UNIVERSITAS COMENIANA
ACTA MATHEMATICA UNIVERSITATIS COMENIANAE
LII—LIII—1987

ON PROOFS OF EXISTENCE OF THE m-th ROOT AND THE LOGARITHM OF POSITIVE NUMBERS

MÁRIA BENEŠOVÁ—TIBOR ŠALÁT, Bratislava

The proof of existence of the m-th root of positive numbers is a basis for constructing the arithmetic in the domain of real numbers. This proof enables us to define the powers of positive numbers with rational and also real exponents and so to create the conditions for proving the existence of the logarithm of positive numbers ([3] pp. 188—196). All this knowledge presents a basis for inroducing differential and integral calculi and therefore all monographs on fondations of real analysis in the domain of real numbers contains them.

This paper consists of two parts. The aim of the first part is to give a survey of various procedures for proving the existence of the m-th root and the logarithm of positive numbers. The aim of the second part of the paper is to give new proofs for the existence of the m-th root and the logarithm of positive numbers. These proofs will be based on some unifying principles for proving fundamental theorems of real analysis that were formulated by H. LKeinfelder [7] and P. Shanahan [13].

1 Survey of various procedures for proving the existence of the m-th root and the logarithm of positive numbers

In this part of the paper we shall describe several procedures for proving the existence of the m-th root and logarithm of positive numbers.

a) In [6] (pp. 51—52; 57—58) a procedure is given which is based on the convergence of decimal expansions of real numbers (this convergence is an easy consequence of the theorem on the convergence of non-decreasing bounded sequences of real numbers).

We shall indiciate the procedure for proving the existence of $\sqrt[m]{a}$, $m \in N$, $a > 0$.

Let us choose an integer $C_0 \geqq 0$ such that $c_0^m \leqslant a < (c_0 + 1)^m$. Divide the interval $\langle c_0, c_0 + 1)$ into ten parts of the same length $\left(\text{equal to } \dfrac{1}{10}\right)$. Among these subintervals of the interval $\langle c_0, c_0 + 1)$ there is such a subinterval

$$\left\langle c_0 + \frac{c_1}{10}, c_0 + \frac{c_1 + 1}{10}\right), \qquad c_1 \in \{0, 1, \dots, 9\}$$

that

$$\left(c_0 + \frac{c_1}{10}\right)^m \leqslant a\left(c_0 + \frac{c_1 + 1}{10}\right)^m.$$

By induction we can construct two sequences $\{x_k\}_1^\infty$, $\{y_k\}_1^\infty$ of real numbers, where

$$x_k = c_0 + \frac{c_1}{10} + \dots + \frac{c_k}{10^k} \qquad (k = 1, 2, \dots),$$

$$y_k = c_0 + \frac{c_1}{10} + \dots + \frac{c_{k-1}}{10^{k-1}} + \frac{c_k + 1}{10^k} \qquad (k = 1, 2, \dots),$$

$$c_j \in \{0, 1, \dots, 9\} \qquad (j = 1, 2, \dots)$$

with

$$x_k^m \leqslant a < y_k^m \qquad (k = 1, 2, \dots). \tag{1}$$

Let $z = \lim\limits_{k \to \infty} x_k = \lim\limits_{k \to \infty} y_k$. Then by $k \to \infty$ we get from (1) the equality $z^m = a$, i.e. $z = \sqrt[m]{a}$.

The idea for proving the existence of the logarithm is analogous. Let $b > 1$, $t > 0$. We shall show the existence of $\log_b t$.

