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TRANSFORMATIONS OF SETS IN TOPOLOGICAL GROUP

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Abstract. In this paper, along with some new results, we prove generalisations of some of the results proved in [4] in a topological group. Steinhaus' theorem on distance sets has been generalised by many mathematicians in various directions. We prove further generalisations of Steinhaus' theorem in a topological group.

1 Introduction

Let X be a locally compact Hausdorff topological group and S be the σ-ring generated by all compact subsets of X. Lot m be a regular Haar (left) measure on S. In the sequel, we shall assume also that X is compact, because in that case when we restrict our considerations to Baire sets only, the left Haar measure coinsides with the right Haar measure {p 262, [2]}. We shall need this fact very often. If E be any subset of X, then the outer measure, $m^*(E)$ of E is defined by

$$m^*(E) = \inf\{m(F) \colon E \subset F \in S\}.$$

In Section 4 we prove some lemmas which are vital in the proof of some of the subsequent theorems. Section 5 contains some convergence theorems which are motivated by and include as particular cases some convergence theorems proved in [4]. In Sections 6 and 7 we prove that certain sets in X and in the product space $X \times X$ are open. These are motivated by a basic result, proved in [4], that if F is a compact set of positive measure, then the set of points $a \in X$ for which $m[F \cap aF] > 0$ forms an open set. To prove theorems in these sections we need the equaivalence of sequential continuity and continuity in a topological-space. For this, we further assume that X is first countable. Equivalence under this assumption is known {p 131, [5]}. In one of the lemmas, we prove the equivalence in the product space $X \times X$, which we need to prove some theorems.

If F is a set of real numbers which is closed and is of positive measure (Lebesgue), then a fundamental theorem of Steinhaus [7] tells that the distance set of F fills up an interval. This theorem has been generalised, by Kestelman [3] in an n-dimensional Euclidean space and by one of the authors [4], to a topological group. Also, Ray [6] generalised the theorem of Kestelman by considering more than one set in an n-dimensional Euclidean space. In Section 8, we obtain further generalisations of Steinhaus theorem in a topological group. In one of the generalisations, we require the notion of density of sets as introduced in [4] in a topological group and also the density theorem {Theorem 5.1, [4]}. The proof of density theorem, however, needs Vitali theorem for invariant measures which has been proved in [1].

2 Known definitions

Definition 1 [1]. Let $\{S_n\}$ be a sequence of compact subsets of X. The sequence $\{S_n\}$ is said to be a sequence of demi-spheres if

(i) $$\lim m(S_n) = 0$$

and for some $\alpha > 0$ and each positive integer n,

(ii) $$S_{n+1}^{-1} S_{n+1} \subset S_n,$$
(iii) $$m(S_{n+1}) > \alpha \cdot m(S_n).$$

An example of a demi-sphere may be seen in {Ex. 4.2 [i]}. As a consequence, it follows that $m(S_n) > 0$ for $n > 1$.

Definition 2 [4]. Let E bo a subset of X and $x \in X$. The upper outer density of E at x, denoted by $\bar{D}^*(E, x)$, and the lower outer density of E at x, denoted by $\underline{D}^*(E, x)$, are defined as follows:
suppose that $\{S_n\}$ is a sequence of demi-spheres, then

$$\bar{D}^*(E, x) = \sup \left\{ \limsup_n \frac{m^*(E \cap g_n S_n)}{m(S_n)} \right\}$$

and

$$\underline{D}^*(E, x) = \inf \left\{ \liminf_n \frac{m^*[E \cap g_n S_n)}{m(S_n)} \right\},$$

where the supremum and the infimum are taken over all sequences $\{g_n S_n\}$, $g_n \in X$, $x \in g_n S_n$ for all values of n.

In $\bar{D}^*(E, x) = \underline{D}^*(E, x) = 1$, we say that E has outer density one at x. If $E \in S$, then x is said to be a density point of E.

Definition 3 [4]. Let $A_r \in S$, $r = 1, 2, 3, \ldots$. If there exists a set $A \in S$ such that $m[A_r \triangle A] \to 0$ as $r \to \infty$, then the sequence of sets $\{A_r\}$ is said to converge to the set A, and in symbol we write $A_r \to A$ (where the notation $\triangle$ stands for the symmetric difference of sets).

