UNIVERSITAS COMENIANA
ACTA MATHEMATICA UNIVERSITATIS COMENIANAE
LIV—LV—1988

CRITERIA FOR DISCONJUGACY OF A DIFFERENTIAL EQUATION WITH DELAY

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In [2], the autor has introduced the notion of a disconjugate differential equation with delay

$$x''(t) + N(t)x(t) + M(t)x(t - \Delta(t)) = 0 \tag{1}$$

and has proved that (1) is disconjugate on an interval I if and only if each boundary value problem associated with this equation has exactly one solution. In this paper we shall prove some criteria for disconjugacy of a differential equation with delay.

In the sequel we shall need the following theorem, which will enable us to compare the solution of (1) satisfying the initial conditions

$$\begin{gathered} x(A) = x_A, \quad x'(A) = x'_A \\ x(t - \Delta(t)) = \phi(t - \Delta((t)), \quad \text{if} \quad t - \Delta(t) < A \end{gathered} \tag{2}$$

with the solution of the differential equation

$$y''(t) + N(t)y(t) + M(t)y(t - \tau(t)) = 0, \tag{3}$$

satisfying the initial conditions

$$\begin{gathered} y(A) = y_A, \quad y'(A) = y'_A \\ y(t - \tau(t)) = \psi(t - \tau(t)), \quad \text{if} \quad t - \tau(t) < A. \end{gathered} \tag{4}$$

We shall suppose that the coefficients $N(t)$, $M(t)$ and the delays $\Delta(t) \geqslant 0$, $\tau(t) \geqslant 0$ are continuous functions on an interval $\langle a, b)$, $b \leqslant +\infty$ and that the functions $\phi(t)$, $\psi(t)$ are defined and continuous on the initial sets

$$E_A = \{t - \Delta(t)\colon t - \Delta(t) < A \text{ and } t \in \langle A, b)\} \cup \{A\}$$

and

$$E'_A = \{t - \tau(t)\colon t - \tau(t) < A,\ t \in \langle A, b)\} \cup \{A\}$$

respectively. This theorem is proved in [3] for the case of the differential equations of the form

$$x_i''(t) = \int_0^{\sigma_i(t)} x_i(t - \tau)\,\mathrm{d}r_i(t, \tau), \quad a \leqslant t < +\infty, \quad i = 1, 2$$

under some additional assumptions. The proof of our theorem is analogous to that of the above mentioned result.

Theorem 1. Suppose that $x(t)$ and $y(t)$ are solutions of (1) and (3) respectively, satisfying the initial conditions (2) and (4), respectively.
Let

$$M(t) \geqslant 0 \quad \text{for} \quad t \in \langle A, b), \tag{5}$$

$$\Delta(t) \geqslant \tau(t) \geqslant 0 \quad \text{for } t \in \langle A, b) \tag{6}$$

and

$$\phi(t) \leqslant \psi(\bar{t}) \quad \text{for} \quad t \leqslant \bar{t} \quad (t, \bar{t} \in E_A \cap E'_A), \tag{7}$$

$$x(A) = \phi(A) = \psi(A) = y(A) \geqslant 0, \quad x'(A) \geqslant y'(A). \tag{8}$$

Finally, let

$$y'(t) > 0 \quad \text{for} \quad t \in \langle A, b).$$

Then

$$\frac{x'(t)}{x(t)} \geqslant \frac{y'(t)}{y(t)} \quad \text{for} \quad t \in (A, b), \tag{9}$$

and

$$x'(t) \geqslant y'(t), \quad \text{for} \quad t \in \langle A, b).$$

Now we are able to prove the following criterion.

Theorem 2. Suppose that the differential equation

$$y''(t) + p(t)y(t) = 0 \tag{10}$$

is disconjugate on an interval $\langle a, +\infty)$ and the solution satisfying the initial conditions

$$y_o(a, a) = 0, \quad y'_o(a, a) = 1$$

satisfies the inequality

$$y'_o(t, a) > 0 \quad \text{for } t \in \langle a, +\infty).$$

Then the differential equation

$$x''(t) + N(t)x(t) + M(t)x(t - \Delta(t)) = 0,$$

where

$$M(t) \geqslant 0, \quad N(t) + M(t) = p(t), \tag{11}$$

is disconjugate on $\langle a, +\infty)$.

Proof. Let the assumptions of the theorem be satisfied. Then the equation (2), with $\tau(t) \equiv 0$ is disconjugate on the interval $\langle a, +\infty)$ and

$$y_o'(t, a) > 0, \quad \text{for} \quad t > a$$

holds.

Further,

$$y_o(t, A) = y_o(A, a)\left[y_o(t, a)\int_A^t \frac{\mathrm{d}s}{y_o^2(s, a)}\right], \quad \text{for} \quad t \geqslant A\,(a < A < +\infty).$$

Thus

$$y_o'(t, A) = y_o(A, a)\left[y_o'(t, a)\int_A^t \frac{\mathrm{d}s}{y_o^2(s, a)} + y_o(t, a)\frac{1}{y_o^2(t, a)}\right] > 0 \quad \text{for} \quad t \geqslant A.$$

The assertion of the theorem follows now from Theorem 1.

