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ON THE p-CENTER AND p-MEDIAN PROBLEMS IN CACTUS GRAPHS

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1 Introduction

The terminology used in this paper is standard and follows that of [1].

Let $G = G(V, E)$ be a finite connected graph with a nonnegative number $w(v)$ (called the weight of a vertex v) associated with each of its $|V| = n$ vertices, and a positive number $l(e)$ (called the length of an edge e) associated with each of its $|E|$ edges (i.e., $l(e) = c(v_r, v_s)$, where $e = (v_r, v_s)$).

Let $X_p = \{x_1, x_2, \ldots, x_p\}$ be a set of p points on G, where by a point on G we mean a poing along any edge of G which may or not be a vertex of G. We define the distance $d(v, X_p)$ between a vertex v of G and a set X_p on G by

$$(1.1) \qquad d(v, X_p) = \min_{1 \le i \le p} \{d(v, x_i)\},$$

where $d(v, x_i)$ is the length of a shortest path in G between vertex v and point x_i (x_i can be considered as a new vertex inserted into the edge e. Let

$$(1.2) \qquad F_G(X_p) = \max_{v \in V} \{w(v) \cdot d(v, X_p)\}.$$

Let X_p^* be such that

$$(1.3) \qquad F_G(X_p^*) = \min_{X_p \text{ on } G} \{F(X_p)\}.$$

then X_p^* is called an absolute p-center of G and $F_G(X_p^*)$ is called the absolute p-radius of G. If X_p and X_p^* in (1.3) are restricted to be sets of p verticles of G, then X_p^* is called a vertex p-center and $F_G(X_p^*)$ is called the vertex p-radius of G. Further we define:

$$(1.4) \qquad H_G(X_p) = \sum_{v \in V} w(v) \cdot d(v, X_p).$$

We call $H_G(X_p)$ the distance-sum of the set X_p. If X_p^* on G is such that

$$(1.5) \qquad H_G(X_p^*) = \min_{X_p \text{ on } G} \{H_G(X_p)\},$$

then X_p^* is called a p-median of G [7], [8]. Hakimi [8] has shown that there exists a set of p vertices $V_p^* \subset V$, such that $H_G(V_p^*) = H_G(X_p^*)$. If all the vertices of G have the same weigt c then we shall assume that $c = 1$ and we refer to this case as the wertex-unweight case. Otherwise, we say that G is a vertex-weighted graph. We shall assume that $p < n$, since if $p = n$, then $V_p^* = V$, $F_G(V_p^*) = 0$ and $H_G(V_p^*) = 0$, while $p > n$ has no mathematical meaning. Further assume that the graph G contains neither loops nor multiple edges. Finally, we assume that for each edge $e = (v_r, v_s)$ the length of e is equal to the distance between v_r and v_s (i.e., $l(e) = d(v_r, v_s)$), because otherwise the edge e could be eliminated without affecting the p-radius, the p-median, or the p-center of G.

The "inverse" of the vertex p-center problem is defined as follows: Given a graph $G = G(V, E)$ and a positive integer r, find the smallest positive integer p such that the p-radius of G is not greater than r. This number p is called the vertex domination number of radius r of G while a corresponding p-center is called a vertex dominating set of radius r.

The problems of finding the domination number and a dominating set are NP-hard [6]. However, the problem of finding the domination number and a dominating set when G is a tree was solved in linear time by Cockayne, Goodman, and Hedetniemi [3]. The problem of finding a p-center of G was originated by Hakimi [7], [8] and is discussed in a number of papers [4], [5], [10], [11], [12], [17]. In [9] Hakimi, Schmeichel and Pierce discussed improvements and generalizations of various existing algorithms for finding p-centers of graphs and gave the corresponding orders of complexity.

Obviously, if one knows how to find a p-center, then by performing a binary search over the n possible values of the domination number, one can find a dominating set of radius r. On the other hand, information on the domination number and on dominating sets can by used to draw conclusions on p-centers and the p-radii.

Kariv and Hakimi in [15] showed that the problem of finding a vertex p-center (for $1 < p < n$) of a vertex-weighted network, and the problem of finding a dominating set of radius r, are NP-hard even in the case where the network has a simple structure (e.g. a planar graph of maximum vertex degree 3), and obtained the following algorithms, when the network is a vertex-weighted tree: an $0(n)$ algorithm for finding a vertex dominating set of radius r; an $0(n \cdot \lg n)$ algorithm for finding a 1-center; an $0(n^2 \cdot \lg n)$ algorithm for finding a vertex p-center for any $1 < p < n$.

Also, the problem to find a near optimal vertex (or absolute) p-center is NP-hard [19].

Moreover it is known that the problem of finding a p-median of a network is an NP-hard problem even when the network has a simple structure [16]. Since the publication of [7], there have been many attempts to efficiently compute a p-median of a network [13], [22], [23] (see also [18], [21]). In [16] Kariv and Hakimi described an algorithm which finds a p-median of a tree (for $p > 1$) in time $0(n^2 \cdot p^2)$. An $0(n)$ algorithm for finding a 1-median of a tree was also described by Goldman [5].