3 Standing conventions and assumptions

(i) $\{S_n\}$ is a fixed sequence of demi-spheres, (ii) if $x \in X$, there exists a sequence $\{g_n\}$, $g_n \in X$ such that $x \in g_n S_n$ for all n, (iii) if $x \in X$ and V is open, $x \in V$ and $x \in g_n S_n$ for all n, then $g_n S_n \subset V$ for all large n, (iv) small and capital letters with or without suffixes denote respectively the elements and sets in X, (V) o denotes the identity element of X.

4 Lemmas

The following lemma is needed in the proof of several theorems.

Lemma 1. Let f be a mapping from the product space $X \times X$ into Y, where Y is a topological space. Then for the function f to be continuous at $(a, b) \in X \times \times X$, it is necessary that for every sequence $\{(a_n, b_n)\}$ of points in $X \times X$,

$$(a_n, b_n) \to (a, b)$$

in $X \times X$ implies $f(a_n, b_n) \to f(a, b)$. This condition is also sufficient.

Proof Necessity. Let N be a neighbourhood of $f(a, b)$ in Y. Since f is continuous at (a, b), there exists an open set G in $X \times X$ with $(a, b) \in G$ such that $f[G] \subset N$.

Since $(a_n, b_n) \to (a, b)$, there exists a positive integer n_o such that for $n > n_o$, $(a_n, b_n) \in G$. Thus for $n > n_o$, $f(a_n, b_n) \in f[G) \subset N$. Consequently, $f(a_n, b_n) \to \to f(a, b)$.

Sufficiency. Since X is the first countable, $X \times X$ is also so. There exists therefore a nested local base

$$\mathscr{B}(a, b) = \{B_1, B_2, B_3, \ldots\}$$

at the point (a, b).

If possible, assume that f is not continuous at (a, b). Then there exists an open set H in Y with $f(a, b) \in H$ such that for every open set G in $X \times X$ containing (a, b), $f[G)$ is not contained in H. Consequently, $f[B_n]$ is not contained in H for every n. So, for each n, there exists a point $(a_n, b_n) \in B_n$ such that $f(a_n, b_n)$ is not contained in H. Thus $\{f(a_n, b_n)\}$ cannot converge to $f(a, b)$. But we show that $(a_n, b_n) \to (a, b)$ and then we obtain a contradiction.

Let G be an open set containing (a, b). Since $\mathscr{B}(a, b)$ is a local base at (a, b), there exists a positive integer n_0 such that

$$(a, b) \in B_{n_0} \subset G .$$

Since $\mathscr{B}(a, b)$ in nested, $B_n \subset B_{n_0}$ for $n > n_0$. So, $(a_n, b_n) \in B_n \subset B_{n_0} \in G$ if $n > n_0$. Thus $(a_n, b_n) \to (a, b)$. So f must be continuous at (a, b). This proves the lemma.

Note 1. The above lemma can be extended to the product of a finite number of spaces.

Note 2. If the product space $X \times X$ is replaced by the single space and the assumptions are changed accordingly, then also the above lemma remains true{p 131, [5]}.

The following lemma is vital in the proof of Theorem 12.

Lemma 2. Let C_1 and C_2 be two compact sets of positive measures. Then the set H of all those points $a \in X$ such that

$$m[C_1 \cap a^{-1}C_2] > 0$$

forms a nonempty open set.

Proof. We first show that H is nonempty. Since m is regular, there are open sets U_1 and U_2 containing C_1 and C_2 respectively such that $m(U_1) < \infty$ and $m(U_2) < \infty$. So by Theorem 5.1 [4] almost all points of C_1 and C_2 are points of densities of the sets C_1 and C_2 respectively. Let α, β be points of densities of the sets C_1 and C_2 respectively. Then corresponding to $\varepsilon = 1/4$, there exist $h_n \in X$, $g_n \in X$ and a positive integer N such that for $n \geqslant N$,

$$m[C_1 \cap h_n S_n] > 3/4\, m(S_n)\,, \ \alpha \in h_n S_n$$

and

$$m[C_2 \cap g_n S_n] > 3/4\, m(S_n)\,, \ \beta \in g_n S_n\,.$$

So, in particular,

$$m[C_1 \cap h_N S_N] > 3/4\, m(S_N)$$

and

$$m[C_2 \cap g_N S_N] > 3/4\, m(S_N)\,. \looparrowright m[h_N^{-1}C_1 \cap S_N] =$$