Choose an integer c_0 such that

$$b^{c_0} \leqslant t < b^{c_0 + 1}.$$

Divide the interval $\langle c_0, c_0 + 1)$ into ten parts of the same length. Choose the part

$$\left\langle c_0 + \frac{c_1}{10}, c_0 + \frac{c_1 + 1}{10}\right) \qquad (c_1 \in \{0, 1, \dots, 9\})$$

with

$$b^{c_0 + \frac{c_1}{10}} \leqslant t < b^{c_0 + \frac{c_1 + 1}{10}}$$

Using the previous considerations we get

$$b^{x_k} \leqslant t < b^{y_k} \qquad (k = 1, 2, \dots),$$

$\{x_k\}_1^\infty$, $\{y_k\}_1^\infty$ have the previous meaning. By $k \to \infty$ we get $b^z = t$, hence $z = \log_b t$, where $z = \lim\limits_{k \to \infty} x_k = \lim\limits_{k \to \infty} y_k$.

b) The proof of the existence of $\sqrt[m]{a}$ and $\log_b t$ in [9] (pp. 22—31) is based on the fact that the set R is continuously ordered, i.e. that every non-empty set bounded from above has the least upper bound (supremum) in R. More precisely, this is made only in the proof of the existence of $\sqrt[m]{a}$ $(a > 0)$, since the proof of the existence of the logarithm is omited.

We sketch the proof of the existence of $\sqrt[m]{a}$, $a > 0$, $m \in N$. Put $E = \{u \in R: u > 0 \wedge u^m < a\}$. Then evidently $E \neq \emptyset$ and E is bounded from above. Hence there exists $y = \sup E$. We prove that $y^m = a$. The proof is realized by eliminating the possibilities: $y^m < a$, $y^m > a$.

Let $y^m < a$ (in the case $y^m > a$ we can proceed analogously). Choose an $h \in (0, 1)$ such that

$$h < \frac{a - y^m}{(1 + y)^m - y^m}$$

Then using the binomial theorem we have

$$(y + h)^m = y^m + \binom{m}{1} y^{m-1} h + \ldots + \binom{m}{m} h^m < y^m + h\left[\binom{m}{1} y^{m-1} + \right.$$

$$\left. + \binom{m}{2} y^{m-2} + \ldots + \binom{m}{m}\right] = y^m + h[(1 + y)^m - y^m] < y^m + (a - y^m) = a.$$

Hence $(y + h)^m < a$ and therefore $y + h \in E$. This contradicts the definition of y.

c) An in our country very popular textbook on mathematical analysis [4] uses in the mentioned proofs a procedure similar to that used in [9] (cf. [4], pp. 60—64; 115—116). The little modification of this procedure is based on the following two auxiliary results:

(P1) Let $m \in N$, $a > 0$, $x > 0$ and $x^m < a$. Then there exists a $y > x$ such that $y^m < a$.

(P2] Let $m \in N$, $a > 0$, $x > 0$ and $x^m > a$. Then there exists a z $0 < z < x$ such that $z^m > a$.

d) We outline the proof of the existence of $\sqrt[m]{a}$ $(a > 0)$ given in [2] (pp. 35—37; 39).

Choose an $n \in N$ such that $\frac{1}{n} < a < n$ and put $X = \{u \in Q: u \leqslant 0 \vee [(u > 0) \wedge \wedge (u^m < a)]\}$, $Y = Q - X$ (Q being the set of all rational numbers). It is easy to see that (X, Y) is a cut in Q. If for each $q > 0$, $q \in Q$ we have $q^m \neq a$, then $\gamma = (X, Y)$ is a cut of the third kind (a gap) and for each $x \in X$, $x > 0$, $y \in Y$ we have $x < \gamma < y$. But then $x^m < \gamma^m < y^m$ and since $x^m < a < y^m$, we get $|\gamma^m - a| < < y^m - x^m$. Since $y^m - x^m$ can be made arbitrarily small by suitable choose x and y, we get $\gamma^m = a$.

The proof of the existence of the logarithm proceeds in an analogous way.

e) Let us remember the methods contained in [14] (pp. 85—87; 140—142). These methods are similar to those used in [6].

The proof of the existence of $\sqrt[m]{a}$ $(a > 0)$ proceeds in the following way. For each $n \in N$ we chose the greatest integer $k_n > 0$ with $k_n^m \leqslant 2^{m \cdot n} \cdot a$. Then we have

$$k_n^m \leqslant 2^{m \cdot n} a < (k_n + 1)^m. \tag{2}$$

Let us construct the sequence $\{u_n\}_1^\infty$, where $u_n = \frac{k_n}{2^n}$ $(n = 1, 2, \ldots)$. Since $(2k_n)^m \leqslant \leqslant 2^{m(n+1)}$, then according to the definition of k_{n+1} we get $2k_n \leqslant k_{n+1}$, hence $u_n \leqslant u_{n+1}$. Therefore the sequence $\{u_n\}_1^\infty$ is non-decreasing.

