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ON THE MULTIPLICITY OF $(X_1^m, X_2^n, X_1^k X_2^l)$

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Let $A = K[X_1, X_2]_{(X_1, X_2)}$ be a local ring over an algebraically closed field K with the maximal ideal $M = (X_1, X_2)$, and Q be an M-primary ideal in A. The length $L_A(A/Q)$ of the A-module A/Q is the number of a maximal chain of M-primary ideals beginning at Q and ending in M. The multiplicity $e_0(Q, A)$ of Q in A is defined to be the leading coefficient of the Hilbert—Samuel polynomial $L_A(A/Q^t)$, $t \gg 0$. It is known that $e_0(Q, A) \geqslant L_A(A/Q)$ with equality if and only if Q is an parametric ideal, i.e. Q is generated by two polynomials. For more details see e.g. [4].

In this short note we give a formula for the calculation of the multiplicity and length of certain class of M-primary ideals in A. More exactly, we prove the following.

Theorem. Let $Q = (X_1^m, X_2^n, X_1^k X_2^l)$ be an ideal in the local ring $A = K[X_1, X_2]_{(X_1, X_2)}$. Then

(a) $$e_0(Q, A) = \min\{m \cdot n,\ m \cdot l + n \cdot k\}$$

(b) $$L_A(A/Q) = m \cdot l + n \cdot k - k \cdot l.$$

Remark. In the above theorem it is naturally assumed that $m \geqslant k \geqslant 1$ and $n \geqslant l \geqslant 1$.

Before proving the Theorem, we formulate and prove a lemma using the notion "reduction of an ideal" introduced by Northcott and Rees, see [3].

Definition. An ideal $J \subseteq I$ of A is called a reduction of I, if $J \cdot I^{t-1} = I^t$ for an integer $t > 1$.

Lemma. Let A, Q be the same as in the Theorem. Put $Q_1 = (X_1^m + X_2^n, X_1^k X_2^l)$ and $Q_2 = (X_1^m, X_2^n)$. Then Q_1 is a reduction of Q if

$$m \cdot l + n \cdot k \leqslant m \cdot n \tag{1}$$

and Q_2 is a reduction of Q if

$$m \cdot l + n \cdot k \geqslant m \cdot n. \tag{2}$$

Proof of the lemma. Let's prove the case (1) first. Since $Q_1 \cdot Q^{t-1} \subseteq Q^t$ for all $t > 1$, we need to prove the opposite inclusion $Q^t \subseteq Q_1 \cdot Q^{t-1} = ((X_1^m + X_2^n) \cdot Q^{t-1}, (X_1^k X_2^l) \cdot Q^{t-1})$ for a certain $t > 1$.
Let $F \in Q^t$. Then $F = X_1^k X_2^l \cdot F'$, with $F' \in Q^{t-1}$ (and clearly $F \in Q_1 \cdot Q^{t-1}$ for all $t > 1$) or $F \in (X_1^m, X_2^n)^t = (\ldots, F_j = X_1^{m(t-j)} X_2^{nj}, \ldots), j = 0, 1, \ldots, t$.
For $(X_1^m, X_2^n)^{t-1} \cdot (X_1^m + X_2^n) = (\ldots, X_1^{m(t-i)} X_2^{ni} + X_1^{m(t-i-1)} X_2^{n(i+1)}, \ldots)_{i=0,1,\ldots,t-1} =$
$= (F_0 + F_1, F_1 + F_2, \ldots, F_j + F_{j+1}, \ldots, F_{t-1} + F_t) \subseteq Q_1 \cdot Q^{t-1}$, it is clear that $F_j \in$
$\in Q_1 \cdot Q^{t-1}$ for all $j = 0, 1, \ldots, t$ if it is so for at least one j.