$$= m[h_N^{-1}(C_1 \cap h_N S_N)] = m[C_1 \cap h_N S_N] > 3/4\, m(S_N)\,.$$

Similarly,

$$m[g_N^{-1}C_2 \cap S_N] > 3/4\, m(S_N)\,.$$

Let

$$Z = S_N \cap h_N^{-1}C_1 \cap g_N^{-1}C_2\,.$$

Then

$$Z = S_N - [(S_N - h_N^{-1}C_1) \cup (S_N - g_N^{-1}C_2)]\,.$$

$$\looparrowright m(Z) \geqslant m(S_N) - m(S_N - h_N^{-1}C_1) - m(S_N - g_N^{-1}C_2) >$$

$$> m(S_N) - 1/4\, m(S_N) - 1/4\, m(S_N) > 0\,.$$

Since,

$$h_N^{-1} C_1 \cap g_N^{-1} C_2 \supset Z,$$

we get

$$m(h_N^{-1} C_1 \cap g_N^{-1} C_2 \geqslant m(Z) > 0.$$

So,

$$m[C_1 \cap (g_N h_N^{-1})^{-1} C_2] = m[C_1 \cap h_N g_N^{-1} C_2] =$$
$$= m[h_N(h_N^{-1} C_1 \cap g_N^{-1} C_2)] = m[h_N^{-1} C_1 \cap g_N^{-1} C_2] > 0.$$

Thus the element $a = g_N h_N^{-1} \in X$ is such that

$$m(C_1 \cap a^{-1} C_2) > 0.$$

So, H is nonempty.

It now remains to show that H is open. We define a function

$$f\colon X \to R,$$

where R is the set of real numbers with the usual topology by

$$f(a) = m(C_1 \cap a^{-1} C_2), \quad a \in X.$$

We show that f is continuous and then the result follows.

Let $a_r \to a$. Since the mapping

$$g\colon X \to X$$

defined by $g(x) = x^{-1}$ for all $x \in X$ is continuous,

$$g(a_r) \to g(a), \quad \text{i.e.} \quad a_r^{-1} \to a^{-1}.$$

Since $C_1 \in S$, by lemma 4.2 [4] we get

$$C_1 \cap a_r^{-1} C_2 \to C_1 \cap a^{-1} C_2.$$

Consequently,

$$m(C_1 \cap a_r^{-1} C_2) \to m(C_1 \cap a^{-1} C_2),$$

by Lemma 4.1 [4]. So, $f(a_r) \to f(a)$ and f is continuous by Note 2. This proves the lemma.

The proof of the following lemma is omitted as its technique is similar to Lemma 2 except for long calculations.

Lemma 3. Let C_1 and C_2 be compact sets of positive measures. Let $p > 1$ be an integer. Then the set G of all those points,

$$(a_1, a_2, \dots, a_p) \in X \times X \times \dots \times X (p\text{-times})$$

such that

$$m[C_1 \cap a_1^{-1} C_2 \cap a_2^{-1} C_2 \ldots \cap ap^{-1} C_2] > 0,$$

forms a nonempty open set.

5 Convergence theorems

Theorem 1. Let F be a compact Baire set and $a_r, a, b_r, b \in X$. If $a_r \to a$ and $b_r \to b$, then

$$a_r F b_r \to aFb.$$

Proof. Let $\varepsilon > 0$ be arbitrary. Since m is regualar, there exists an open set U such that $aFb \subset U$ and

$$m(U - aFb) < \varepsilon/3.$$

So, there exists a (symmetric)neighbourhood V of o such that

$$VaFbV \subset U.$$

Since $a_r \to a$ and $b_r \to b$ and since Va and bV are open sets containing a and b respectively, there exists a positive integer N such that $a_r \in Va$ and $b_r \in bV$ for $r \geqslant N$. So, for $r \geqslant N$, $a_r F b_r \subset U$.