Example 1. The differential equation

$$y''(t) = 0$$

is disconjugate on an interval $\langle a, +\infty)$, $a \in R$ and the coefficients of the equation

$$x''(t) - \frac{1}{2}x(t) + \frac{1}{2}x(t - 3\pi) = 0 \tag{12}$$

fulfil (11). Thus the assumptions of Theorem 2 are fulfilled, therefore equation (12) is disconjugate on $\langle a, +\infty)$.

Example 2. The differential equation

$$x''(t) + \frac{1}{2}x(t) - \frac{1}{2}x(t - 3\pi) = 0, \tag{13}$$

for $a = 0$ has the solution

$$x_o(t, 0) = \sqrt{2}\sin\frac{\sqrt{2}}{2}t, \quad \text{for } t \geqslant 0$$

and thus (13) is disconjugate at most on the interval $\langle 0, \sqrt{2}\pi)$. (On this interval it is equivalent to the differential equation without delay. Thus it is really disconjugate on $\langle 0, \sqrt{2}\pi)$.) But in this case the inequality from (11) is not fulfilled. Thus the assumption $M(t) \geqslant 0$ in Theorem 2 cannot be omitted.

In the next theorem we shall consider the differential equation of the form

$$x''(t) + k(t)x'(t) + l(t)x(t) + m(t)x(t - \Delta(t)) = 0, \tag{15}$$

where $k(t)$, $l(t)$, $m(t)$ and $\Delta(t) \geqslant 0$ are continuous functions on an interval $\langle a, b)$, $b \leqslant +\infty$. The initial value problem for (15) as well as the definition of the disconjugacy are the same as for (1).

Theorem 3. Suppose that

$$l(t) \leqslant 0, \quad m(t) \leqslant 0 \quad \text{for} \quad t \in \langle a, b).$$

Then the differential equation (15) is disconjugate on $\langle a, b)$.

Proof. Let us multiply the equation (15) by the function $\exp\left(\int_a^t k(s)\,\mathrm{d}s\right) > 0$. Then we get

$$\left[\exp\left(\int_a^t k(s)\,\mathrm{d}s\right)x'(t)\right]' + N(t)x(t) + M(t)x(t - \Delta(t)) = 0, \tag{16}$$

where

$$N(t) = l(t)\exp\left(\int_a^t k(s)\,\mathrm{d}s\right) \leqslant 0 \tag{17}$$

$$M(t) = m(t)\exp\left(\int_a^t k(s)\,\mathrm{d}s\right) \leqslant 0.$$

It is easy to see that it is sufficient to show that the equation (16) with the coefficients satisfying (17) is disconjugate on the interval $\langle a, b)$.

We have to prove that no solution $x_o(t, A)$, $A \in \langle a, b)$ of (16) has any zero on (A, b). For this purpose, let us multiply the equation (16) by $x_o(t, A)$ and then integrate it by parts from A to t. We get

$$x_o(t, A)x_o'(t, A)\exp\left(\int_a^t k(s)\,\mathrm{d}s\right) =$$

$$= -\int_A^t [N(s)x_o^2(s, A) + M(s)x_o(s, A)x_o(s - \Delta(s), A)]\,\mathrm{d}s +$$

$$+ \int_A^t \exp\left(\int_a^s k(\tau)\,\mathrm{d}\tau\right)(x_o'(s, A))^2\,\mathrm{d}s. \tag{18}$$

Suppose now, on the contrary, that $x_o(t, A)$ has a zero in (A, b). Then there is a point $T \in (A, b)$ such that

$$x_o'(T, A) = 0, \quad x_o'(t, A) > 0 \quad \text{for} \quad t \in \langle A, T)$$

and with respect to the initial conditions fulfilled by $x_o(t, A)$ we have

$$x_o(t, A) > 0, \quad x_o(t - \Delta(t), A) \geqslant 0 \quad \text{for} \quad t \in (A\ , T\rangle.$$

By this fact and by the assumptions of the theorem we get that the left-hand side of (18) is equal to zero at the point $t = T$, while the right-hand side of (18) at $t = T$ is a positive number. This is a contradiction.

With the aid of Theorem 3 we shall prove

Theorem 4. Suppose that the equation

$$x''(t) + N(t)x(t) = 0 \tag{19}$$

is disconjugate on an interval $\langle a, b)$ and

$$M(t) \leqslant 0 \quad \text{for } t \in \langle a, b). \tag{20}$$

Then the equation (1) is disconjugate on $\langle a, b)$.

