UNIVERSITAS COMENIANA
ACTA MATHEMATICA UNIVERSITATIS COMENIANAE
LIV—LV—1988

ON CONCERNING A CERTAIN QUASIVARIATIONAL INEQUALITY

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One of the fundamental problems of renewal theory is to secure a smooth progress of production in such a way that the costs are minimal. In the renewal process with preventive replacements this means finding an optimal strategy which minimizes the average cost.

The optimality of such a replacement strategy, its construction and its asymptotic behaviour in a nonparametric situation (i.e. when the distribution function of failure times is unknown) were treated in [1], [2], [3]. The parametric situation (i.e. when the distribution function of failure times is specified up to an unknown parameter) was investigated in [5], [6].

This paper is connected with [5], [6], but uses another mathematical method for the minimization of the average cost. The average cost and the optimal stationary management of the replacement process are given here as a solution of a system of quasivariational inequalities.

1 Fundamental model

Consider two types of replacements:

1) *service replacement* after failure at the cost c_1 (type 1),

2) *preventive replacement* at the cost c_2 (type 2).

We assume $c_1 > c_2 > 0$.

The *total cost* accumulated up to time t is

$$C_t = c_1 N_t^1 + c_2 N_t^2, \tag{1}$$

where N_t^i, $i = 1, 2$ is the total number of replacements of type i up to time t.

The *average cost per unit time* $\Theta(x)$ corresponding to the policy with a constant critical age $x \in [0, \infty)$ (age at which a component is replaced by a new one) is

$$\Theta(x) = (c_1 F(x) + c_2 \bar{F}(x)) \Big/ \int_0^x \bar{F}(y)\,\mathrm{d}y, \tag{2}$$

where $F(x)$ is the distribution function of failure times,

$$\bar{F}(x) = 1 - F(x) = \exp\left\{-\int_0^x g(y)\,dy\right\}, \tag{3}$$

$g(x)$ is the *failure rate* and the denominator in (2) is the mean time between replacements. We denote by $f(x)$ the probability density of failure times. Then $f(x) = g(x)\bar{F}(x)$. About $g(x)$ we assume the following.

Assumption 1.

(i) $g(x)$ is differentiable on $[0, \infty)$ and has no inflexion point,

(ii) $g(x)$ has at most one extreme and $g(\infty) = \infty$.

In general, the critical age is not a constant but a nonanticipative random function $\{Z_t, t \geq 0\}$ which assumes positive values and is left-continuous.

2 The Bellman equation for the expected discounted cost

The *impulsive control of a Markov process* consists in shifting instantaneously its trajectory to the selected position. In our case the process $\{X_t, t \geq 0\}$, the age of the component at time t, is a Markov process and the impulsive control defined by $\{Z_t, t \geq 0\}$ consists in the choice of an increasing sequence of stopping times at which the trajectory of the process is shifted to the zero-position (see Figure 1). $\{X_t, t \geq 0\}$ is assumed to be left-continuous.

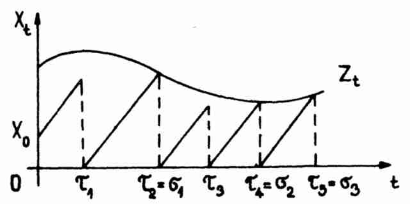

Fig. 1

$\tau_j, j = 1, 2, \ldots$ is the time of the j-th replacement (of type 1 or 2) $\sigma_j, j = 1, 2, \ldots$ is the time of the j-th preventive replacement given by the policy $\{Z_t, t \geq 0\}$.

The *optimality criterion* is given by the expected discounted cost

$$\mathrm{E}_x^Z \int_0^\infty \mathrm{e}^{-\beta t}\,\mathrm{d}C_t, \tag{4}$$

where $\beta > 0$ is the *discount-factor* and x is the *initial age*. The minimum satisfies the quasivariational inequalities which we obtain assuming a random stopping of the process with a rate $\bar{r} < \infty$ and then letting $\bar{r} \to \infty$. The policy, here the selected value of the stopping rate,

$$R = \{R_t,\ t \geq 0\}, \quad 0 \leq R_t \leq \bar{r},$$

defines the process of preventive replacements.