Let

$$Y_r = (aFb) \cap (a_r F b_r),\ r \geqslant N.$$

Then

$$Y_r \subset U \quad \text{for} \quad r \geqslant N \quad \text{and} \quad Y_r = U - [(U - aFb) \cup (U - a_r F b_r)].$$

So,

$$m(Y_r) \geqslant m(U) - m(U - aFb) - m(U - a_r F b_r) > m(U) - \varepsilon/3 - m(U) + \\ + m(a_r F b_r) = m(aFb) - \varepsilon/3 > m(U) - 2\varepsilon/3 > m(U) - \varepsilon.$$

This implies that

$$m[U - Y_r] < \varepsilon \quad \text{for} \quad r \geqslant N,$$

i.e. $m[(U - aFb) \cup (U - a_r F b_r)] < \varepsilon, \quad r \geqslant N.$

Since

$$a_r F b_r \Delta aFb \subset (U - aFb) \cup (U - a_r F b_r) \quad \text{for} \quad r \geqslant N,$$

we obtain that

$$m(a_r F b_r \triangle aFb) < \varepsilon \quad \text{if} \quad r \geqslant N.$$

So, $a_r F b_r \to aFb$.

This proves the theorem.

Note 3. A particular case of the above theorem has been proved in Theorem 4.1 of [4] by taking $b_r = 0$ for each r.

Corollary 1. Let F be a compact Baire set and a_r, a, b_r, $b \in X$. If $a_r \to a$ and $b_r \to b$, then for $A \in S$

$$a_r F b_r \cap A \to aFb \cap A.$$

Proof. By Theorem 1 we have

$$m[(a_r F b_r \cap A) \triangle (aFb \cap A)] = m[(a_r F b_r \triangle aFb) \cap A] \leq$$

$$\leq m(a_r F b_r \triangle aFb) \to 0.$$

Corollary 2. Let F be a compact Baire set and A be compact. Let a_r, a, b_r, b and c_r, $c \in X$. If $a_r \to a$, $b_r \to b$ and $c_r \to c$, then

$$(a_r F b_r) \cap c_r A \to (aFb) \cap cA.$$

Proof. By Theorem 1, $a_r F b_r \to aFb$.
Also, $c_r A \to cA$, by Theorem 4.1 [4].
So, $(a_r F b_r) \cap c_r A \to (aFb) \cap cA$, by Lemma 4.4[4].

Corollary 3. Let F be a compact Baire set and a_r, a, b_r, $b \in x$. If $a_r \to a$ and $b_r \to b$, and U is an open set containing aFb, $a_r F b_r$, $r = 1, 2, 3, \ldots$ such that $m(U) < \infty$, then

$$U - a_r F b_r \to U - aFb.$$

Proof. Wo have

$$[(U - a_r F b_r) - (U - aFb) \cup [(U - aFb) - (U - a_r F b_r)] = (a_r F b_r \Delta aFb)$$

and so the Corollary follows from Theorem 1.

Corollary 4. Let F be a compact Baire set and b_r, $b \in X$. If $b_r \to b$, then $F b_r \to Fb$.

Proof. In Theorem 1, assume $a_r = e$ for all r.

Corollary 5. Let F be acompact Baire set and a_r, $a \in X$. If $a_r \to a$, then for $A \in S$

$$F a_r \cap A \to Fa \cap A.$$

Proof. By Corollary 4 we have

$$m[(F a_r \cap A) \triangle (Fa \cap A)] = m[(F a_r \triangle Fa) \cap A] \leq m[F a_r \triangle Fa] \to 0$$

Theorem 2. Suppose that $\{A_r\}$ is a sequence of Baire sets and A is a compact Baire set. Let $a_r, a, b_r, b \in X$. If $a_r \to a$, $b_r \to b$ and $A_r \to A$, then

$$a_r A_r b_r \to aAb.$$

Proof. We have

$$a_r A_r b_r - aAb \subset (a_r A_r b_r - a_r A b_r) \cup (a_r A b_r - aAb)$$

and so

$$m(a_r A_r b_r - aAb) \leq m(a_r A_r b_r - a_r A b_r) + m(a_r A b_r - aAb) =$$

$$= m[a_r(A_r - A)b_r] + m(a_r A b_r - aAb) = m(A_r - A) + m(a_r A b_r - aAb) \to 0$$

as $r \to \infty$ by Theorem 1 and because $A_r \to A$.
Similarly, $m(aAb - a_r A_r b_r) \to 0$ as $r \to \infty$.
We now conclude that

$$a_r A_r b_r \to aAb.$$

This proves the theorem.

Note 4. In Theorem 4.2 of [4], a particular case of the above theorem has been obtained by taking $b_r = e$ for each r.

