev. Math. Pures at Appl. 4 (1959), 341—344.
8. Šalát, T.: Vybrané časti z matematiky III (skriptá), UK, Bratislava 1983.
9. Waterhouse, W. C.: Uniform convergence and graph convergence. Amer. Math. Monthly 83 (1976), 641—643.

Author's address:

Ľubica Holá,
Katedra teórie pravdepodobnosti
a matematickej štatistiky MFF UK
Matematický pavilón
Mlynská dolina
842 15 Bratislava

Tibor Šalát
Katedra algebry a teórie čísel MFF UK
Matematický pavilón
Mlynská dolina
842 15 Bratislava

Received: 27. 11. 1986

РЕЗЮМЕ

СХОДИМОСТЬ ГРАФИКОВ, РАВНОМЕРНАЯ, КВАЗИРАВНОМЕРНАЯ И НЕПРЕРЫВНАЯ СХОДИМОСТЬ И ХАРАКТЕРИСТИКА КОМПАКТНОСТИ

Любица Гола — Тибор Шалат, Братислава

Работа исходит из статьи [9] В. Ц. Ватергоуса, в которой тоже находится понятие сходимости графиков функциональных последова — тельностей. Для этого типа сходимости в работе доказана теорема типа «Дини» (теорема 1.1.). Кроме того, в работе даны некоторые характеристики компактности метрических пространств при помощи разных типов сходимости функциональных последовательностей.

SÚHRN

GRAFOVÁ KONVERGENCIA, ROVNOMERNÁ, KVÁZIROVNOMERNÁ A SPOJITÁ KONVERGENCIA A CHARAKTERIZÁCIE KOMPAKTNOSTI

Ľubica Holá — Tibor Šalát, Bratislava

Práca nadvazuje na článok [9] W. C. Waterhousea, z ktorého je prevzatý pojem grafovej konvergencie. Pre tento typ konvergencie konkcionálnych postupností je v práci dokázaná veta Diniho typu (veta 1.1). Okrem toho sú v práci podané viaceré charakterizácie kompaktnosti metrických priestorov pomocou rôznych typov konvergencie funkcionálnych postupností.