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DISCONJUGACY OF DIFFERENTIAL EQUATIONS WITH DELAY

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The notion of disconjugacy plays an important role in the theory of ordinary differential equations. There are many papers and one nice monograph (see [1] and the References in it) devoted to these topics. The corresponding theory for the differential equations with delay has not built up yet. The purpose of this paper is to generalize the notions of a conjugate point, and a disconjugate differential equation, and to show that the interval of disconjugacy of each differential equation with delay does not degenerate into a one-point set (a generalization of de la Valée Poussion's theorem) and to show the connection between the disconjugacy of the differential equation with delay and the solvability of the boundary value problem.

In this paper we shall deal with the differential equation

$$x''(t) + N(t)x(t) + M(t)x(t - \Delta(t)) = 0, \tag{1}$$

where $N(t)$, $M(t)$ and $\Delta(t) \geqslant 0$ are defined and continuous on the interval $\langle a, b)$, $(b \leqslant \infty)$.

The underlying initial value problem for this differential equation is defined as follows (see [11]):

Let the continuous function $\phi(t)$ be deffined on the initial set

$$E_a = \{t - \Delta(t)\colon t - \Delta(t) < a \quad \text{and} \quad t \in \langle a, b)\} \cup \{a\}.$$

Let $x_a = \phi(a)$ and let x'_a be an arbitrary real number. We have to find the solution $x(t)$ of (1) satisfying

$$\begin{gathered} x(a) = x_a, \quad x'(a+) = x'_a, \\ x(t - \Delta(t)) = \phi(t - \Delta(t)), \quad \text{as} \quad t - \Delta(t) < a. \end{gathered} \tag{2}$$

By the derivative at the end point a of the interval $\langle a, b)$ we shall mean the right-hand point derivative and instead of $x'(a+)$ we shall simply write $x'(a)$.

Under the above assumptions, the initial value problem (1), (2) has exactly one solution on the interval $\langle a, b)$ (see [11] p. 20), which we shall denote by $x_\phi(t, a, x_a, x'_a)$.

Since some special solutions will often occur, we shall use a short notation:

By $x_1(t, a)$ we shall denote the solution of (1) which satisfies the initial conditions

$$x_1(a, a) = 1, \quad x'_1(a, a) = 0,$$
$$x_1(t - \Delta(t), a) \equiv 1, \quad \text{if} \quad t - \Delta(t) < a.$$

Further, by $x_0(t, a)$ we shall denote the solution of (1) which satisfies the initial conditions

$$\begin{aligned} x_o(a, a) = 0, \quad x'_o(a, a) = 1, \\ x_o(t - \Delta(t), a) \equiv 0, \quad \text{if} \quad t - \Delta(t) < a. \end{aligned} \tag{2'}$$

Finally, by $x_\varphi(t, a)$ we shall denote the solution of (1) which satisfies the initial conditions

$$x_\varphi(a, a) = 0, \quad x'_\varphi(a, a) = 0,$$
$$x(t - \Delta(t), a) \equiv \varphi(t - \Delta(t)) \equiv \phi(t - \Delta(t)) - \phi(a),$$

if $t - \Delta(t) < a$.

It is easy to see that the relation

$$x_\phi(t, a, x_a, x'_a) \equiv x_a x_1(t, a) + x'_a x_o(t, a) + x_\varphi(t, a) \tag{3}$$

holds.

Further, it is easy to prove

Theorem 1. The set $B(a)$ of all solutions of (1) with the initial function $\phi(t) \equiv 0$ and $x'_a \in (-\infty, +\infty)$ is a one-dimensional vector space.

Definition 1. Let $A \in \langle a, b)$. By conjugate points to the point A with respect to (1) we shall mean the zeros of the solution $x_o(t, A) \in B(A)$ which are on the right of A. A point c^i_A $(A < c^i_A)$ will be called the i-th conjugate point to the point A with respect to (1) iff it is the i-th zero of the solution $x_o(t, A)$ on the right of A.

The following theorem gives an estimate for the distance between the point A and its first conjugate point c^1_A, and is a generalization of the well-known de la Valle'e Poussin's theorem from the theory of ordinary differential equations.

Theorem 2. Let J be a subinterval of the interval $\langle a, b)$ and assume that the inequalities

$$|N(t)| \leq N, \ |M(t)| \leq M, \ 0 \leq \Delta(t) \leq \Delta \quad \text{for} \quad t \in \langle a, b) \tag{4}$$

hold.