2 Construction of a tree from a given graph

In many practical situations we do not need to know the exact location of the p-center and the p-median (obtained after long calculations); we need to know only some bounds of the p-radius or the minimum distance sum of the given graph (obtained with less difficulties). Our aim in this paper will be to determine some bounds of the p-radius and the minimum distance sum of the p-median.

Definition. Cactus is a connected graph, every cyclic blok of which is a cycle [1].

Lemma 2.1. Let $G = (V, E)$ be a connected graph with edge length $l(e) > 0$ and vertex weight $w(v) \geqslant 0$. Let there exist a dominating set of p points of radius r on G. Then there exists a dominating set $X_p = (x_1, x_2, \ldots, x_p)$ of radius r on G such that (2.1) if x_i for some i is an internal point of some edge (v_{i_1}, v_{i_2}), then

$$d(v_{i_1}, (X_p - \{x_i\})) > d(v_{i_1}, x_i)$$

and also

$$d(v_{i_2}, (X_p - \{x_i\})) > d(v_{i_2}, x_i).$$

Proof. We prove this lemma by contradiction. Assume that there is no dominating set X_p of radius r on G fulfilling (2.1) and let $Y_p = (y_1, y_2, \ldots, y_p)$ be a dominating set of radius r on G such that this set Y_p has the least number k of points not satisfying condition (2.1) from all the dominating sets of p points of radius r on G.

Let $k \neq 0$ and let $y_s \in Y_p$ be an internal point of edge (v_1, v_2) which does not satisfy (2.1). Without loss of generality we can assume that

$$d(v_1, y_s) \geqslant d(v_1, (Y_p - \{y_s\})) = d(v_1, y_t),$$

where y_t is some vertex of $(Y_p - \{y_s\})$. We define $Y'_p = (y'_1, \ldots, y'_p)$ in such a way, that $y'_i = y_i$ for all $i \neq s$ and $y'_s = v_2$.
We divide $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$ so that

$$V_1 = \{v \in V;\ d(v, y_s) = d(y_s, v_1) + d(v_1, v) < d(y_s, v_2) + d(v, v_2)\}$$

and

$$V_2 = \{v \in V;\ d(v, y_s) = d(y_s, v_2) + d(v_2, v) \leqslant d(y_s, v_1) + d(v, v_1)\}.$$

(2.2) For all $i \neq s$ and every vertex $v \in V$ it holds

$$d(y'_i, v) = d(y_i, v).$$

(2.3) For every vertex $v \in V_2$ it holds

$$d(y'_s, v) = d(v_2, v) \leq d(v_2, v) + d(v_2, y_s) = d(y_s, v).$$

(2.4) For every vertex $v \in V_1$ it holds

$$d(y'_t, v) \leqslant d(y'_t, v_1) + d(v_1, v) = d(y_t, v_1) + d(v_1, v) \leqslant$$
$$\leqslant d(y_s, v_1) + d(v, v_1) = d(v, y_s).$$

From (2.2), (2.3) and (2.4) it follows that for all $v \in V$ it holds that $d(v, Y'_p) \leqslant \leqslant d(v, Y_p)$, i.e. Y'_p is a dominating set of radius r, too, and there are less points in Y'_p not satisfying (2.1) than in Y_p. This is a contradiction. Hence there exists a dominating set X_p of radius r, all points of which satisfy the condition (2.1).

Lemma 2.2. Let $G = (V, E)$ be a connected graph with a given weight w of vertices and a length l of edges. Let there exist a dominating set of p points of radius r on G. Then there exists a dominating set $X_p = (x_1, x_2, \ldots, x_p)$ of radius r on G such that

(2.5) if x_i, x_j for some i, j are internal points of some edges e_i, e_j respectively, then e_i and e_j are not adjacent.

Proof. From Lemma 2.1 it follows that there exists a dominating set $X_p = = (x_1, x_2, \ldots, x_p)$ of radius r on G with the property (2.1). We shall show that then X_p has the property (2.5), too. We shall prove this fact by contradiction.

Let X_p have not the property (2.5). Then there exist internal points $x_s \in e_s$, $x_t \in e_t$ in G such that e_s and e_t are adjacent, i.e. there exist vertices $v_0, v_s, v_t \in V$ such that $e_s = (v_0, v_s)$, $e_t = (v_0, v_t)$. Without loss of generality we can assume that $d(x_s, v_0) \leqslant d(x_t, v_0)$, and then $d(v_0, (X_p - \{x_t\})) \leqslant d(v_0, x_s) \leqslant d(v_0, x_t)$. Thus X_p does not satisfy (2.1), which is a contradiciton proving the lemma.