Further, if l is a positive integer such that $a < l^m$, then $k_n^m \leqslant 2^{mn} a < 2^{mn} l^m$, $k_n < 2^n l$, $u_n < 1$ $(n = 1, 2, \ldots)$. Hence $\{u_n\}_1^\infty$ is non-decreasing sequence and so there exists

$$y = \lim_{n \to \infty} u_n \tag{3}$$

From (2) it is easy to see that

$$u_n^m \leqslant a < \left(u_n + \frac{1}{2^n}\right)^m \qquad (n = 1, 2, \ldots) \tag{4}$$

From (3), (4) we get $y^m = a$.

The proof of the existence of $\log_b t$ $(t > 0,\ b > 1)$ we begin by the definition of such greatest integer k_n that

$$b^{k_n} \leqslant t^{2^n} \qquad (n = 1, 2, \ldots).$$

The further considerations are analogous to those used in the proof of the existence of $\sqrt[m]{a}$.

f) In [10] (pp. 176—178; 191—193) the existence of $\sqrt[m]{a}$ $(a > 0)$ is proved by the construction of two sequences $\{x_k\}_1^\infty$, $\{y_k\}_1^\infty$ of real numbers, where $x_1 = \frac{a}{1 + a}$, $y = 1 + a$. The construction of these sequences begins by dividing the interval $\langle x_1, y_1 \rangle$ into two parts:

$$\left\langle x_1, \frac{x_1 + y_1}{2} \right\rangle, \qquad \left\langle \frac{x_1 + y_1}{2}, y_1 \right\rangle.$$

As the interval $\langle x_2 y_2 \rangle$ we choose the first of the intervals (5) if $a \leqslant \left(\frac{x_1 + y_1}{2}\right)^m$, and the second if $\left(\frac{x_1 + y_1}{2}\right)^m < a$.

In this way (by induction) we get the sequences $\{x_k\}_1^\infty$, $\{y_k\}_1^\infty$ such that $\{x_k\}_1^\infty$ is non-decreasing and $\{y_k\}_1^\infty$ non-increasing.

$$y_k - x_k = \frac{y_1 - x_1}{2^{k-1}} \qquad (k = 1, 2, \ldots) \tag{6}$$

and

$$x_k^m < a \leqslant y_k^m \qquad (k = 1, 2, \ldots) \tag{7}$$

Since $\{x_k\}_1^\infty$ is non-decreasing and bounded from above (by y_1), there exists $y = \lim\limits_{k \to \infty} x_k$. On account of (6) we get also $y = \lim\limits_{k \to \infty} y_k$ and from (7) we obtain $a = y^m$.

The proof of the existence of logarithm is based on an analogous procedure.

g) In the monograph [15] (pp. 242—244) on real number system the following method for proving the existence of $\sqrt[m]{a}$ is used:

For each $n \in N$ we choose the greatest integer $k_n \geq 0$ such that $k_n^m \leq n^m a$. From the definition of k_n we have

$$k_n^m \leq n^m a < (k_n + 1)^m$$

and the further procedure is analogous to that used in [14].

At the end of this part of the paper we can state that there are many procedures for proving the existence of $\sqrt[m]{a}$ $(a > 0)$ and $\log_b t$ $(t > 0,\ b > 0,\ b \neq 1)$, but each of these procedures uses such a property of real numbers which is equivalent to the continuous ordering of R, see [1], pp. 95—101; [12] (e.g. the continuous ordering is directly used in proofs given in [2], [4], [9]; the existence of the limit of a non-decreasing bounded sequence of real numbers is used in proofs given in [6], [10], [14]). Some of the previous procedures have in a sense a constructive character and therefore their use in the pedagogical process is very suitable. On the other hand, the variety of procedures for proving the existence of $\sqrt[m]{a}$ and $\log_b t$ may make vague the fact that the existence $\sqrt[m]{a}$ and $\log_b t$ is a consequence of the continuity of the ordering of R. This last fact seems to be more evident if we use some of the unifying principles that are known in contemporary real analysis.