To minimize the expected discounted cost (4) we apply the *Bellman principle*. We assume a suitable choice of the stopping rate on the interval $[0, \Delta]$ and then an optimal procedure with respect to the situation in time Δ, $\Delta \to 0_+$. Denote

$$\bar{u}(x) = \inf_R \mathrm{E}_x^R \int_0^\infty \mathrm{e}^{-\beta t}\,\mathrm{d}C_t, \quad R = \{R_t,\ t \geq 0\}, \quad 0 \leq R_t \leq \bar{r}.$$

Then

$$\bar{u}(x) = \min_{0 \leq r \leq \bar{r}} [(1 - g(x)\Delta - r\Delta)\,\mathrm{e}^{-\beta\Delta}\,.\,\bar{u}(x + \Delta) +$$

$$+ g(x)\Delta\,.\,(c_1 + \bar{u}(0)) + r\Delta(c_2 + \bar{u}(0)) + o(\Delta)].$$

Hence,

$$0 = \Delta[\bar{u}'(x) + g(x)(c_1 + \bar{u}(0) - \bar{u}(x)) - \beta\bar{u}(x) + \min_{0 \leq r \leq \bar{r}} r(c_2 + \bar{u}(0) - \bar{u}(x))],$$

where

$$\bar{u}(x + \Delta) = \bar{u}(x) + \Delta\bar{u}'(x) + o(\Delta),\ \mathrm{e}^{-\beta\Delta} = 1 - \beta\Delta + o(\Delta).$$

Thus, we get the *Bellman equation*

$$\bar{u}'(x) + g(x)(c_1 + \bar{u}(0) - \bar{u}(x)) + \min_{0 \leq r \leq \bar{r}} r(c_2 + \bar{u}(0) - \bar{u}(x)) - \beta\bar{u}(x) = 0. \tag{5}$$

Next we find the limit solution of the Bellman equation (5) as $\bar{r} \to \infty$. Let a be a point where the term $c_2 + \bar{u}(0) - \bar{u}(x)$ changes its sign, i.e. $\bar{u}(a) = c_2 + \bar{u}(0)$. The function $\bar{u}(x)$ has at most one extreme and satisfies the inequalities

$$c_2 + \bar{u}(0) - \bar{u}(x) \geq 0 \quad \text{for} \quad x \in [0, a],$$

$$c_2 + \bar{u}(0) - \bar{u}(x) \leq 0 \quad \text{for} \quad x \geq a.$$

Denote $\bar{u}(0) = \bar{k}$, where $\bar{k}$ is a constant and $\bar{v}(x) = \bar{u}(x) - \bar{u}(0)$. Then, from the Bellman equation (5) we get

1. For $x \geq a$

$$\begin{cases} \bar{v}'(x) - (g(x) + \beta + \bar{r})\bar{v}(x) = \beta\bar{k} - c_2\bar{r} - c_1 g(x) \\ \bar{v}(a) = c_2. \end{cases} \tag{6}$$

By solving (6) with initial condition $\bar{v}(a) = c_2$ we get

$$\bar{v}(x) = \left[c_2 + \int_a^x (\beta\bar{k} - c_2\bar{r} - c_1 g(s))\, \mathrm{e}^{-\int_a^s (g(y) + \beta + \bar{r})\,\mathrm{d}y} \mathrm{d}s \right] \mathrm{e}^{\int_a^x (g(y) + \beta + \bar{r})\,\mathrm{d}y},$$

i.e.

$$\bar{v}(x) = \frac{c_2 \bar{F}(a)\,\mathrm{e}^{-(\beta+\bar{r})a} - c_1 \int_a^x f(s)\,\mathrm{e}^{-(\beta+\bar{r})s}\,\mathrm{d}s + (\beta\bar{k} - c_2\bar{r}) \int_a^x \bar{F}(s)\,\mathrm{e}^{-(\beta+\bar{r})s}\,\mathrm{d}s}{\bar{F}(x)\,.\,\mathrm{e}^{-(\beta+\bar{r})x}}. \quad (7)$$