Lemma 2.3. Let $G = (V, E)$ be a connected graph with a vertex weight w and edge length l. Let $X_p = (x_1, x_2, \ldots, x_p)$ be a set of p points on G such that for every internal point $x_i \in X_p$ of an edge (v_{i_1}, v_{i_2}) it holds that

$$d(v_{i_1}, (X_p - \{x_i\})) > d(v_{i_1}, x_i)$$

and also

$$d(v_{i_2}, (X_p - \{x_i\})) > d(v_{i_2}, x_i).$$

Then there exists a spanning tree T of G such that all the points of X_p lie on T, too, and that $d_G(v, X_p) = d_T(v, X_p)$ for all $v \in V$.

Proof. We shall construct the spanning tree T using induction in finite number of steps.

STEP 0: We define $T_0 = (V(T_0), E(T_0))$, where
$V(T_0) = \{v \in V; \exists x_i \in X_p, x_i \equiv v$ or $\exists x_j \in X_p$ and $v' \in V$ so that x_j is an internal point of the edge $(v, v')\}$;
$E(T_0) = \{e \in E; \exists x_i \in X_p$ such that x_i is an internal point of the edge $e\}$.
It follows from the proof of Lemma 2.2. that T_0 is a forest with maximum vertex degree 1, because no two edges in T_0 are adjacent. It follows from the assumption that for every $v \in V(T_0)$ it holds that $d_{T_0}(v, X_p) =$
$= d_G(v, X_p)$.

STEP k: Let T_{k-1} be such a forest that for all $v \in V(T_{k-1})$ we have $d_{T_{k-1}}(v, X_p) =$
$= d_G(v, X_p)$. Let v' be such a vertex of $V(G) - V(T_{k-1})$ that for all $v \in V(G) - V(T_{k-1})$ it holds that $d_G(v', X_p) \leqslant d_G(v, X_p)$. Let $(x_i = v_1, v_2, \ldots,$
$v'', v')$ be a path of the length $d_G(v', X_p)$ from the set X_p to v' in G. Then we define

$$V(T_k) = V(T_{k-1}) \cup \{v'\}$$

$$E(T_k) = E(T_{k-1}) \cup \{(v'', v')\}.$$

As we have added only an end vertex with its edge by this opeation, the graph T_k remains a forest. From the choice of vertex v' it follows that $v'' \in V(T_{k-1})$ and from the inductional assumption it follows that

$$d_G(v'', X_p) = d_{T_{k-1}}(v'', X_p).$$

From $E(T_k) \subset E(G)$ it follows that

$$d_{T_k}(v', X_p) \geqslant d_G(v', X_p).$$

Further we have

$$d_{T_k}(v', X_p) \leqslant d_{T_k}(v', v'') + d_{T_k}(v'', X_p) = d_G(v', v'') +$$

$$+ d_G(v'', X_p) = d_G(v', X_p).$$

This implies that for all $v \in V(T_k)$ it holds that

$$d_{T_k}(v, X_p) = d_G(v, X_p).$$

End of step k.

Now we put $m = |V(G) - V(T_0)|$. After m steps we obtain the forest T_m such that $V(T_m) = V(G)$ and $E(T_m) \subset E(G)$ and for all $v \in V$ it holds that

$$d_{T_m}(v, X_p) = d_G(v, X_p).$$

Now in the end we add to the forest T_m arbitrary edges from $E(G)$ to obtain a connected graph without cycles. In this way we obtain the spanning tree T of G which we wanted to find.

Theorem 2.1. Let $G = (V, E)$ be a connected graph with a given weight w of vertices and a length l of edges. Let there exist a dominating set of p points of radius r on G. Then there exists a spanning tree T of G with a dominating set $X_p = (x_1, x_2, \ldots, x_p)$ on T of the same radius r.

Proof. According to Lemma 2.1 there exists a dominating set $X_p = (x_1, x_2, \ldots, \ldots, x_p)$ on G such that $d_G(v, X_p) \leqslant r$ for every $v \in V$ and that for every internal point $x_i \in X_p$ of an edge (v_{i_1}, v_{i_2}) it holds that

$$d(v_{i_1}, (X_p - \{x_i\})) > d(v_{i_1}, x_i)$$

and also

$$d(v_{i_2}, (X_p - \{x_i\})) > d(v_{i_2}, x_i).$$

Then according to Lemma 2.3 there exists a spanning tree T of G such that all points of X_p lie on T, and that

$$d_T(v, X_p) = d_G(v, X_p) \leqslant r$$

holds for all $v \in V$.

From this directly follows that X_p is a dominating set of radius r on T.

Lemma 2.4. Let $G = (V, E)$ be a connected graph with the vertex weight w and the edge length l and let $G_1 = (V_1, E_1)$ be a connected subgraph of G, such that $V = V_1$, $E_1 \subset E$. Then the value of the (absolute or vertex) p-center or the minimum distance sum of the p-median of G_1 is equal to or greater than the value of the same parameter of G.

Proof. The proof of this lemma follows immediately from the fact that $d_{G_1}(x, v) \geqslant d_G(x, v)$ for arbitrary vertex $v \in V$ and arbitrary point x on G_1.

Theorem 2.2. Let $G = (V, E)$ be a connected graph with the vertex weight w and with the edge length l. Let the (absolute or vertex) p-radius of G be r_p. Then there exists a spanning tree T of G with the same p-radius r_p.