6 Certain open sets in X

If F is a closed set of Lebesgue positive measure in the n-dimensional Euclidean space E^n then it is known [3] that the set of all translates of F those intersect F in a set of positive measure is a nonvoid open set containing the origin. In fact this is a generalisation of Steinhaus theorem [7]. Guided by this fact, in this section and in the following section we prove that certain sets in X and in the product space $X \times X$ are open where our basic sets are compact Baire sets of positive measures.

Theorem 3. Let F be a compact Baire set of positive measure. Then the set of all those points $a \in X$ such that

$$m(aF \cap Fa) > 0$$

forms a nonempty open set containing e.

Proof. We define a function

$$\Phi: X \to R,$$

where R is the set of real numbers with the usual topology by

$$\Phi(a) = m(aF \cap Fa), \quad a \in X.$$

We show that Φ is continuous and then the proof follows.

Let $a_r \to a$. Then, $a_r F \to aF$ and $F a_r \to Fa$, by Theorem 4.1 [4] and Corollary 4. So,

$$a_r F \cap F a_r \to aF \cap Fa$$

by Lemma 4.4 [4].

Consequently, $m(a_r F \cap Fa_r) \to m(aF \cap Fa)$, by Lemma 4.1 [4], i.e. $\Phi(a_r) \to \Phi(a)$. The proof is now complete with the help of Note 2.

The proofs of the following theorems are omitted as these can be constructed by a method similar to Theorem 3.

Theorem 4. Let F be a compact Baire set of positive measure. Then the set of all those points $a \in X$ for which

$$m(F \cap aFa) > 0,$$

forms a nonempty open set containing e.

Theorem 5. Let F be a compact Baire set of positive measure. Then the set of all those points $a \in X$ for which

$$m(F \cap Fa) > 0 \quad \text{or} \quad m(F \cap aF) > 0$$

forms a nonempty open set containing e.

Note 5. The case $m(F \cap aF) > 0$ has been considered in Theorem 4.3 [4].

7 Certain open sets in $X \times X$

Theorem 6. Let F be a compact set of positive measure. Then the set of all those points $(a, b) \in X \times X$ for which

$$m(aF \cap bF) > 0$$

forms a nonempty open set containing the point (e, e).

Proof. We define a function

$$\psi: X \times X \to R,$$

where R is the set of all real numbers with the usual topology, by

$$\psi(a, b) = m(aF \cap bF), \quad (a, b) \in X \times X.$$

We show that ψ is continuous. Let $(a_r, b_r) \to (a, b)$. Then $a_r \to a$ and $b_r \to b$. So, $a_r F \to aF$ and $b_r F \to bF$, by Theorem 4.1 [4]. By Lemma 4.4 [4] $a_r F \cap b_r F \to aF \cap bF$.

Consequently, $m(a_r F \cap b_r F) \to m(aF \cap bF)$, by Lemma 4.1 [4].

So, $\psi(a_r, b_r) \to \psi(a, b)$.

Continuity of ψ now follows from Lemma 1 and hence the proof.

Note 6. The above theorem generalises Theorem 4.3 [4] to product spaces.

Theorem 7. Let C_1 and C_2 be two compact Baire sets of positive measures. Then the set of all those points $(a, b) \in X \times X$ for which

$$m(aC_1b \cap bC_2a) > 0$$

forms an open set in $X \times X$.

Proof. We define a function

$h: X \times X \to R$, where R is the set of all real numbers with the usual topology, by

$h(a, b) = m(aC_1b \cap bC_2a)$ for all $(a, b) \in X \times X$ and prove its continuity as in Theorem 6, from which the result follows.

Theorem 8. Let C_1 and C_2 be two compact sets of positive measures. Then the set of all those points $(a, b) \in X \times X$ for which

$$m(aC_1 \cap bC_2) > 0$$

forms an open set in $X \times X$.

Proof. We define a function

$$\chi: X \times X \to R,$$

where R is the set of real numbers with the usual topology, by

$\chi(a, b) = m(aC_1 \cap bC_2)$, $(a, b) \in X \times X$, and prove its continuity as in Theorem 6. THe proof follows from the continuity property.

We mention the following theorems without proof.

Theorem 9. Let F be a compact Baire set of positive measure. Then the set of all those points $(a, b) \in X \times X$ for which $m(F \cap aFb) > 0$ forms a nonempty open set containing (e, e).

Theorem 10. Let F be a compact Baire set of positive measure. Then the set of all those points $(a, b) \in X \times X$ for which

$$m(aFb \cap bFa) > 0$$

forms a nonempty open set containing (e, e).