To have the function $\bar{v}(x)$ bounded, it is necessary that

$$0 = c_2 \bar{F}(a)\,\mathrm{e}^{-(\beta+\bar{r})a} - c_1 \int_a^\infty f(s)\,\mathrm{e}^{-(\beta+\bar{r})s}\,\mathrm{d}s + (\beta\bar{k} - c_2\bar{r}) \int_a^\infty \bar{F}(s)\,\mathrm{e}^{-(\beta+\bar{r})s}\,\mathrm{d}s. \quad (8)$$

Since

$$\int_a^\infty f(y)\,\mathrm{e}^{-(\beta+\bar{r})y}\,\mathrm{d}y = \mathrm{e}^{-(\beta+\bar{r})a}\bar{F}(a) - \int_a^\infty (\beta+\bar{r})\,\mathrm{e}^{-(\beta+\bar{r})y}\,.\,\bar{F}(y)\,\mathrm{d}y,$$

from (8) we have for the constant

$$\bar{k} = \beta^{-1}\left[(c_2 - c_1)\bar{r} - c_1\beta + (c_1 - c_2)\left(\int_a^\infty \mathrm{e}^{-(\beta+\bar{r})(y-a)}\,.\,\frac{\bar{F}(y)}{\bar{F}(a)}\,\mathrm{d}y \right)^{-1} \right]. \quad (9)$$

Further,

$$\int_a^\infty \mathrm{e}^{-(\beta+\bar{r})(y-a)} \frac{\bar{F}(y)}{\bar{F}(a)}\,\mathrm{d}y = \frac{1}{\beta+\bar{r}}\left(1 - \frac{1}{\bar{F}(a)} \int_a^\infty \mathrm{e}^{-(\beta+\bar{r})(y-a)} f(y)\,\mathrm{d}y \right)$$

and

$$\frac{1}{\bar{F}(a)} \int_a^\infty \mathrm{e}^{-(\beta+\bar{r})(y-a)} f(y)\,\mathrm{d}y = \frac{1}{\beta+\bar{r}}\left(\frac{f(a)}{\bar{F}(a)} + o(1) \right) \quad \text{as } \bar{r} \to \infty.$$

From (9) we have

$$\bar{k} = \beta^{-1}\left\{ (c_1 - c_2)\left[(\beta+\bar{r}) \frac{1}{1 - \frac{1}{\beta+\bar{r}}\left(\frac{f(a)}{\bar{F}(a)} + o(1) \right)} - \bar{r} \right] - c_1\beta \right\},$$

i.e.

$$\bar{k} = \beta^{-1}(c_1 - c_2)\left[(\beta+\bar{r})\left(1 + \frac{1}{\beta+\bar{r}}\left(\frac{f(a)}{\bar{F}(a)} + o(1) \right) \right) - \bar{r} \right] - c_1 =$$

$$= \beta^{-1}(c_1 - c_2) g(a) - c_2 + o(1). \quad (10)$$

Denote $k = \lim_{\bar{r}\to\infty} \bar{k}$, then for $\bar{r} \to \infty$ we have

$$k = \beta^{-1}(c_1 - c_2)g(a) - c_2. \tag{11}$$

Now we modify the term (7)

$$\bar{v}(x) = \underbrace{\frac{c_1}{\bar{F}(x)} \int_x^\infty f(s)\, \mathrm{e}^{-(\beta+\bar{r})(s-x)}\, \mathrm{d}s}_{\text{(i)}} - \underbrace{\frac{\beta\bar{k}}{\bar{F}(x)} \int_x^\infty \bar{F}(s)\, \mathrm{e}^{-(\beta+\bar{r})(s-x)}\, \mathrm{d}s}_{\text{(ii)}} +$$

$$+ \underbrace{\frac{c_2\bar{r}}{\bar{F}(x)} \int_x^\infty \bar{F}(s)\, \mathrm{e}^{-(\beta+\bar{r})(s-x)}\, \mathrm{d}s}_{\text{(iii)}}.$$