Proof. The proof of this theorem follows directly from Theorem 2.1. Let Y_p be an absolute (or vertex) p-center on G with the p-radius r_p. Then Y_p is the dominating set on G of the radius r_p. From Theorem 2.1 it follows that there exists a spanning tree T of G with a dominating set X_p of radius r_p.

From the proof of Theorem 2.1 it follows that if $Y_p \subset V$, then $X_p \equiv Y_p$. From this fact and from Lemma 2.4. it directly follows that X_p is the absolute (or vertex) p-center of T with the radius r_p.

Theorem 2.3. Let $G = (V, E)$ be a connected graph with the vertex weight w and the edge length l. Let $X_p = (x_1, x_2, \ldots, x_p)$, $X_p \in V$ be a vertex p-median of G with the distance sum $H(X_p)$. Then there exists a spanning tree T of G such that X_p is a p-median of T with the same distance sum, i.e.

$$H_T(X_p) = H_G(X_p).$$

Proof. As the set of vertices $X_p \subset V$ satisfies the conditions of Lemma 2.3, it follows that there exists a spanning tree T of G such that $d_G(v, X_p) = d_T(v, X_p)$ for all $v \in V$. From this it immediately follows that

$$H_T(X_p) = \sum_{v \in V} w(v) \cdot d_T(v, X_p) = \sum_{v \in V} w(v) \cdot d_G(v, X_p) = H_G(X_p).$$

From this and from Lemma 2.4 we have that X_p is a p-median of T with the same distance sum as the graph G.

Definition 2.1. Let G be a connected vertex-weighted graph. Then a spanning tree T of the graph G will be called the absolute (vertex, respectively) p-central spanning tree of a graph G when the absolute (or vertex) p-radii of G and T are equal.

The existence of such a spanning tree for every graph follows from Theorem 2.2, and we can find it using the next algorithm.

Algorithm 2.1. Determining the p-central spanning tree of a graph G

Step 1: Determine some p-center X_p of $G = (V, E)$.

Step 2: Every vertex of p-center which is an internal vertex of some edge and does not satisfy the condition of Lemma 2.3 transform to one of the end vertices of the edge and obtain a p-center X'_p.

Step 3: We define the forest $T_0 = (V(T_0), E(T_0))$ so that $V(T_0) = \{v \in V;\ \exists x \in X'_p,\ x \equiv v$ or $\exists x' \in X'_p$ and $v' \in V$ so that x' is an iternal vertex of $(v, v') \in E\}$ and $E(T_0) = \{e \in E;\ \exists x \in X'_p$ such that x is an internal vertex of $e\}$.

Step 4: We have the forest $T_{k-1} = (V(T_{k-1}), E(T_{k-1}))$. We choose $v_* \in V(G) - V(T_{k-1})$ such that for all $v \in V(G) - V(T_{k-1})$ it holds that $d_G(v_*, X'_p) \leqslant d_G(v, X'_p)$. Let $(x_i = v_1, v_2, \ldots, v_e, v_*)$ be a path of length $d_G(v, X'_p)$ from the set X'_p to the vertex v_* in the graph G. Then we define

$$V(T_k) = V(T_{k-1}) \cup \{v_*\}$$

and

$$E(T_k) = E(T_{k-1}) \cup \{(v_e, v_*)\}.$$

Step 5: If $|V(T_k)| < |V(G)|$, then go to step 4 else go to step 6.

Step 6: Add arbitrary edges from $E(G)$ to the last obtained tree T_k in order to obtain a connected graph T without cycles. Then T is the p-central spanning tree of the graph G.

The correctness of this algorithm follows from the proof of Theorem 2.2. The complexity of Algorithm 2.1 is determined by its most complicated step 1 and so it is the same as that of the algorithm determining the p-center of a graph. The complexity of Algorithm 2.1 in the simplest case, for $p = 1$, is $0(|E|\, n \lg n)$, and we will use the 1-central spanning tree of some graph for determining the upper bound of the p-radius and the minimum distance sum of G.

In order to determine the lower bound of these invariants we will use another tree obtained from G using algorithm 2.2 for all cycles of G. In this way we decrease the complexity of these problems.

Algorithm 2.2. Construction of a star from a vertex-weighted m-cycle.

Let $G = (V, E)$ be a m-cycle with $V = \{v_1, v_2, \ldots, v_m\}$, $E = \{(v_1, v_2), (v_2, v_3), \ldots, \ldots, (v_m, v_1)\}$, with a vertex weight w_G and an edge length l_G. We define $T = (V(T), E(T))$ so that $V(T) = V(G) \cup \{v_0\}$, $E(G) = \{(v_0, v_1), \ldots, (v_0, v_m)\}$, $v_0 \notin V(G)$, with a vertex weight $w_T(v_i) = w_G(v_i)$ for $i = 1, 2, \ldots, m$ and $w_T(v_0) = 0$ and the edge length l_T such that

$$l_T(v_0, v_i) \geqslant 0 \qquad \text{for all } i = 1, 2, \ldots, m$$

$$l_T(v_0, v_i) + l_T(v_0, v_{i+1}) \leqslant l_G(v_i, v_{i+1}) \qquad \text{for all } i = 1, \ldots, m-1$$

$$l_T(v_0, v_m) + l_T(v_0, v_1) \leqslant l_G(v_m, v_1)$$

and

$$\sum_{1 \leqslant i < j \leqslant m} d_G(v_i, v_j) - [l_T(v_0, v_i) + l_T(v_0, v_j)]$$

is minimum.