Suppose that $A \in J$ and $c_A^1 \in J$. Then for the distance $h = c_A^1 - A$ of the points A, c_A^1 we have

$$h \geqslant \frac{-3M\Delta + \sqrt{9\Delta^2 M^2 + 24(N+M)}}{2(N+M)} = d \tag{5}$$

Proof. Let us observe that, for an arbirary function $u(t)$ with a continuous derivative, the identity

$$h\,u(t) = \int_0^t s\,u'(s)\,\mathrm{d}s - \int_t^h (h-s)u'(s)\,\mathrm{d}s + \int_o^h u(s)\,\mathrm{d}s \tag{6}$$

holds on the interval $\langle 0, h\rangle$.

To prove (6), it suffices to calculate the first and second integrals on the right-hand side of the identity by parts.

Without loss of generality we shall assume that $A = 0$. Then $c_A^1 = h$. Apply now the identity (6) to the function $u(t) = x_o'(t, 0)$. Since

$$\int_o^h x_o'(s, 0)\,\mathrm{d}s = x_o(h, 0) - x_o(0, 0) = 0,$$

by (6) we get

$$h\,x_o'(t, 0) = \int_o^t s\,x_o''(s, 0)\,\mathrm{d}s - \int_t^h (h-s)x_o''(s, 0)\,\mathrm{d}s$$

and, because $x_o(s, 0)$ is a solution of (1), we have

$$\begin{aligned} h\,x_o'(t, 0) = &- \int_o^t s(N(s)x_o(s, 0) + M(s)x_o(s - \Delta(s), 0))\,\mathrm{d}s + \\ &+ \int_t^h (h-s)(N(s)x_o(s, 0) + M(s)x_o(s - \Delta(s), 0))\,\mathrm{d}s. \end{aligned} \tag{7}$$

Denote by

$$k = \max_{t \in \langle 0,\, h\rangle} |x_o'(t, 0)| > 0.$$

Then, since $t = 0$ and $t = h$ are zeros of $x_o(t, 0)$, the inequalities

$$|x_o(s, 0)| \leqslant k\,s, \quad \text{for} \quad s \in \langle 0, h\rangle$$

and

$$|x_o(s, 0)| \leqslant k(h - s), \quad \text{for}\, s \in \langle 0, h\rangle$$

hold. From this fact we have

$$0 \leqslant s\,x_o(s, 0) \leqslant s\,k(h - s) \quad \text{for} \quad s \in \langle 0, h\rangle \tag{8}$$

and

$$0 \leqslant (h - s)x_o(s, 0) \leqslant sk(h - s) \quad \text{if} \quad s \in \langle 0, h\rangle. \tag{9}$$

Now, as the solution $x_o(t, 0)$ satisfies the initial conditions (2′) with $a = 0$, we get

$$0 \leqslant s\,x_o(s - \Delta(s), 0) \leqslant s\,k(h - s + \Delta(s)) \quad \text{for} \quad s \in \langle 0, h\rangle \tag{10}$$

and

$$0 \leqslant (h - s)x_o(s - \Delta(s), 0) \leqslant (h - s)k|s - \Delta(s)| \quad \text{for}\, s \in \langle 0, h\rangle.$$

From this we deduce:

If $s - \Delta(s) \geqslant 0$ for $s \in \langle 0, h\rangle$, then

$$0 \leqslant (h - s)x_o(s - \Delta(s), 0) \leqslant s\,k(h - s + \Delta(s)) \quad \text{for} \quad s \in \langle 0, h\rangle. \tag{11}$$

Because of (2′), this inequality also holds for $s - \Delta(s) \leqslant 0$. Now, by (7) owing to inequalities (8), (9), (10), (11) and (4), we obtain

$$h|x_o'(t, 0)| \leqslant k\,N \int_o^t s(h - s)\,\mathrm{d}s + k\,M \int_o^t s(h - s + \Delta)\,\mathrm{d}s +$$

$$+ k\,N \int_t^h s(h - s)\,\mathrm{d}s + k\,M \int_t^h s(h - s + \Delta)\,\mathrm{d}s =$$

$$= k\,N \int_o^h s(h - s)\,\mathrm{d}s + k\,M \int_o^h s(h - s)\,\mathrm{d}s + k\,M\Delta \int_o^h s\,\mathrm{d}s,$$

or

$$h|x_o'(t, 0)| \leqslant k\,(N + M)\frac{h^3}{6} + k\Delta M\frac{h^2}{2}, \quad \text{for} \quad t \in \langle 0, h\rangle.$$