We have (for $\bar{r} \to \infty$) : (i) $\to 0$, (ii) $\to 0$, (iii) $\to c_2$.
If we denote

$$v(x) = \lim_{\bar{r}\to\infty} \bar{v}(x), \quad u(x) = \lim_{\bar{r}\to\infty} \bar{u}(x), \tag{12}$$

we get the *limit solution of the Bellman equation* (for $\bar{r} \to \infty$) in the form

$$u(x) = c_2 + k, \quad x \geq a. \tag{13}$$

2. For $x \in [0, a]$

$$\min_{0 \leq r \leq \bar{r}} r(c_2 + \bar{u}(0) - \bar{u}(x)) = 0.$$

From (11), (12) and from the Bellman equation we get

$$\begin{cases} v'(x) - (g(x) + \beta)v(x) = -g(x)c_1 + g(a)(c_1 - c_2) - \beta c_2 \\ v(0) = 0. \end{cases} \tag{14}$$

By solving (14) we get for $x \in [0, a]$

$$v(x) = \mathrm{e}^{\int_0^x (g(y)+\beta)\mathrm{d}y} \left(\int_0^x \left[-c_1 g(s) + g(a)(c_1 - c_2) - \beta c_2 \right] \mathrm{e}^{-\int_0^s (g(y)+\beta)\mathrm{d}y}\, \mathrm{d}s \right) =$$

$$= \frac{-c_1 \int_0^x f(s)\mathrm{e}^{-\beta s}\, \mathrm{d}s + [g(a)(c_1 - c_2) - \beta c_2] \int_0^x \bar{F}(s)\mathrm{e}^{-\beta s}\, \mathrm{d}s}{\bar{F}(x)\mathrm{e}^{-\beta x}}, \quad x \in [0, a].$$

Thus, the *limit solution of the Bellman equation* (5) is

$$u(x) = \begin{cases} k + \dfrac{-c_1 \int_0^x f(s)\mathrm{e}^{-\beta s}\, \mathrm{d}s + \beta k \int_0^x \bar{F}(s)\mathrm{e}^{-\beta s}\, \mathrm{d}s}{\bar{F}(x)\mathrm{e}^{-\beta x}} & \text{for} \quad x \in [0, a] \\ k + c_2 \quad \text{for} \quad x \geq a, \end{cases} \tag{15}$$

where k is given by (11).

3 Quasivariational inequalities for the expected discounted cost

Theorem 1. *The quasivariational inequalities for the expected discounted cost*

$$\begin{aligned}
&(\mathrm{I}')\ u'(x) + g(x)(c_1 + u(0) - u(x)) - \beta u(x) \geq 0 \\
&(\mathrm{II}')\ c_2 + u(0) - u(x) \geq 0 \qquad (16)\\
&(\mathrm{III}')\ (c_2 + u(0) - u(x))[u'(x) + g(x)(c_1 + u(0) - u(x)) - \beta u(x)] = 0
\end{aligned}$$

have solution (15), *where a is the unique solution of the equation*

$$(c_1 - c_2)[\bar{F}(a)\,\mathrm{e}^{-\beta a} + (g(a) + \beta)\int_0^a \bar{F}(y)\,\mathrm{e}^{-\beta y}\,\mathrm{d}y] = c_1 \tag{17}$$

and the constant is

$$k = \beta^{-1}(c_1 - c_2)g(a) - c_2\,.$$

Proof. The proof will be performed in two steps:

1. *The unicity of the solution of* (17)

From (17) we get

$$\bar{F}(a)\,\mathrm{e}^{-\beta a} + (g(a) + \beta)\int_0^a \bar{F}(y)\,\mathrm{e}^{\beta y}\,\mathrm{d}y = \frac{c_1}{c_1 - c_2}.$$

Now, we investigate the function

$$Q(x) = \bar{F}(x)\,\mathrm{e}^{-\beta x} + (g(x) + \beta)\int_0^x \bar{F}(y)\,\mathrm{e}^{-\beta y}\,\mathrm{d}y,\ x \in [0, \infty).$$