This is a linear programming problem which can be solved by the simplex method [20], [21].

Note 2.1. This construction needs some comments in the case of $m = 3$. Then it is easy to prove that the optimal solution l_T of the LP problem has the property

$$d_G(v_i, v_j) = l_T(v_0, v_i) + l_T(v_0, v_j) \qquad \text{for all } 1 \leqslant i < j \leqslant 3.$$

Note 2.2. We can apply this construction also to cycles on general graphs and in some cases it can happen that we obtain a graph with some edges with zero length. In such a case we can delete the zero edges in the following manner which has no influence on the p-radius or the minimum distance sum:

Let $l_T(v_1, v_2) = 0$ for some $v_1, v_2 \in E(T)$ and let $w_T(v_2) \geqslant w_T(v_1)$. Then we define

$$T' = (V(T'), E(T'))$$

so that

$$V(T') = V(T) - \{v_1\}$$

$$E(T') = E(T) - \{e \in E;\ \exists\, v,\ e = (v_1, v)\} \cup \{(v_2, v);\ (v_1, v) \in E\}$$

$$w_{T'}(v) = w_T(v) \qquad \text{for all } v \in V(T'),\ v \neq v_2$$

and

$$w_{T'}(v_2) = w_T(v_1) + w_T(v_2);$$

$$l_{T'}(u, v) = l_T(u, v) \qquad \text{for all } u, v \in V(T'),\ u, v \neq v_2$$

$$l_{T'}(v_2, v) = \min\{l_T(v_1, v), l_T(v_2, v)\} \qquad \text{for all } v.$$

Lemma 2.5. Let G be a cactus with t cycles. Let G' be a graph formed from G using Algorithm 2.2 for every cycle of G. Then G' is a tree.

Proof. It can be easily seen that any time after using Algorithm 2.2 on some cycle of G the resulting graph remains connected and the number of cycles is less than before. So after using Algorithm 2.2 t-times, we obtain the graph G' which is a tree.

Lemma 2.6. Let G be a graph with an m-cycle $C = (V(C), E(C))$ and let G' be a graph obtained from G using Algorithm 2.2. Then

$$d_{G'}(v_i, v_j) \leqslant d_G(v_i, v_j) \qquad \text{for all } v_i, v_j \in V(G).$$

Proof. The proof follows immediately from the facts that $l_{G'}(v_i, v_j) = l_G(v_i, v_j)$ for arbitrary edge $(v_i, v_j) \in E(G)$ such that $v_i \in V(G)$ and $v_j \in V(G) - V(C)$, and that

$$d_{G'}(v_i, v_j) \leqslant d_G(v_i, v_j) \qquad \text{for all } v_i, v_j \in V(C).$$

Corollary 2.1. Let G a cactus and T be a tree formed from G using Algorithm 2.2 for every cycle of G. Then

$$d_T(v_i, v_j) \leqslant d_G(v_i, v_j) \qquad \text{for all } v_i, v_j \in V(g).$$

Lemma 2.7. Let $G = (V(G), E(G))$ be a cactus with cycles with maximum length 3 and let T be a tree formed from G using Algorithm 2.2 for every cycle of G. Then

$$d_T(v_i, v_j) = d_G(v_i, v_j) \qquad \text{for all } v_i, v_j \in V(G).$$

Proof. Let $(v_1, v_2) \in E(G)$ be an arbitrary edge.

A. If there exists a vertex $v_3 \in V(G)$ such that $\{(v_1, v_3), (v_2, v_3)\} \subset E(G)$, then $\{(v_1, v_2), (v_2, v_3), (v_1, v_3)\} \cap E(T) = \emptyset$, but there exists $v_0 \in V(T)$ such that $\{(v_1, v_0), (v_2, v_0)\} \subset E(T)$ and $d_T(v_1, v_2) = l_T(v_1, v_0) + l_T(v_2, v_0) = l_G(v_1, v_2)$.

B. If there does not exist a vertex $v_3 \in V(G)$ such that $\{(v_1, v_3), (v_2, v_3)\} \subset E(G)$, then $(v_1, v_2) \in E(T)$ and $l_T(v_1, v_2) = l_G(v_1, v_2)$.