8 Generalisation of Steinhaus' Theorem in topological group

Theorem 11. Let C be a compact Baire set of positive measure. Let p be any positive integer and λ be any number such that $0 < \lambda < m(C)$. There exists a neighbourhood V of e such that if $\xi_r \in V$, $\eta_r \in V$, $r = 1, 2, \ldots, p$ and $\lambda \in V$, then the set Y of all points x such that

$$\lambda\chi \in C \quad \text{and} \quad \xi_r\chi\eta_r \in C, \quad r = 1, 2, \ldots, p$$

is a compact set of positive measure such that

$$m(Y) > m(C) - \lambda$$

Proof. Let λ be such that $0 < \lambda < m(C)$. Since m is regular, there exists an open set U with $C \subset U$ such that

$$m(U - C) < \frac{\lambda}{p+1}.$$

There exists a symetric neighbourhood V of e such that

$$VCV \subset U.$$

Let $\xi_r \in V$, $\eta_r \in V$, $r = 1, 2, \ldots, p$ and let also $\lambda \in V$.
Let $Y = (\lambda^{-1}C) \cap (\xi_1^{-1}C\eta_1^{-1}) \cap \ldots \cap (\xi_p^{-1}C\eta_p^{-1})$.
Then Y is compact and if $x \in Y$, then

$$x \in \lambda^{-1}C \quad \text{and} \quad x \in \xi_r^{-1}C\eta_r^{-1}\ r = 1, 2, \ldots, p,$$

i.e. $\lambda\chi \in$ C, *and* $\xi_r\chi\eta_r \in C$, $r = 1, 2, \ldots, p$.
So, Y is the set of points as desired in the theorem. The proof will, therefore, be completed if we show that $m(Y) > m(C) - \lambda$.
Now

$$Y = U - \left[(U - \lambda^{-1}C) \cup \left\{\bigcup_{r=1}^{p} (U - \xi_r^{-1}C\eta_r^{-1})\right\}\right]$$

$$\hookrightarrow \quad m(Y) \geqslant m(U) - m(U - \lambda^{-1}C) - \sum_{r=1}^{p} m(U - \xi_r^{-1}C\eta_r^{-1}).$$

Since $\lambda^{-1}C \subset U$ and $\xi_r^{-1}C\eta_r^{-1} \subset U$, $r = 1, 2, \ldots, p$, we have

$$m(U - \lambda^{-1}C) = m(U - C)$$

and $m(U - \xi_r^{-1}C\eta_r^{-1}) = m(U - C)$.

$$\hookrightarrow \quad m(Y) > (U) - \frac{\lambda}{p+1} - p \cdot \frac{\lambda}{p+1} = m(U) - \lambda \geqslant m(C) - \lambda.$$

Note 7. The above theorem coincides with Theorem 6.1 of [4] if $\eta_r = e$ and $\lambda = e$, where of course it is only shown that Y is of positive measure. We, however, show that the measure of the set Y may be made arbitrarily near to the measure of the set C.

The following theorems are generalisations of Steinhaus Theorem to more than one set.

Theorem 12. Let C_1 and C_2 be two compact sets of positive measures. Then there exists a nonempty open set G such that if $a \in G$, then the set of all those points x such that

$$x \in C_1 \quad \text{and} \quad ax \in C_2$$

forms a compact set of positive measure.

Proof. Let $G = \{\xi\colon \xi \in X$ and $m(C_1 \cap \xi^{-1}C_2) > 0$. By Lemma 2, G is a nonempty open set. Let $a \in G$ and $A = C_1 \cap a^{-1}C_2$, then $m(A) > 0$ and A is compact. If $x \in A$, then $x \in C_1$ and $ax \in C_2$. This proves the theorem.

The above theorem can easily be generalised to the following one whose proof is omitted.

Theorem 13. Let C_1 and C_2 be two compact sets of positive measures and p be any positive integer. Then here exists a nonempty open set G such that if ξ_1, $\xi_2, \ldots, \xi_p \in G$, then the set of all those points x such that

$$x \in C_1 \quad \text{and} \quad \xi_r x \in C_2, \quad r = 1, 2, \ldots, p$$

forms a compact set of positive measure.