By Assumption 1 the function $g(x)$ has a derivative so that

$$Q'(x) = g'(x) \cdot \int_0^x \bar{F}(y)\,\mathrm{e}^{-\beta y}\,\mathrm{d}y.$$

Thus the functions $Q'(x)$ and $g'(x)$ have the same sign. By Assumption 1 there are two possibilities:

a) If $g(x)$ has no local extreme, then $g'(x) > 0$ everywhere and $Q(x)$ is increasing on $(0, \infty)$. Since $\lim_{x \to 0_+} Q(x) = 1$ and $\frac{c_1}{c_1 - c_2} > 1$ $(c_1 > c_2 > 0)$, the graph of the function $Q(x)$ crosses the line $y = \frac{c_1}{c_1 - c_2}$ in exactly one point.

b) If $g(x)$ has an extreme, it is (by Assumption 1) a local minimum in some point $b \in [0, a]$. Thus $Q(x)$ decreases on $[0, b]$, increases on $[b, \infty)$ and its graph crosses the line $y = \frac{c_1}{c_1 - c_2}$ also in exactly one point.

2. *The fulfilment of the quasivariational inequalities* (I′)—(III′)

Ad (I′)

a) On the interval $[0, a]$ in (I′) equality holds, because for $x \in [0, a]$

$$u(x) = k + \frac{-c_1 \int_0^x f(y)\,\mathrm{e}^{-\beta y}\,\mathrm{d}y + \beta k \int_0^x \bar{F}(y)\,\mathrm{e}^{-\beta y}\,\mathrm{d}y}{\bar{F}(x)\,\mathrm{e}^{-\beta x}}.$$

Substituting this into (I′) we get

$$u'(x) + g(x)(c_1 + u(0) - u(x)) - \beta u(x) = 0.$$

b) On the interval $[a, \infty)$we have

$$u(x) = k + c_2 = \beta^{-1}(c_1 - c_2)g(a),$$

so that

$$g(a)(c_1 + u(0) - u(x)) - \beta u(x) = g(a)(c_1 - c_2) + \beta(k + c_2) = 0.$$

Since $g(x)$ is increasing for $x \geq a$ (i.e. $g(x) \geq g(a)$ for $x \geq a$), it holds

$$u'(x) + g(x)(c_1 + u(0) - u(x)) - \beta u(x) = g(x)(c_1 - c_2) - \beta(c_2 + k) \geq 0,$$

i.e. (I′) holds.

Ad (II′)

a) On the interval $[0, a]$

$$c_2 + u(0) - u(x) \geq 0$$

is equivalent to

$$c_2 \bar{F}(x)\,\mathrm{e}^{-\beta x} + c_1 \int_0^x f(y)\,\mathrm{e}^{-\beta y}\,\mathrm{d}y - [g(a)(c_1 - c_2) - \beta c_2] \int_0^x \bar{F}(y)\,\mathrm{e}^{-\beta y}\,\mathrm{d}y \geq 0. \quad (18)$$

Since

$$\int_0^x f(y)\,\mathrm{e}^{-\beta y}\,\mathrm{d}y = 1 - \bar{F}(x)\,\mathrm{e}^{-\beta x} - \beta \int_0^x \bar{F}(y)\,\mathrm{e}^{-\beta y}\,\mathrm{d}y,$$

from (18) we get the condition

$$c_1 - (c_1 - c_2)\left[\bar{F}(x)\,\mathrm{e}^{-\beta x} + (g(a) + \beta)\int_0^x \bar{F}(y)\,\mathrm{e}^{-\beta y}\,\mathrm{d}y\right] \geq 0,$$

or

$$\frac{c_1}{c_1 - c_2} - \left[\bar{F}(x)\,\mathrm{e}^{-\beta x} + (g(a) + \beta)\int_0^x \bar{F}(y)\,\mathrm{e}^{-\beta y}\,\mathrm{d}y\right] \geq 0. \quad (19)$$

Denote the left part of (19) by $T(x)$. Then

$$T'(x) = f(x)e^{-\beta x} + \bar{F}(x)\beta e^{-\beta x} - (g(a) + \beta)\bar{F}(x)e^{-\beta x} = (g(x) - g(a))\bar{F}(x)e^{-\beta x},$$

i.e. $T(x)$ is nonincreasing on $[0, a]$, and nonnegative in virtue of $T(a) = 0$. Hence (II′) holds on $[0, a]$.
b) On the interval $[a, \infty)$, *where* $u(x) = c_2 + k$, we have $c_2 + u(0) - u(x) = 0$, i.e. in (II′) the equality holds.