From the cases A and B now it immediately follows that

$$d_T(v_i, v_j) = d_G(v_i, v_j) \qquad \text{for all } v_i, v_j \in V(G).$$

Lemma 2.8. Let $G = (V(G), E(G))$ be a cactus and let $T = (V(T), E(T))$ be a tree formed from G using Algorithm 2.2 for every cycle of G. Let $Y_p = (y_1, y_2, \ldots, \ldots, y_p)$ be an arbitrary set of points on G; then there exists a set $X_p = (x_1, x_2, \ldots, x_p)$ on T such that

$$d_T(X_p, v) \leqslant d_G(Y_p, v) \qquad \text{for all } v \in V(G).$$

Proof. We define the points $x_i \in X_p$ for $i = 1, 2, \ldots, p$ as follows:

1. First assume that $y_i \in V(G)$ or y_i is an internal point of some edge $e \in (E(T) \cap E(G))$. Then we define $x_i = y_i$.

2. Now assume that y_i is an internal point of some edge $e \in E(G) - E(T)$. This implies that $e = (v_1, v_2)$ is an edge of some cycle C of G. Let $v_0 \in V(T) -$

$- V(G)$ be the central point of the star formed from the cycle C using Algorithm 2.2, then $\{(v_0, v_1), (v_0, v_2)\} \subset E(T)$.

2.1. If $d_T(v_1\, v_0) > d_G(v_1, y_i)$, then we define the point x_i on the edge (v_1, v_0) in the distance $d_G(v_1, y_i)$ from the point v_1.

2.2. If $d_T(v_1, v_0) \leqslant d_G(v_1, y_i)$ then we define the point x_i on the edge (v_2, v_0) in the distance $d_T(x_i, v_2) = \min\{d_G(v_2, y_i), d_T(v_2, v_0)\}$ from the point v_2.

As $d_T(v_1, v_0) + d_T(v_2, v_0) \leqslant d_G(v_1, v_2)$, it follows from the definition of x_i that $d_T(x_i, v_1) \leqslant d_G(y_i, v_1)$ and $d_T(x_i, v_2) \leqslant d_G(y_i, v_2)$.

Now let $v \in V(G)$ be an arbitrary vertex and let $y_i \in Y_p$ be an arbitrary point.

A. First assume that $y_i \in e, e \in E(T)$. Let $e = (v_{i_1}, v_{i_2})$ and $d_G(v, v_{i_1}) + d_G(v_{i_1}, y_i) = d_G(v, y_i)$. From Corollary 2.1 we obtain $d_T(v_1, v_2) \leqslant d_G(v_1, v_2)$ for all $v_1, v_2 \in V(G)$, hence $d_G(v, y_i) = d_G(v, v_{i_1}) + d_G(v_{i_1}, y_i) = d_G(v, v_{i_1}) + d_G(v_{i_1}, x_i) \geqslant d_T(v, v_{i_1}) + d_T(v_{i_1}, x_i) = d_T(v, x_i)$.

B. Let us assume that $y_i \in e$, $e \notin E(T)$ now, and let $e = (v_1, v_2)$ and $d_G(v, y_i) = d_G(v, v_1) + d_G(v_1, y_i)$. From definition of x_i and from Corollary 2.1 we have $d_G(v, y_i) = d_G(v, v_1) + d_G(v_1, y_i) \geqslant d_T(v, v_1) + d_T(v_1, x_i) \geqslant d_T(v, x_i)$.

Now from A and B it follows that for all $v \in V(G)$ and for all i, $1 \leqslant i \leqslant p$ it holds that $d_T(v, x_i) \leqslant d_G(v, y_i)$. This implies that $d_T(v, X_p) \leqslant d_G(v, Y_p)$, which we wanted to prove.

Lemma 2.9. Let $G = (V(G), E(G))$ be a cactus with cycles with the maximum length 3 and let T be a tree formed from G using Algorithm 2.2 for every cycle of G. Let $X_p = (x_1, x_2, \ldots, x_p)$ be an arbitrary set of points T; then there exists a set $Y_p = (y_1, y_2, \ldots, y_p)$ of points on G such that

$$d_T(X_p, v) \geqslant \frac{1}{2}\, d_G(Y_p, v) \qquad \text{for all } v \in V(G).$$

Proof. We define the points $y_i \in Y_p$ for $i = 1, 2, \ldots, p$ as follows:

1. First assume that $x_i \in V(G)$ or that x_i is an internal point of some edge $e \in (E(T) \cap E(G))$. Then we define $y_i = x_i$.

2. Assume now that x_i is an internal point of some edge $e \in E(T)$; $e \notin E(G)$. This implies that $e = (v_0, v_1)$, where $v_0 \notin V(G)$, and that there exists $v_2, v_3 \in V(G)$ such that $(v_0, v_2) \in E(T)$, $(v_0, v_3) \in E(T)$ and $\{(v_1, v_2), (v_2, v_3), (v_3, v_1)\} \subset E(G) - E(T)$.

Without loss of generality we may assume that $l_G(v_1, v_2) \geqslant l_G(v_1, v_3)$. If $d_T(x_i, v_1) \leqslant d_T(x_i, v_3)$, then we define $y_i = v_1$, else we put $y_1 = v_3$. From the geometrical properties of the triangle it then follows that

$$d_T(x_i, a) \geqslant \frac{1}{2}\, d_G(y_i, a) \qquad \text{for all } a = v_1, v_2, v_3. \tag{2.6}$$

Now let $v \in V(G)$ be an arbitrary vertex and let $x_i \in X_p$ be an arbitrary point.