Theorem 14. Let $A, A_1, A_2, \ldots, A_{m-1} (m > 1)$ be compact sets of positive measures. Then there exist open sets $G_1, G_2, \ldots, G_{m-1}$ such that if $\xi_r \in G_r$, $r = 1, 2, \ldots, m-1$, then the set of points x such that

$$x \in A \quad \text{and} \quad \xi_r x \in A_r \quad (r = 1, 2, \ldots, m-1)$$

is a compact set of positive measure.

Proof. Let $Y = X \times X \times \ldots \times X$ ($m - 1$ times). We define a function

$$f\colon Y \to R,$$

where R is the set of real numbers with the usual topology, by

$$f(a_1, a_2, \ldots, a_{m-1}) = m[A \cap a_1^{-1}A_1 \cap \ldots \cap a_{m-1}^{-1}A_{m-1}]$$

for all $(a_1, a_2, \ldots, a_{m-1}) \in Y$.

We show that f is continuous and then it will follow that the set H of all those points $(a_1, a_2, \ldots, a_{m-1}) \in Y$ for which

$$m[A \cap a_1^{-1}A_1 \cap \ldots \cap a_{m-1}^{-1}A_{m-1}] > 0$$

forms an open set in Y.

Let $(a_1^{(r)}, a_2^{(r)}, \ldots, a_{m-1}^{(r)}) \to (a_1, a_2, \ldots, a_{m-1})$. Then, $a_i^{(r)} \to a_i$, $i = 1, 2, 3, \ldots, m-1$.

Since the mapping

$$g\colon X \to X$$

given by $g(x) = x^{-1}$ for all $x \in X$ is continuous, it follows that

$$a_i^{(r)-1} \to a_i^{-1}, \quad i = 1, 2, \ldots, m-1.$$

So, by Theorem 4.1 [4], Lemma 4.2 [4] and Lemma 4.4 [4], we get

$$A \cap a_i^{(r)^{-1}} A_i \cap \ldots \cap a_{m-1}^{(r)^{-1}} A_{m-1} \to A \cap a_1^{-1} A_1 \cap \ldots \cap a_{m-1}^{-1} A_{m-1}.$$

Consequently, by Lemma 4.1 [4]

$$m(A \cap a_1^{(r)^{-1}} A_1 \cap \ldots \cap a_{m-1}^{(r)^{-1}} A_{m-1}) \to m(A \cap a_1^{-1} A_1 \cap \ldots \cap a_{m-1}^{-1} A_{m-1}),$$

i.e. $f(a_1^{(r)}, a_2^{(r)}, \ldots, a_{m-1}^{(r)}) \to f(a_1, a_2, \ldots, a_{m-1})$.
So, f is continuous because of Note 1.

We now show that H is nonempty. Since m is regular, there are open sets U, $U_1, U_2, \ldots, U_{m-1}$ containing $A, A_1, A_2, \ldots, A_{m-1}$ respectively such that $m(U) < \infty$ and $m(U_i) < \infty$, $i = 1, 2, \ldots, m-1$. So, by Theorem 5.1 [4] almost all points of $A, A_1, A_2, \ldots, A_{m-1}$ are points of densities of the respective sets. Let $\alpha, \beta_1, \beta_2, \ldots, \beta_{m-1}$ be points of densities of $A, A_1, A_2, \ldots, A_{m-1}$ respectively. Then there exist $g_n \in X$, $h_n^{(1)} \in X$, $h_n^{(2)} \in X, \ldots, h_n^{(m-1)} \in X$, and a positive integer N such that

$$m[A \cap g_N S_N] > (1 - 1/4m) m(S_N)$$

and

$$m[A_i \cap h_N^{(i)} S_N] > (1 - 1/4m) m(S_N), \quad i = 1, 2, \ldots, m-1,$$

where

$$\alpha \in g_n S_n \quad \text{and} \quad \beta_i \in h_n^{(i)} S_n, \quad i = 1, 2, \ldots, m-1; n = 1, 2, \ldots.$$

So,

$$m[g_N^{-1} A \cap S_N] = m[g_N^{-1}(A \cap g_N S_N)] = m[A \cap g_N S_N] > (1 - 1/4m) m(S_N).$$

Similarly,

$$m[h_N^{(i)^{-1}} A_i \cap S_N] > (1 - 1/4m) m(S_N), \quad i = 1, 2, \ldots, m-1.$$