Ad (III′)

(III′) holds, because on $[0, a]$ the equality holds in (I′) and on $[a, \infty)$ in (II′). □

Theorem 2. *The bounded solution of quasivariational inequalities* (I′)—(III′) *for the expected discounted cost is unique. It holds*

$$u(x) = \inf_Z \mathrm{E}_x^Z \int_0^\infty e^{-\beta t} \mathrm{d}C_t, \quad x \in [0, \infty).$$

For the replacement age $\hat{Z}_t = a$, $t \geq 0$, where a satisfies (17)

$$\mathrm{u}(x) = \mathrm{E}_x^{\hat{Z}} \int_0^\infty e^{-\beta t} \mathrm{d}C_t, \quad x \in [0, \infty). \tag{20}$$

Proof. Consider an arbitrary bounded solution $u(x)$ of the quasivariational inequalities (I′)—(III′). Assume that $u(x)$ has continuous derivatives and consider the integral

$$\int_0^T e^{-\beta t}(u'(X_t) - \beta u(X_t))\,\mathrm{d}t, \quad T > 0.$$

At the points τ_i, $i = 1, 2, \ldots$, $\tau_i \in [0, T]$, the trajectory of the process $\{X_t, t \geq 0\}$ jumps to the zero-position. Taking this into account we get

$$\int_0^T e^{-\beta t}(u'(X_t) - \beta u(X_t))\,\mathrm{d}t = e^{-\beta T}u(X_t^+) - u(X_0) +$$
$$+ \int_0^T e^{-\beta t}(u(X_t) - u(0))\,\mathrm{d}(N_t^1 + N_t^2),$$

where X_t^+ is the right-continuous version of the age of the component at time t,

$$X_t^+ = X_t \, . \, \chi_{\{N_t^i = N_{t^-}^i,\, i = 1, 2\}}.$$

For $T \to \infty$ we get

$$u(X_0) - \int_0^\infty e^{-\beta t}(u(X_t) - u(0))\,\mathrm{d}(N_t^1 + N_t^2) + \int_0^\infty e^{-\beta t}(u'(X_t) - \beta u(X_t))\,\mathrm{d}t = 0. \tag{21}$$

By the definition of C_t and by adding the left part of (21) we obtain for the expected value

$$\mathrm{E}_x^Z \int_0^\infty \mathrm{e}^{-\beta t}\,\mathrm{d}C_t = \mathrm{E}_x^Z \int_0^\infty \mathrm{e}^{-\beta t}\,\mathrm{d}(c_1 N_t^1 + c_2 N_t^2) = \mathrm{E}_x^Z u(X_0) +$$
$$+ \mathrm{E}_x^Z \int_0^\infty \mathrm{e}^{-\beta t}(c_1 + u(0) - u(X_t))(\mathrm{d}N_t^1 - g(X_t)\,\mathrm{d}t) +$$
$$+ \mathrm{E}_x^Z \int_0^\infty \mathrm{e}^{-\beta t}(c_2 + u(0) - u(X_t))\,\mathrm{d}N_t^2 +$$
$$+ \mathrm{E}_x^Z \int_0^\infty \mathrm{e}^{-\beta t}[u'(X_t) + g(X_t)(c_1 + u(0) - u(X_t)) - \beta u(X_t)]\,\mathrm{d}t. \tag{22}$$

We have $\mathrm{E}_x^Z(u(X_0)) = u(x)$ for $X_0 = x$. By the martingale properties

$$\underbrace{\mathrm{E}_x^Z \int_0^\infty \mathrm{e}^{-\beta t}(c_1 + u(0) - u(X_t))(\mathrm{d}N_t^1 - g(X_t)\,\mathrm{d}t)}_{\text{(iv)}} = 0,$$

namely, $N_t^1 - \int_0^t g(X_s)\,\mathrm{d}s$, $s \in [0, t]$ is a martingale (see [6], Lemma 1) and (iv) is a martingale, too (see [6], Lemma 2). Since $u(x)$ is a bounded solution of quasivariational inequalities (I′)—(III′), the last two integrals in (22) are non-negative and

$$\mathrm{E}_x^Z \int_0^\infty \mathrm{e}^{-\beta t}\,\mathrm{d}C_t \geq u(x). \tag{23}$$

The equality in (23) is satisfied for the replacement age x such that $c_2 + u(0) - u(x) = 0$, i.e. for the age $x = a$.