A. First assume that $x_i \in e$, $e \in E(G)$. Let $e = (v_{i_1}, v_{i_2})$ and let $d_T(v, x_i) =$

$d_T(v, v_{i_1}) + d_T(v_{i_1}, x_i)$. From Lemma 2.7 it follows that $d_T(v_1, v_2) = d_G(v_1, v_2)$ for all $v_1, v_2 \in V(G)$ and so $d_T(v, x_i) = d_T(v, v_{i_1}) + d_T(v_{i_1}, x_i) = d_T(v, v_{i_1}) + d_T(v_{i_1}, y_i) = = d_G(v, v_{i_1}) + d_G(v_{i_1}, y_i) = d_G(v, y_i)$.

B. Let us assume now that $x_i \in e$, $e \notin E(G)$. Let $e = (v_0, v_1)$, $v_0 \notin V(G)$, $v_1 \in V(G)$ and let the two other vertices of the triangle in G be v_2 and v_3.

If $v \in \{v_1, v_2, v_3\}$, then from (2.6) it follows that

$$d_T(x_i, v) \geqslant \frac{1}{2} \cdot d_G(y_i, v).$$

Let $v \notin \{v_1, v_2, v_3\}$ and let $d_T(v, x_i) = d_T(v, a) + d_T(a, x_i)$, where $a \in \{v_1, v_2, v_3\}$. Then

$$d_T(v, x_i) = d_T(v, a) + d_T(a, x_i) \geqslant d_G(v, a) + \frac{1}{2} d_G(a, y_i) >$$

$$> \frac{1}{2} d_G(v, a) + \frac{1}{2} d_G(a, y_i) = \frac{1}{2} d_G(v, y_i).$$

Now from A and B, we obtain that for all $v \in V(G)$ and for all i, $1 \leqslant i \leqslant p$ it holds that

$$d_T(v, x_i) \geqslant \frac{1}{2} d_G(v, y_i).$$

This implies that

$$d_T(v, X_p) \geqslant \frac{1}{2} d_G(v, Y_p),$$

which we wanted to prove.

Note 2.3. From Lemma 2.4 it follows that for the upper bound of the p-radius or the minimum distance sum of a graph G we can take the values of these parameters of any spaning subgraph G_1 of G.

3 The bounds of the p-radius of a cactus

In this section we use the constructions and assertions proved in Section 2, and determine some bounds of the p-radius for the class of cactus-graphs.

Theorem 3.1. Let $G = (V, E)$ be a connected graph with a vertex weight w and an edge length l. Let T_1 be an 1-central spanning tree of G, V_p be an (absolute or vertex) p-center of G, and W_p be a p-center of T_1. Then

$$F_G(V_p) \leqslant F_{T_1}(W_p).$$

Proof. As T_1 is a spanning tree of G, then the proof follows immediately from Lemma 2.4.

Theorem 3.2. Let $G = (V(G), E(G))$ be a weighted cactus and $T = (V(T), E(T))$ be a tree obtained from G by Algorithm 2.2. Let U_p be a p-center of T and V_p be a p-center of G. Then

$$F_T(U_p) \leqslant F_G(V_p).$$

Proof. From Lemma 2.8 it follows that there exists a set of points X_p on T such that $d_T(v, X_p) \leqslant d_G(v, X_p)$ for all $v \in V(G)$. Then we have

$$\begin{aligned} F_T(X_p) &= \max_{v \in V(T)} w(v) \cdot d_T(v, X_p) = \\ &= \max\{\max_{v \in V(G)} w(v) \cdot d_T(v, X_p), \max_{v \in V(T) - V(G)} w(v) \cdot d_T(v, X_p)\} = \\ &= \max\{\max_{v \in V(G)} w(v) \cdot d_T(v, X_p), \max_{v \in V(T) - V(G)} 0 \cdot d_T(v, X_p)\} = \\ &= \max_{v \in V(G)} w(v) \cdot d_T(v, X_p) \leqslant \max_{v \in V(G)} w(v) \cdot d_G(v, V_p) = F_G(V_p). \end{aligned}$$

As U_p is a p-center of T, then $F_T(U_p) \leqslant F_T(X_p)$, and thus $F_T(U_p) \leqslant F_G(V_p)$.