2 Proofs of the existence of the *m*-th root and the logarithm of positive numbers based on unifying principles for proving fundamental theorems of real analysis

In the 20th century some efforts to prove the fundamental theorems of real analysis have led to formulations of some unifying principles (cf. [5], [7], [8], [13]). In the first part of this paper we have given a survey of various procedures leading to the proof of the existence of $\sqrt[m]{a}$ and $\log_b t$. In this part se shall use the principles formulated in [7] and [13] for these proofs and we shall evaluate such a procedure from the point of view of the didactics of mathematics.

The principles formulated in [7] and [13] are very effective and enable us to give the proofs of several fundamental theorems of real analysis from the unified point of view. This fact has a great meaning in teaching mathematics. The extension of the application of these principles for the proofs of existence of $\sqrt[m]{a}$ and $\log_b t$ means extending the applicability of these principles to a domain which is a basis for constructing real analysis.

At first we shall introduce the principles of Leinfelder [7] and Shanahan [13].

Theorem L. Let $I \subseteq R$ be an interval (considered as an ordered set with the usual ordering $<$). Let L be a binary relation on I satisfying the following conditions:

(A1) $xLy \wedge yLz \Rightarrow xLz$ (transitivity)

(A2) $L \subseteq <$

(A3) The relation L is locally valid, i.e. if $c \in I$, then there exists a neighbourhood $V(c) \subseteq I$ (in the relative topology of I) such that:

$$x \in V(c), \quad x < c \Rightarrow xLc,$$

$$x \in V(c), \quad c < x \Rightarrow cLx$$

Then we have $L = <$.

Theorem S. Let $a, b \in R$, $a < b$. Let S be a system of closed intervals $I \subseteq \langle a, b\rangle$ satisfying the following conditions:

(B1) The system S is additive, i.e. if $\langle c, d\rangle$, $\langle e, f\rangle \in S$ and $\langle c, d\rangle \cap \langle e, f\rangle \neq \emptyset$, then $\langle c, d\rangle \cup \langle e, f\rangle \in S$.

(B2) The system S is local, i.e. to every $x \in \langle a, b\rangle$ there exists an interval $I \in S$ such that $x \in \operatorname{Int} I$ (in the relative topology of $\langle a, b\rangle$).

Then we have $\langle a, b\rangle \in S$.

Let us remark that if we formulate Theorems L and S for a chain (totally ordered set) with the minimal and the maximal element, then Theorem L is equivalent to Theorem S (cf. [11]).

Using theorems S and L we shall give the proofs of the following Theorems.

Theorem A. Let $a \in R$, $a > 0$, $m \in N$, $m > 1$. Then there exists a $y \in R$, $y > 0$ such that $y^m = a$.

Theorem B. Let $b \in R$, $b > 1$ and $t \in (0, +\infty)$. Then there exists a $v \in R$ such that $a^v = t$.

At first we shall introduce the proofs of Theorems A and B by using Theorem L.

Proof of Theorem A. Let us assume that there is no $y > 0$ with $y^m = a$. Put $I = (0, +\infty)$ in Theorem L and let $<$ denote the usual ordering. Define $L \subseteq I \times I$ in the following way:

$$xLy \Leftrightarrow (x < Y) \wedge [(x^m < y^m < a) \vee (a < x^m < y^m)]. \tag{8}$$

Evidently, L satisfies the condition (A2) (see (8)). We shall show that it satisfies (A1), too.

Let xLy, yLz. Then $x < y$, $y < z$ and hence $x < z$. Since xLy, one of the following possibilities:

a) $x^m < y^m < a$, b) $a < x^m < y^m$

occurs.