Take $H = X_1^{m+k(t-1)} X_2^{l(t-1)} \in Q_1 \cdot Q^{t-1}$. Then $F_j \in (H) \subseteq Q_1 \cdot Q^{t-1}$ when

$$l(t-1) \leqslant jn$$

$$m + k(t-1) \leqslant m(t-j)$$

and this is equivalent to

$$\frac{l \cdot (t-1)}{n} \leqslant j \leqslant \frac{(m-k) \cdot (t-1)}{m}.$$

From the assumption (1), i.e. $m \cdot l + n \cdot k \leqslant m \cdot n$, and putting $t = n + 1, j = l$, we get $F_j \in Q_1 \cdot Q^{t-1} = Q_1 \cdot Q^n$ for $j = l$. Thus for $t = n + 1$ there is $Q_1 \cdot Q^{t-1} = Q^t$ as required.
Let us go to the case $m \cdot l + n \cdot k \geqslant m \cdot n$ now. It is clear that $Q^t \subseteq Q_2 \cdot Q^{t-1}$ if and only if $(X_1^k X_2^l)^t \in Q_2 \cdot Q^{t-1}$ and this holds if and only if there is $u \leqslant t$ such that $(X_1^k X_2^l)^u \in Q_2^u$. Then $(X_1^k X_2^l)^u = X_1^{m(u-j)} X_2^{nj} \cdot G$, $G \in A$. Now put $u = n$, $j = l$. Then $m \cdot l + n \cdot k \geqslant m \cdot n$ implies $Q^u \subseteq Q_2 \cdot Q^{u-1}$, i.e. Q_2 is a reduction of Q in the case (2).

Proof of the Theorem.

(a) In the case $m \cdot l + n \cdot k \leqslant m \cdot n$ the ideal $Q_1 = (X_1^m + X_2^n, X_1^k X_2^l)$ is a reduction of Q by virture of the above lemma. Then $e_0(Q, A) = e_0(Q_1, A)$, see [3]. Putting $A_i = K[X_i]_{(X_i)}$, $i = 1, 2$ and using the Associativity Formula (see [1], Theorem 24.7) for the parametric ideal Q_1, we get

$$\begin{aligned} e_0(Q_1, A) &= e_0(Q_1 \cdot A/(X_1), A/(X_1)) \cdot e_0((X_1^k X_2^l) \cdot A_{(X_1)}, A_{(X_1)}) + \\ &+ e_0(Q_1 \cdot A/(X_2), A/(X_2)) \cdot e_0((X_1^k X_2^l) \cdot A_{(X_2)}, A_{(X_2)}) = \\ &= e_0((X_2^n) \cdot A_2, A_2) \cdot e_0((X_1^k) \cdot A_1, A_1) + \\ &+ e_0((X_1^m) \cdot A_1, A_1) \cdot e_0((X_2^l) \cdot A_2, A_2) = n \cdot k + m \cdot l. \end{aligned}$$

In the case $m \cdot l + n \cdot k \geqslant m \cdot n$ the Lemma implies that the ideal $Q_2 = (X_1^m, X_2^n)$ is a reduction of Q, therefore $e_0(Q, A) = e_0(Q_2, A) = m \cdot n$ (see [2], Chap. 7, Corollary 1 of Theorem 7).

(b) In order to prove (b) we use the exact sequence

$$0 \to A/I_1 \cap I_2 \to A/I_1 \oplus A/I_2 \to A/(I_1, I_2) \to 0,$$

where $I_1 = (X_1^m, X_2^l)$, $I_2 = (X_1^k, X_2^n)$. For $Q = I_1 \cap I_2$ and $(I_1, I_2) = (X_1^k, X_2^l)$, the addivity of length of A-modules yields

$$L_A(A/Q) = L_A(A/I_1) + L_A(A/I_2) - L_A(A/(I_1, I_2)) = m \cdot l + n \cdot k - k \cdot l,$$

as required. The proof is now complete.

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SÚHRN

O NÁSOBNOSTI $(X_1^m, X_2^n, X_1^k X_2^l)$

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Nech $Q = (X_1^m, X_2^n, X_1^k X_2^l)$ je ideál v lokálnom okruhu $A = K[X_1, X_2]_{(X_1, X_2)}$. V práci je ukázané, že pre Samuelovu násobnosť primárneho ideálu Q platí

$$e_0(Q, A) = \min\{m \cdot n, m \cdot l + n \cdot k\}.$$

РЕЗЮМЕ

О КРАТНОСТИ ИДЕАЛА $(X_1^m, X_2^n, X_1^k X_2^l)$

Эдуард Бодя и Штефан Солчан

Пусть $Q = (X_1^m, X_2^n, X_1^k X_2^l)$ — идеал в локальном кольце $A = K[X_1, X_2]_{(X_1, X_2)}$. В работе показывается, что для кратности Самюеля примарного идеала Q имеет место следующее равенство

$$e_0(Q, A) = \min\{m \cdot n, m \cdot l + n \cdot k\}.$$