Proof. We shall prove this theorem by contradiction. Let the equation (1) be not disconjugate on $\langle a, b)$. Then there is a point $t = A$ such that for the first conjugate point c_A to the point A (i.e. for the first zero of $x_o(t, A)$) with respect to the equation (1) we have

$$c_A < b. \tag{21}$$

On the other hand, since the equation (19) is disconjugate on $\langle A, B\rangle$ ($c_A < B < b$), there is a solution $v(t)$ of (19) such that

$$v(t) \neq 0 \quad \text{for} \quad t \in \langle A, B\rangle.$$

Let $u(t)$ be the function defined by

$$u(t) = \begin{cases} v(t) \text{ for } t \in \langle A, B\rangle \\ \max(0, v(t)) \quad \text{for} \quad t < A. \end{cases} \tag{22}$$

Then the differential edquation (1) is transformed by the substitution

$$x = u(t)y$$

into the differential equation

$$y''(t) + 2\frac{u'(t)}{u(t)}y'(t) + M(t)\frac{u(t - \Delta(t))}{u(t)}y(t - \Delta(t)) = 0 \tag{23}$$

for $t \in \langle A, B)$.

By the definition of $u(t)$ and by (20), we have

$$M(t)\frac{u(t - \Delta(t))}{u(t)} \leqslant 0 \quad \text{for} \quad t \in \langle A, B\rangle.$$

Thus the assumptions of Theorem 3 are fulfilled and hence the equation (23) is disconjugate on $\langle A, B)$. But this is a contradiction with the definition of the point B.

Theorem 5. Suppose that $N(t)$, $M(t)$ are nonnegative functions defined on the interval $\langle a, +\infty)$ and

$$\int_a^{+\infty} [(s-a)N(s) + (s - \Delta(s) - a)^+ M(s)]\,\mathrm{d}s < 1, \tag{24}$$

where $(s - \Delta(s) - a)^+ = \max(0, s - \Delta(s) - a)$.
Then the equation (1) is disconjugate on the interval $\langle a, +\infty)$.

Proof. By integrating the equation (1) from A to t we get

$$x_o'(t, A) = 1 - \int_A^t [N(s)x_o(s, A) + M(s)x_o(s - \Delta(s), A)]\,\mathrm{d}s$$

for $t \geqslant A$ and hence follows the inequality

$$x_o(t, A) \leqslant (t - A) \quad \text{for} \quad a \leqslant A \leqslant t \leqslant c_A \tag{25}$$

if there is a conjugate point c_A to the point A. If such a point does not exist, then (25) holds for $t \geqslant A$.
Further, by (25) and the initial condition fulfilled by $x_o(t, A)$, we have

$$x_o(t - \Delta(t), A) \leqslant (t - \Delta(t) - A)^+. \tag{26}$$

Now we shall prove the theorem by contradiction. Suppose that the assumptions of the theorem are fulfilled and the equation (1) is not disconjugate on the interval $\langle a, +\infty)$. Then there is a point $A \in \langle a, +\infty)$ such that its conjugate point c_A lies in the interval $(A, +\infty)$. By Rolle's Theorem, there is a point

$$T \in (A, c_A)$$

such that

$$x_o'(T, A) = 0. \tag{27}$$

On the other hand,

$$x_o'(T, A) = 1 - \int_A^T [N(s)x_o(s, A) + M(s)x_o(s - \Delta(s), A)]\,\mathrm{d}s,$$

which by (25) and (26) gives

$$x_o'(T, A) \geqslant 1 - \int_A^T [N(s)(s - A) + M(s)(s - \Delta(s) - A)^+]\,\mathrm{d}s.$$

From this fact, since $a \leqslant A < T < c_A$, it follows that

$$x_o'(T, A) \geqslant 1 - \int_a^T (N(s)(s - a) + M(s)(s - \Delta(s) - a)^+] \,ds \geqslant$$

$$\geqslant 1 - \int_a^{+\infty} [N(s)(s - a) + M(s)(s - \Delta(s) - a)^+] \,ds.$$

The last inequality together with the inequality (24) yields

$$x_o'(T, A) > 0$$

which is a contradiction with (27).

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Received: 11. 6. 1986

SÚHRN

KRITÉRIA DISKONJUGOVANOSTI PRE DIFERENCIÁLNU ROVNICU S ONESKORENÍM

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V práci sú dokázané kritéria diskonjugovanosti pre diferenciálnu rovnicu tvaru

$$x''(t) + N(t)x(t) + M(t)x(t - \Delta(t)) = 0. \quad (1)$$

Pojem diskonjugovanej diferenciálnej rovnice tvaru (1) bol zavedený v práci [2], kde je aj dokázaný vzťah medzi diskonjugovanosťou diferenciálnej rovnice (1) a existenciou riešenia okrajových úloh pre túto rovnicu.

РЕЗЮМЕ

ДОСТАТОЧНЫЕ УСЛОВИЯ ДЛЯ ТОГО, ЧТОБЫ ДИФФЕРЕНЦИАЛЬНОЕ УРАВНЕНИЕ С ЗАПАЗДЫВАНИЕМ БЫЛО БЕЗ СОПРЯЖЁННЫХ ТОЧЕК

Александер Хащак, Братислава

В работе даны достаточные условия для того, чтобы дифференциальное уравнение вида

$$x''(t) + N(t)x(t) + M(t)x(t - \Delta(t)) = 0 \tag{1}$$

было без сопряжённых точек. Понятие дифференциального уравнения вида (1) без сопряжённых точек было введено в работе [2], в которой также доказана связь между свойством, что дифференциальное уравнение (1) является без сопряжённых точек, и существованием решения краевых задач для этого уравнения.