If a) is valid, then since yLz and $y^m < a$, we get

$$y^m < z^m < a. \tag{9}$$

But then from a) and (9) we get $x^m < z^m < a$, hence xLz.

If b) occurs, then we proceed analogously as in the case a).

We shall check that L satisfies the condition (A3).

Let $c \in I = (0, +\infty)$. On account of the assumption we have $c^m \neq$ a. Therefore one of the following cases:

α) $c^m < a$ β) $c^m > a$

occurs. If α) occurs, then choose a $\delta > 0$ such that

$$V(c) = (c - \delta, c + \delta) \subset I \tag{10}$$

$$\delta < \frac{a - c^m}{(1 + c)^m - c^m}, \qquad \delta < 1 \tag{11}$$

Hence (see (10)) $V(c)$ is neighbourhood of c. If $x \in V(c)$, $x < c$, then $x^m < c^m$ and so (see α)) we have $x^m < c^m < a$. Hence xLc.

If $x \in V(c)$, $c < x$, then $c < x < c + \delta$. Using the binomial theorem according to (11) we get

$$\begin{aligned}(c + \delta)^m &= c^m + \binom{m}{1} c^{m-1}\delta + \ldots + \binom{m}{m}\delta^m < \\ &< c^m + \delta\left[\binom{m}{1}c^{m-1} + \binom{m}{2}c^{m-2} + \ldots + \binom{m}{m}\right] = \\ &= c^m + \delta[(1 + c)^m - c^m] < c^m + (a - c^m) = a.\end{aligned}$$

Hence $(c + \delta)^m < a$ and therefore $x^m < (c + \delta)^m < a$.

So we get $c^m < x^m < a$, hence cLx.

Analogously we proceed in the case β). Since L satisfies the conditions (A1)—(A3), we have

$$L = <. \tag{12}$$

(see Theorem L).

Choose $n \in N$ such that

$$\frac{1}{n^m} < a < n^m. \tag{13}$$

Then we have $\frac{1}{n} < n$ and hence accordint to (12) we get $\frac{1}{n} Ln$. But then on account of the definition of L (see (8)) we get

$$\frac{1}{n^m} < n^m < a \quad \text{or} \quad a < \frac{1}{n^m} < n^m.$$

Both these inequalities contradict (13).
The proof is finished.

For the proof of Theorem B we shall use the following well-known result.

Lemma 1. Let $d \in R$, $d > 0$. Then $\lim_{n \to \infty} \sqrt[n]{d} = 1$ (see e.g. 14, p. 113).

Proof of Theorem B. Let us suppose that $t \in (0, +\infty)$ and there is no $v \in R$ with $b^v = t$. Put in Theorem L: $I = R = (-\infty, +\infty)$,

$$xLy \Leftrightarrow (x < y) \wedge [(b^x < b^y < t) \vee (t < b^x < b^y)] \tag{14}$$

The relation L satisfies the condition (A2) in Theorem L and analogously to the previous proof we can check that it satisfies (A1), too.

We prove that L satisfies (A3). Let $c \in R$. According to our assumption we have $b^c \neq t$. Therefore one of the following assertions:

1) $b^c < t$, 2) $t < b^c$

is valid.

Let 1) be valid (in the 2) we proceed analogously). Then according to Lemma 1 there exists a $k \in N$ such that

$$b^{c+\frac{1}{k}} < t. \tag{15}$$

Put $V(c) = \left(c - \frac{1}{k}, c + \frac{1}{k}\right)$ (a neighbourhood of c). If $x \in V(c)$, $x < c$, then $b^x < b^c < t$ and so xLc. If $x \in V(c)$, $c < x$, then $c < x < c + \frac{1}{k}$ and according to (15) we obtain

$$b^c < b^x < b^{c+\frac{1}{k}} < t,$$

hence $b^c < b^x < t$, so we have cLx.