From this we have

$$|x_o'(t, 0)| \leqslant \frac{k(N + M)}{6}h^2 + \frac{k\Delta M}{2}h, \quad \text{for} \quad t \in \langle 0, h\rangle. \tag{12}$$

Inequality (12) also holds at the point at which

$$|x_o'(t, 0)| = k$$

(such a oint exists in $\langle 0, h\rangle$) and if we divide this inequality by $k > 0$, we get

$$1 \leqslant \frac{N + M}{6}h^2 + \frac{\Delta M}{2}h,$$

i. e.

$$\frac{N + M}{6}h^2 + \frac{\Delta M}{2}h - 1 \geqslant 0.$$

Therefore, h must not be from the interior of the interval the end point of which are the roots of the equation

$$\frac{N+M}{6}t^2 + \frac{\Delta M}{2}t - 1 = 0,$$

i. e.

$$\frac{-3\Delta M - \sqrt{9\Delta^2 M^2 + 24(N+M)}}{2(N+M)},$$

$$\frac{-3\Delta M + \sqrt{9\Delta^2 M^2 + 24(N+M)}}{2(N+M)}.$$

From this fact we have the conclusion of Theorem 2.

Definition 2. The equation (1) is said to be disconjugate on the interval $J(J \subset \langle a, b))$, iff

$$A \in J \Rightarrow c_A^1 \notin J.$$

As a consequence of Theorem 2, we have

Theorem 3. The differential equation (1) is disconjugate on every interval $J \subset \langle a, b)$ whose length is less that $d(d$ is defined in (5)).

Suppose now that $A \in \langle a, b)$ and $\varphi(t)$ is a given initial function defined on E_A and such that $\varphi(A) = 0$. Let us consider the following set of solutions of (1)

$$\{x(t)\colon x(t) = x(t, \alpha)\colon = \ x_\varphi(t, A) + \alpha x_o(t, A),\ \alpha \in (-\infty, +\infty)\}. \tag{13}$$

It is easy to see that $x(t) \in (13)$ satisfies the conditions

$$x(A) = 0, \quad x'(A) = \alpha.$$

Lemma 1. Two different solutions from (13) intersect each other in the points

$$(c_A^i,\ x(x_A^i, A))$$

only.

Lemma 2. By every point of the stripe

$$P_i = \{(t, x)\colon c_A^i < t < c_A^{i+1},\ -\infty < x < +\infty\},$$

$i = 0, 1, 2, \ldots$ (where $c_A^o = A$)

passes exactly one solution of (1) from (13).

Proof. Let $(t_o, x_o) \in P$. This point lies on such a solution of (1) of the form (13) for which

$$\alpha = \frac{x_o - x_\varphi(t_o, A)}{x_o(t_o, A)}$$

$(x_o(t_o, A) \neq 0$ since $t_o \neq c_A^i)$. By Lemma 1 there is only one such solution.

Let us define now the boundary value problem for (1):

Let $a \leqslant a_1 < a_2 < b$ and x_1, x_2 be real numbers. Suppose that $\phi(t)$ is a continuous function defined on the initial set E_{a_1} and such that $\phi(a_1) = x_1$. We have to find a solution $x(t)$ of (1) defined on $\langle a_1, b)$ and satisfying

$$\begin{aligned} x(a_1) = x_1, \quad x(a_2) = x_2, \\ x(t - \Delta((t)) \equiv \phi(t - \Delta(t)), \quad \text{if} \quad t - \Delta(t) < a_1 . \end{aligned} \tag{14}$$

Lemma 3. Let $c^1_{a_1}$ be the first conjugate point to a_1 with respect to (1).

Suppose that $a \leqslant a_1 < a_2 < c_{a_1}$ and $\phi(t)$ is a continuous function defined on the initial set E_{a_1} satisfying $\phi(a_1) = x_1$. Then there is exactly one solution $x(t)$ of (1) satisfying (14).

Proof. By (3) each solution of (1) which fulfils the condition

$$x(a_1) = x_1; \quad x(t - \Delta(t)) = \phi(t - \Delta(t)), \quad \text{if} \quad t - \Delta(t) < a_1, \tag{14'}$$

can be written in the form

$$x(t) = x_1 x_1(t, a_1) + \gamma x_o(t, a_1) + x_\varphi(t, a_1), \tag{3'}$$

where γ is a real number.

