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A NOTE ON GENERALIZED TOPOLOGIES

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By a generalized topology we understand any topology obtained by replacing the Kuratowski closure axioms by some weaker ones. These generalized topologies occur in various branches of mathematics and have been studied by many authors — see [2]. In the present contribution some systems of generalized topologies on a given set are investigated from the point of view of the theory of lattices. Although the results of this note are obtained in quite a simple way from those of [2], it is useful to publish them explicitly because of their general set-theoretical signification.

By a *topology without axioms* (briefly a *topology*) u on a set P we mean a mapping u: $\exp P \to \exp P$. In the literature, topologies fulfilling some of the following axioms are often studied:

$u\emptyset = \emptyset$	*O-axiom,*
$X \subseteq P \Rightarrow X \subseteq uX$	*I-axiom,*
$X \subseteq Y \subseteq P \Rightarrow uX \subseteq uY$	*M-axiom,*
$X, Y \subseteq P \Rightarrow u(X \cup Y) \subseteq uX \cup uY$	*A-axiom,*
$\emptyset \neq X \subseteq P \Rightarrow uX \subseteq \bigcup_{x \in X} u\{x\}$	*S-axiom,*
$X \subseteq P \Rightarrow uuX \subseteq uX$	*U-axiom.*

If $f \in \{O, I, M, A, S, U\}$ and if a topology u fulfils the f-axiom, then u is called an *f-topology*. If also $g \in \{O, I, M, A, S, U\}$ and u is both an f-topology and g-topology, then it is called an *fg-topology*, etc. The system of all topologies on P is denoted by $\mathscr{P}$. By $\mathscr{P}_f$ we denote the system of all f-topologies on P, by $\mathscr{P}_{fg}$ the system of all fg-topologies on P, etc. All these systems will be considered as ordered by the relation $\leqq$ defined as usual, i.e. $u \leqq v \Leftrightarrow uX \subseteq vX$ for any subset $X \subseteq P$. It is well known that $\mathscr{P}$ is a complete lattice (even a completely distributive complete Boolean algebra). By $\bigvee$ and $\bigwedge$ we denote the join and meet in $\mathscr{P}$. Clearly, if $\emptyset \neq \mathscr{U} \subseteq \mathscr{P}$, then $(\bigvee \mathscr{U})X = \bigcup_{u \in \mathscr{U}} uX$ and $(\bigwedge \mathscr{U})X = \bigcap_{u \in \mathscr{U}} uX$ for every subset $X \subseteq P$. Next, for $u \in \mathscr{P}$ by $(u]$ and $[u)$ we denote the principal and principal dual ideals of $\mathscr{P}$ generated by u, i.e. $(u] = \{v \in \mathscr{P} | v \leqq u\}$ and $[u) = \{v \in \mathscr{P} | u \leqq v\}$.

Theorem 1. Let $u \in \mathscr{P}$ be a topology. Then $u \in \mathscr{P}_A$ iff $\mathscr{P}_A \cap [u)$ is a complete lattice.

Proof. It can be easily seen that $\mathscr{P}_A$, and thus also $\mathscr{P}_A \cap [u)$, are complete join subsemilattices of $\mathscr{P}$. In [2] it is shown that $u = \bigwedge (\mathscr{P}_A \cap [u))$. Thus, we have $u \in \mathscr{P}_A \Leftrightarrow u \in \mathscr{P}_A \cap [u) \Leftrightarrow \bigwedge (\mathscr{P}_A \cap [u)) \in \mathscr{P}_A \cap [u) \Leftrightarrow \mathscr{P}_A \cap [u)$ has the least element $\Leftrightarrow \mathscr{P}_A \cap [u)$ is a complete lattice. The proof is complete.

Theorem 2. Let $u \in \mathscr{P}_M$. Then

(1) $u \in \mathscr{P}_S$ iff $\mathscr{P}_S \cap [u)$ is a complete lattice,

(2) $u \in \mathscr{P}_U$ iff $\mathscr{P}_{MU} \cap (u]$ is a complete lattice.

Proof. Again, it can bee easily seen that $\mathscr{P}_S$, and thus also $\mathscr{P}_S \cap [u)$, are complete join subsemilattices of $\mathscr{P}$ and that $\mathscr{P}_{MU}$, and thus also $\mathscr{P}_{MU} \cap (u]$, are complete meet subsemilattices of $\mathscr{P}$. In [2] it is shown that $u = \bigwedge (\mathscr{P}_S \cap [u))$ and that $u = \bigvee (\mathscr{P}_{MU} \cap (u])$. Hence, the proof of Theorem 2 is analogous to that of Theorem 1.

Theorem 3. Let $f \in \{O, I, M, OI, OM, IM, OIM, OIMA\}$ and let $u \in \mathscr{P}_f$. Then

(1) $u \in \mathscr{P}_{fA}$ iff $\mathscr{P}_{fA} \cap [u)$ is a complete lattice, whenever $f \in \{O, I, OI, OIM\}$,

(2) $u \in \mathscr{P}_{fS}$ iff $\mathscr{P}_{fS} \cap [u)$ is a complete lattice, whenever $f \in \{M, OM, IM, OIM\}$,

(3) $u \in \mathscr{P}_{fU}$ iff $\mathscr{P}_{fU} \cap (u]$ is a complete lattice, whenever $f \in \{M, OM, OIM, OIMA\}$.

Proof. Excluding the case $f = OIMA$ of (3), the proofs of the assertions of Theorem 3 can be performed analogously to the proof of Theorem 1 by the help of the results of [2]. For $f = OIMA$ the assertion (3) immediately follows from [1], 2.4.

Remark. Obviously $\mathscr{P}_A$, $\mathscr{P}_S$, $\mathscr{P}_{MU}$ and for $f \in \{O, I, M, OI, OM, IM, OIM, OIMA\}$ also $\mathscr{P}_{fA}$, $\mathscr{P}_{fS}$, $\mathscr{P}_{fMU}$ are complete lattices and the joins and meets of them are described in [2]. In [2] it is also shown that $\mathscr{P}_U$ is not even a semilattice in general.

REFERENCES

1. Rosický, J.: Topologies compatible with ordering, Scripta Fac. Sci. Nat. UJEP Brunensis, Mathematica 1, 1971, 9—23.
2. Šlapal, J.: On lattices of generalized topologies, Math. Slovaca, 38 (1988), 51—59.

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SÚHRN

POZNÁMKA O ZOBECNENÝCH TOPOLÓGIÁCH

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Zobecnenou topológiou rozumieme každú topológiu, ktorú získame zoslabením Kuratowského uzáverových axiómov. V článku sa vyšetrujú systémy niektorých zobecnených topológii na danej množine z hľadiska teórie zväzov. Dosiahnuté výsledky majú obecný množinovo-teoretický význam.

РЕЗЮМЕ

ЗАМЕЧАНИЕ ОБ ОБОБЩЕННЫХ ТОПОЛОГИЯХ

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Обобщенной топологией мы понимаем каждую топологию, которую мы получим ослаблением аксиомов замыкания Куратовского. В замечании изучаются системы некоторых обобщенных топологий на данном множестве с точки зрения теории решеток. Достигнутые результаты имеют общее множественно-теоретическое значение.