Thus, we have (20) and simultaneously the unicity. □

4 Quasivariational inequalities for the average cost

The transition from the discounted cost to the average cost is effected by the limit passage in the discount-factor $\beta \to 0_+$.

Denote $w(x) = \lim_{\beta \to 0_+} (u(x) - u(0))$, where $u(x)$ is the limit solution of the Bellman equation (5) by $\bar{r} \to \infty$. Then by (15) we get

$$w(x) = \begin{cases} \dfrac{-c_1 \int_0^x f(y)\,\mathrm{d}y + \left(\int_0^x \bar{F}(y)\,\mathrm{d}y\right) \lim\limits_{\beta \to 0_+} (\beta k)}{\bar{F}(x)\,.\,\mathrm{e}^{-\beta k}} & \text{for } x \in [0, a] \\ c_2 \quad \text{for } x \geq a. & \end{cases} \tag{24}$$

Obviously, $\int_0^x f(y)\,\mathrm{d}y = F(x) = 1 - \bar{F}(x)$.

We calculate $\lim\limits_{\beta \to 0_\mp} (\beta k)$, where $k = \beta^{-1}(c_1 - c_2)g(a) - c_2$ is the unique solution of (17). By $\beta \to 0_+$ from (17) we get

$$(c_1 - c_2)\left[\bar{F}(a_0) + g(a_0)\int_0^{a_0} \bar{F}(y)\,\mathrm{d}y\right] = c_1, \tag{25}$$

where $a_0 = \lim\limits_{\beta \to 0_+} a$.

Namely, from the definition of the average cost $\Theta(x)$ and from the condition that the optimal critical replacement age d minimizes the average cost, we get

$$\Theta'(d) = \frac{\left[(c_1 f(x) - c_2 f(x))\int_0^x \bar{F}(y)\,\mathrm{d}y - (c_1 F(x) + c_2 \bar{F}(x))\right]\bar{F}(x)}{\left(\int_0^x \bar{F}(y)\,\mathrm{d}y\right)^2}\Bigg|_{x=d} = 0.$$

Hence

$$(c_1 - c_2)\left[\bar{F}(d) + g(d)\int_0^d \bar{F}(y)\,\mathrm{d}y\right] = c_1, \tag{26}$$

which is the analogy of the equation (25) for $a_0 = d$. Moreover, we have from (26)

$$(c_1 - c_2)g(d) = \Theta(d) = \Theta$$

and for the limit

$$\lim_{\beta \to 0_+} (\beta k) = (c_1 - c_2)g(d) = \Theta. \tag{27}$$

We can write the relation (24) as

$$w(x) = \begin{cases} \left(-c_1 F(x) + \Theta \int_0^x \bar{F}(y)\,\mathrm{d}y\right)/\bar{F}(x) & \text{for } x \in [0, d] \\ c_2 \quad \text{for } x \geq d, & \end{cases} \tag{28}$$

where d is the unique solution of (26).

Now, we investigate the term $\beta u(x)$ in (I′) by $\beta \to 0_+$, i.e. the limit

$$\lim_{\beta \to 0_+} \beta u(x) =$$

$$= \begin{cases} \lim\limits_{\beta \to 0_+} [\beta k + \beta \dfrac{-c_1 \int_0^x f(y)\,\mathrm{e}^{-\beta y}\,\mathrm{d}y + \beta k \int_0^x \bar{F}(y)\,\mathrm{e}^{-\beta y}\,\mathrm{d}y}{\bar{F}(x)\,.\,\mathrm{e}^{-\beta x}} & \text{for } x \in [0, d] \\ \lim\limits_{\beta \to 0_+} \beta(c_2 + k) \quad \text{for } x \geq d. & \end{cases}$$