Further we have

$$x(t) - x_1 x_1(t, a_1) = x_\varphi(t, a_1) + \gamma x_o(t, a_1) \in (13).$$

Now by Lemma 2 there is exactly one value γ such that $x(t)$ fulfils also the condition $x(a_2) = x_2$.

Theorem 4. The equation (1) is disconjugate on an interval $J \subset \langle a, b)$ if and only if every boundary value problem (1), (14) for $a_1, a_2 \in J$, $a_1 < a_2$ has exactly one solution.

Proof. *i*) If the equation (1) is disconjugate on an interval J, then the existence and the unicity of the solution of the given boundary value problem follows from Lemma 3.

ii) Let every boundary value problem (1), (14) for $a_1, a_2 \in J$ have exactly one solution. Suppose that the equation (1) is not disconjugate on the interval J. Then there exists a point $A \in J$ such that its first conjugate point c_A also lies in J. Let $\phi(t) \not\equiv 0$ be a continuous function defined on the initial set E_A such that $\phi(A) = 0$ (i.e. $\varphi = \phi$). Then the boundary value problem (1), (15)

$$\begin{aligned} x(A) = 0, \quad x(c_A) = x_2 \\ x(t - \Delta(t)) = \phi(t - \Delta(t)), \quad \text{if}\ t - \Delta(t) < A \end{aligned} \tag{15}$$

has infinitely many solutions (if $x_2 = x_\varphi(c_A, A)$) or no solution at all (if $x_2 \neq x_\varphi(c_A, A)$). But this is a contradiction with the assumption.

It is clear from the definition of the disconjugate ordinary differential equation that no solution of the disconjugate linear differential equation can oscillate. But this is not valid for disconjugate linear differential equations with delay as shown by the following example.

Example. The differential equation

$$y''(t) - M(t)y(t - \Delta(t)) = 0, \tag{16}$$

where $M(t) \geqslant 0$ for $t \geqslant a$ is disconjugate on the interval $\langle a, +\infty)$ due to the inequality

$$y_o(t, A) = (t - A) + \int_A^t (t - s)M(s)y_o(s - \Delta(s), A)\,\mathrm{d}s > 0 \quad \text{for} \quad t > A \geqslant a.$$

But if $M(t) \equiv 1$ and $\Delta(t) = \pi$, the equation (16) has oscillating solutions $\sin t$ and $\cos t$.

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SÚHRN

DISKONJUGOVANOSŤ DIFERENCIÁLNEJ ROVNICE S ONESKORENÍM

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Pojem diskonjugovanej obyčajnej lineárnej diferenciálnej rovnice je definovaný pomocou nulových bodov jej riešení. Zo zrejmých dôvodov nemožno formálne zovšeobecniť tento pojem pre rovnice s oneskoreným argumentom (nulové body riešenia závisia od začiatočnej funkcie). V práci je ukázané, že existuje akási množina riešení diferenciálnej rovnice

$$x''(t) + N(t)x(t) + M(t)x(t - \Delta(t)) = 0, \tag{1}$$

pre ktoré platí zovšeobecnená de la Valle'e Poussinova veta (je dokázaná v práci). Pomocou týchto riešení je zavedený pojem konjugovaného bodu ako aj pojem diskonjugovanej lineárnej diferenciálnej rovnice s oneskoreným argumentom na intervale J. V práci je ďalej dokázaná veta: Rovnica (1) je diskonjugovaná na intervale J práve vtedy ak každá okrajová úloha (1), (14) má práve jedno riešenie.

РЕЗЮМЕ

ДИФФЕРЕНЦИАЛЬНЫЕ УРАВНЕНИЯ С ЗАПАЗДЫВАНИЕМ БЕЗ СОПРЯЖЕННЫХ ТОЧЕК

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Понятие обыкновенного линейного дифференциального уравнения без сопряженных точек дается с помощью нулей их решений. По известным причинам невозможно формально расширить это понятие на случай уравнений с запаздывающим аргументом (нули решения зависимы от начальной функции). В работе показано, что существует некоторое множество решений дифференциального уравнения

$$x''(t) + N(t)x(t) + M(t)x(t - \Delta(t)) = 0, \tag{1}$$

которое удовлетворяет обобщенной теореме Валле Пуссена (последняя доказывается в работе). С помощью этих решений дается как понятие сопряженной точки, так и понятие линейного дифференциального уравнения с запаздывающим аргументом без сопряженных точек на интервале J. Кроме того, в роботе доказана теорема: Уравнение (1) является уравнением без сопряженных точек на J тогда и только тогда, когда краевая задача (1), (15) имеет единственное решение.