Let $Z = S_N \cap g_N^{-1} A \cap h_N^{(1)^{-1}} A_1 \cap \ldots \cap h_N^{(m-1)^{-1}} A_{m-1}$.
Then

$$Z = S_N - [(S_N - g_N^{-1} A) \cup (S_N - h_N^{(1)^{-1}} A_1) \cup \ldots \cup (S_N - h_N^{(m-1)^{-1}} A_{m-1}).$$

So,

$$m(Z) \geqslant m(S_N) - m(S_N - g_N^{-1} A) - \sum_{i=1}^{m-1} m(S_N - h^{(i)^{-1}} A_i) >$$

$$> m(S_N) - m \cdot \frac{1}{4m} m(S_N) > 0.$$

Since

$$g_N^{-1} A \cap h_N^{(1)^{-1}} A_1 \cap \ldots \cap h_N^{(m-1)^{-1}} A_{m-1} \supset Z,$$

we get

$$m(g_N^{-1}A \cap h_N^{(1)^{-1}}A_1 \cap \ldots \cap h_N^{(m-1)^{-1}}A_{m-1}) \geqslant m(Z) > 0.$$

So,

$$m[A \cap (h_N^{(1)}g_N^{-1})^{-1}A_1 \cap \ldots \cap (h_N^{(m-1)}g_N^{-1})^{-1}A_{m-1}] =$$
$$= m[A \cap g_N h_N^{(1)^{-1}}A_1 \cap \ldots \cap g_N h_N^{(m-1)^{-1}}A_{m-1}] =$$
$$= m[g_N(g_N^{-1}A \cap h_N^{(1)^{-1}}A_1 \cap \ldots \cap h_N^{(m-1)^{-1}}A_{m-1})] =$$
$$= m[g_N^{-1}A \cap h_N^{(1)^{-1}}A_1 \cap \ldots \cap h_N^{(m-1)^{-1}}A_{m-1}] > 0.$$

Let $a_i = h_N^{(i)}h_N^{-1}$, $i = 1, 2, \ldots, m-1$.
Then the point $(a_1, a_2, \ldots, a_{m-1}) \in Y$ is such that

$$m(A \cap a_1^{-1}A_1 \cap \ldots \cap a_{m-1}^{-1}A_{m-1}) > 0.$$

So, $(a_1, a_2, \ldots, a_{m-1}) \in H$. Therefore, H is nonempty and this gives that H is a nonempty open set. Consequently, there exist open sets G_i, $i = 1, 2, \ldots, m-1$ such that $a_i \in G_i$, $i = 1, 2, \ldots, m-1$ and such that

$$G_1 \times G_2 \times \ldots \times G_{m-1} \subset H.$$

Let $\xi_r \in G_r$, $r = 1, 2, \ldots, m-1$, then

$$(\xi_1, \xi_2, \ldots, \xi_{m-1}) \in H.$$

Let $L = A \cap \xi_1^{-1}A_1 \cap \ldots \cap \xi_{m-1}^{-1}A_{m-1}$.

Then L is compact. Also, if $x \in L$, then $x \in A$ and $x \in \xi_r^{-1}A_r$, $r = 1, 2, \ldots, m-1$, i.e. $x \in A$ and $\xi_r x \in A_r$, $r = 1, 2, \ldots, m-1$. Since $(\xi_1, \xi_2, \ldots, \xi_{m-1}) \in H$,

$$m(A \cap \xi_1^{-1}A_1 \cap \ldots \cap \xi_{m-1}^{-1}A_{m-1}) > 0,$$

i.e. $m(L) > 0$.

This proves the theorem.

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SÚHRN

TRANSFORMATIONS OF SETS IN TOPOLOGICAL GROUP

Sadasiv Chakrabarti — B. K. Lahiri

V práci sú študované niektoré možnosti zovšeobecnenia známej Steinhausovej vety. Tieto otázky sa tu skúmajú pomocou istých transformácií množín v topologických grupách. Získané výsledky sú vlastne zovšeobecnením výsledkov práce [4].

РЕЗЮМЕ

ТРАНСФОРМАТИОНС ОФ СЕТС ИН ТОПОЛОГИЦАЛ ГРОУП

Садасив Хакрабарти — Б. К. Лагири

В этой работе исследуются некоторые возможности обобщения знакомой теоремы таинхауса. Эти вопросы здесь расмотриваются при помощи некоторых трансформации множеств в топологических групах. Полученые результаты являются обобщениями резултатов работы [4].