Hence L satisfies the conditions (A1)—(A3) of Theorem L and therefore

$$L = <. \tag{16}$$

Since $b > 1$, we can choose an $n \in N$ such that

$$b^{-n} < t < b^n \tag{17}$$

According to (16) we have $-nLn$ and therefore by the definition of L (see

(14)) one of the inequalities: $b^{-n} < b^n < t$, $t < b^{-n} < b^n$ occurs. But each of these inequalities contradicts (17). The proof is finished.

We shall now give the proofs of Theorem A and Theorem B by using Theorem S.

Proof of Theorem A. We shall restrict ourselves only to the case if $a > 1$.

Let us suppose that there is no $y > 0$ with $y^m = a$ $(m > 1)$. Put $I_0 = \langle 0, a \rangle$. Then

$$0 < a < a^m. \tag{18}$$

Let us construct the system S of subintervals of I_0 in the following way:

$$\langle c, d \rangle \in S \Leftrightarrow (\forall x \in \langle c, d \rangle)\, x^m < a \vee (\forall x \in \langle c, d \rangle)\, a < x^m.$$

We shall show that S satisfies the conditions (B1), (B2) of Theorem S.

Let $\langle c, d \rangle$, $\langle e, f \rangle \in S$ and $\langle c, d \rangle \cap \langle e, f \rangle \neq \emptyset$.

Choose a $z \in \langle c, d \rangle \cap \langle e, f \rangle$. Assume e.g. that

$$(\forall x \in \langle c, d \rangle)\, x^m < a \tag{19}$$

(in the case if $(\forall x \in \langle c, d \rangle)\, a < x^m$ we proceed analogously). Then we have $z^m < a$ and since $z \in \langle e, f \rangle$ and $\langle e, f \rangle \in S$, we obtain

$$(\forall x \in \langle e, f \rangle)\, x^m < a. \tag{20}$$

But then according to (19), (20) for each $x \in \langle c, d \rangle \cup \langle e, f \rangle$ we have $x^m < a$. Hence $\langle c, d \rangle \cup \langle e, f \rangle \in S$ (the condition (B1)).

We shall show that S satisfies the condition (B2). Let $x_0 \in I_0$ and let e.g. $x_0 < a$ (for $x_0 = a$ we proceed analogously). According to our assumption one of the following cases

1) $x_0^m < a$ 2) $a < x_0^m$

occurs.

Let 1) be valid (in case 2) we proceed analogously). We choose a $\delta > 0$ such that

$$\delta < \frac{a - x_0^m}{(1 + a)^m - a^m}, \qquad \delta < 1. \tag{21}$$

Then $x_0 \in \operatorname{Int} I$, where $I = \langle x_0, - \delta, x_0 + \delta > \cap I_0$.

As in the proof of Theorem A based on using Theorem L here we can also show that for each $x \in I$ we have $x^m < (x_0 + \delta)^m < a$, hence $x^m < a$, and so $I \in S$. Hence S is a local system.

According to Theorem S we have $\langle 0, a \rangle \in S$. It follows from this (since 1) is valid) that for each $x \in \langle 0, a \rangle$ we have $x^m < a$. Especially we have $a^m < a$, which is a contradiction to (18). This ends the proof.

Proof of Theorem B. Let $t \in (0, +\infty)$ and let us assume that there is no $v \in R$ with $b^v = t$.

Choose a $k \in N$ such that

$$b^{-k} < t < b^k \tag{22}$$

Put $I_0 = \langle -k, k \rangle$ and define the system S of subintervals of I_0 in the following way:

$$\langle c, d \rangle \in S \Leftrightarrow (\forall x \in \langle c, d \rangle)\, b^x < t \vee (\forall x \in \langle c, d \rangle)\, t < b^x.$$

The reader can verify that S satisfies the conditions (B1), (B2) of Theorem S. Hence according to this theorem we have $\langle -k, k \rangle \in S$. Since $b^{-k} < t$ (see (22)) we have, with respect to (23), the inequality $b^x < t$ for each $x \in \langle -k, k \rangle$. In particular, we have $b^k < t$, which is a contradiction to (22). This ends the proof.