Obviously, by (27) we have

$$\lim_{\beta \to 0_+} \beta u(x) = \Theta \quad \text{for } x \in [0, \infty). \tag{29}$$

Thus, the following theorem holds:

Theorem 3. *The quasivariational inequalities for the average cost*

$$\text{(I)} \;\; w'(x) + g(x)(c_1 - w(x)) - \Theta \geq 0$$

$$\text{(II)} \;\; c_2 - w(x) \geq 0$$

$$\text{(III)} \;\; (c_2 - w(x))(w'(x) + g(x)(c_1 - w(x)) - \Theta) = 0,\; w(0) = 0,$$

have a solution

$$w(x) = \begin{cases} \left(-c_1 F(x) + \Theta \int_0^x \bar{F}(y)\,\mathrm{d}y\right)/\bar{F}(x) & \text{for } x \in [0, d] \\ c_2 \quad \text{for } x \geq d, & \end{cases}$$

$$\Theta = (c_1 F(d) + c_2 \bar{F}(d)/\int_0^d \bar{F}(y)\,\mathrm{d}y,$$

where d is the (unique) solution of the equation

$$(c_1 - c_2)g(d) = (c_1 F(d) + c_2 \bar{F}(d))/\int_0^d \bar{F}(y)\,\mathrm{d}y.$$

Theorem 4. *Number Θ for which there exists a bounded function $w(x)$ satisfying the conditions* (I)—(III) *of Theorem* 3 *is unique.*

Proof. Let $(w(x), \Theta)$ be a bounded solution of the quasivariational inequalities (I)—(III). Then for any replacement policy Z

$$w(X_T^+) - w(X_0) = \int_0^T w'(X_t)\,\mathrm{d}t - \int_0^T w(X_t)\,\mathrm{d}(N_t^1 + N_t^2). \tag{30}$$

From the definition of the cost and by (30) we have

$$C_T = \Theta T - w(X_T^+) + w(X_0) + \int_0^T (c_1 - w(X_t))(\mathrm{d}N_t^1 - g(X_t)\,\mathrm{d}t) +$$

$$+ \int_0^T [w'(X_t) + g(X_t)(c_1 - w(X_t)) - \Theta]\,\mathrm{d}t + \int_0^T (c_2 - w(X_t))\,\mathrm{d}N_t^2, \quad T \geq 0. \quad (31)$$

Since $w(x)$ is bounded, the first integral in (31) is a martingale which fulfils the law of large numbers (see [6], Lemma 3), i.e.

$$\lim_{T \to \infty} T^{-1} \int_0^T (c_1 - w(X_t))(\mathrm{d}N_t^1 - g(X_t)\,\mathrm{d}t) = 0 \quad \text{a.s.}$$

The second and the third integrals are nonnegative; this follows from the quasivariational inequalities (I)—(III). Hence, from (31) we get

$$\lim_{T \to \infty} T^{-1} C_T \geq \Theta \quad \text{a.s.}$$

Introduce

$$c = \inf\{x\colon w(x) = c_2\} \quad (32)$$

and let Z be the policy with replacement age c. From (31) and from (III) it follows that under Z

$$\lim_{T \to \infty} T^{-1} C_T = \Theta \quad \text{a.s.}$$

We conclude with regard to (32) that $\Theta = \inf_x \Theta(x) = \Theta(d)$. □

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Received: 30. 10. 1986

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SÚHRN

O JEDNEJ KVÁZIVARIAČNEJ NEROVNOSTI

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V tejto práci sa študuje proces obnovy s preventívnou výmenou. Optimálnosť stratégie spočíva v minimalizácii priemerného nákladu metódou kvázivariačných nerovností. Sleduje sa existencia a jednoznačnosť riešenia týchto nerovností.

РЕЗЮМЕ

ОБ ОДНОМ КВАЗИВАРИАЦИОННОМ НЕРАВЕНСТВЕ

Валерия Скриванкова, Кошице

В этой работе изучается процесс восстановления с предворительной заменой. Оптимальность стратегии замены заключается в минимизации средней стоимости методом квазивариационных неравенств.

Исследуется существование и единозначность решения этих неравенств.