Remarks. The wide applicability of Theorem L and Theorem S (cf. [7], [13]) presents a very great advantage for using these unifying principles. Using Theorem L or Theorem S we are able to prove almost all fundamental theorems of real analysis (see [7], [11], [13]). The extension of the applicability of these principles for proving the existence of $\sqrt[m]{a}$ and $\log_b t$, which we have made in this part of our paper, gives teachers a tool to make the students familiar with these principles already at the beginning of their study of mathematics, when they study the fundamental properties of ordered field of real numbers. The principles of Leinfelder and Shanahan can be formulated at once after the introduction of fundamental properties of real numbers. The proofs of these principles are short and it can bee seen in their proofs where the continuity of ordering of R is used. It depends only on the teacher which of these principles he will use in his lectures.

The advantage of using some of the unifying principles consists also in the fact that the existence of the m-th root and the logarithm and also of fundamental theorems of real analysis can be proved by the same method. Such proofs can be remembered more easily than the proofs based on different properties of real numers.

REFERENCES

1. Cohen, L. W.—Ehrlich, G.: The Structure of the Real Number System. D. Van Nostrand Comp., Princeton—New Jersey—Toronto—New York—London, 1963.
2. Fichtengoľc, G. M.: Kurs differencialnogo i integralnogo isčislenija. Moskva—Leningrad, 1958.
3. Grzegorczyk, A.: Zarys arytmetyki teoretycznej. PWN, Warszawa, 1971.
4. Jarník, V.: Diferenciální počet (I). Academia, Praha 1984.
5. Khintchine, A.: The simpliest linear continuum (Russian) Uspechi Mat. Nauk 4 (1949), 180—197.
6. Knopp, K.: Theorie und Anwendung der unendlichen Reihen, 1931.
7. Leinfelder, H.: A unifying principle in real analysis. Real Anal. Exchange, 8 (1982—83), 511—518.

8. Perron, O.: Die vollstandige Induction im Kontinuum. Jhb. Deutsch. Math. Verein 35 (1926), 194—204.
9. Rudin, W.: Principles of Mathematical Analysis (Russian Translation), Mir, Moskva 1964.
10. Šalát, T.: Reálne čísla. Alfa, Bratislava 1982.
11. Šalát, T.: Remarks on unifying Principles in Real Analysis. Real Anal. Exch.
12. Šalát, T.—Tarabová, A.: On properties of Archimedian ordered fields that are equivalent to the completeness. Acta Math. Univ. Com. 42—43 (1983), 299—306.
13. Shanahan, P.: A unified proof of several basic theorems of real analysis. Amer. Math. Monthly 79 (1972), 895—898.
14. Sierpiński, W.: Dzialania nieskończone. Czytelnik 1948.
15. Sierpiński, W.: Arytmetyka teoretyczna. PWN, Warszawa 1959.

Adresses of authors:
Mária Benešová
Tibor Šalát
Katedra algebry a teórie čísel
Matematicko-fyzikálna fakulta
Mlynská dolina
842 15 Bratislava

Received: 21. 11. 1985

SÚHRN

O DÔKAZOCH EXISTENCIE *m*-TEJ ODMOCNINY A LOGARITMU KLADNÉHO ČÍSLA

M. Benešová—T. Šalát, Bratislava

Práca pozostáva z dvoch častí. Prvá časť obsahuje prehľad rôznych dôkazov existencie *m*-tej odmocniny a logaritmu kladného čísla. Druhá časť prináša nové dôkazy existencie *m*-tej odmocniny a logaritmu kladného čísla, založené na istej metóde H. Leinfeldera [7].

РЕЗЮМЕ

ДОКАЗАТЕЛЬСТВА СУЩЕСТВОВАНИЯ *m*-ТОГО КОРНЯ И ЛОГАРИФМА ПОЛОЖИТЕЛЬНЫХ ЧИСЕЛ

М. Бенешова—Т. Шалат, Братислава

Статья состоит из двух частей. В первой предъявляется обзор различных способов доказательств существования *m*-того корня и логарифма. Вторая часть посвещена предъявлению новых доказательств существования *m*-того корня и логарифма положительного числа использующих достижения работы Лейнфельдера [7].