Theorem 3.3. Let $G = (V(G), E(G))$ be a weighted cactus with cycles with the maximum length 3 and $T = (V(T), E(T))$ be a tree formed from G by Algorithm 2.2 applied to each cycle of G. Let U_p be a p-center of T and V_p be a p-center of G. Then

$$F_G(V_p) \leqslant 2 \cdot F_T(U_p).$$

Proof. From Lemma 2.9 it follows that there exists a set of points Y_p on G such that $d_G(v, Y_p) \leqslant 2 \cdot d_T(v, U_p)$ for all $v \in V(G)$. Then we have

$$\begin{aligned} F_T(U_p) &= \max_{v \in V(T)} w(v) \cdot d_T(v, U_p) = \\ &= \max\{\max_{v \in V(G)} w(v) \cdot d_T(v, U_p), \max_{v \in V(T) - V(G)} w(v) \cdot d_T(v, U_p)\} = \\ &= \max\{\max_{v \in V(G)} w(v) \cdot d_T(v, U_p), \max_{v \in V(T) - V(G)} 0 \cdot d_T(v, U_p)\} = \\ &= \max_{v \in V(G)} w(v) \cdot d_T(v, U_p) \geqslant \max_{v \in V(G)} w(v) \cdot \frac{1}{2} d_G(v, Y_p) = \frac{1}{2} F_G(Y_p). \end{aligned}$$

As V_p is a p-center of G, then $F_G(V_p) \leqslant F_G(Y_p)$, and then $F_G(V_p) \leqslant 2F_T(U_p)$.

From Theorems 3.2 and 3.3 we obtain the following corollary.

Corollary 3.1. Let G be a cactus with cycles with the maximum length 3 and let T be a tree obtained from G by Algorithm 2.2 applied on each cycle of G. Let U_p be an (absolute or vertex) p-center of T, and V_p be a p-center of G. Then

$$1 \leqslant \frac{F_G(V_p)}{F_T(U_p)} \leqslant 2.$$

4 The bounds of the minimum distance sum of a cactus

In this section we use the results of Section 3 for determining certain bounds of the minimum distance sum of the p-median for the class of cactus graphs.

Theorem 4.1. Let $G = (V, E)$ be a connected graph with a vertex weight w and an edge length l. Let T_1 be a 1-central spanning tree of G and let V_p be a p-median of G and W_p be a p-median of T_1. Then

$$H_G(V_p) \leqslant H_{T_1}(W_p).$$

Proof. As T_1 is a spanning tree of G, then the proof follows immediately from Lemma 2.4.

Theorem 4.2. Let $G = (V(G), E(G))$ be a weighted cactus, and $T = (V(T), E(T))$ be a tree obtained from G by Algorithm 2.2. Let U_p be a p-median of T and V_p be a p-median of G. Then

$$H_T(U_p) \leqslant H_G(V_p).$$

Proof. From Lemma 2.8 it follows that there exists a set of points X_p on T such that

$$d_T(v, X_p) \leqslant d_G(v, V_p) \qquad \text{for all } v \in V.$$

Then we have

$$\begin{aligned} H_T(X_p) &= \sum_{v \in V(T)} w(v) \cdot d_T(v, X_p) = \\ &= \sum_{v \in V(G)} w(v) \cdot d_T(v, X_p) + \sum_{v \in V(T) - V(G)} w(v) \cdot d_T(v, X_p) = \\ &= \sum_{v \in V(G)} w(v) \cdot d_T(v, X_p) + \sum_{v \in V(T) - V(G)} 0 \cdot d_T(v, X_p) \leqslant \\ &\leqslant \sum_{v \in V(G)} w(v) \cdot d_G(v, V_p) = H_G(V_p). \end{aligned}$$

As U_p is a p-median of T, and it holds that

$$H_T(U_p) \leqslant H_T(X_p)$$

and

$$H_T(U_p) \leqslant H_G(V_p).$$

Theorem 4.3. Let $G = (V(G), E(G))$ be a weighted cactus with cycles with maximum length 3, and $T = (V(T), E(T))$ be a tree formed from G by Algorithm 2.2 aplied to each cycle of G. Let U_p be a p-median of T and V_p be a p-median of G. Then

$$H_G(V_p) \leqslant 2 \cdot H_T(U_p).$$

Proof. From Lemma 2.9 it follows that there exists a set of points Y_p on G such that

$$d_G(v, Y_p) \leqslant 2 \cdot d_T(v, U_p) \qquad \text{for all } v \in V(G).$$

Then we have

$$\begin{aligned}
H_T(U_p) &= \sum_{v\in V(T)} w(v)\cdot d_T(v, U_p) = \\
&= \sum_{v\in V(G)} w(v)\cdot d_T(v, U_p) + \sum_{v\in V(T)-V(G)} w(v)\cdot d_T(v, U_p) = \\
&= \sum_{v\in V(G)} w(v)\cdot d_T(v, U_p) + \sum_{v\in V(T)-V(G)} 0\cdot d_T(v, U_p) \geqslant \\
&\geqslant \sum_{v\in V(G)} w(v)\cdot \frac{1}{2} d_G(v, Y_p) = \frac{1}{2} H_G(Y_p).
\end{aligned}$$

As V_p is a p-median of G, it holds that

$$H_G(V_p) \leqslant H_G(Y_p) \quad \text{and} \quad H_G(V_p) \leqslant 2H_T(U_p).$$

From Theorems 4.2 and 4.3 we obtain the following corollary.

Corollary 3.3.1. Let G be a cactus with cycles with the maximum length 3, and T be a tree obtained from G by Algorithm 2.2 applied to each cycle of G. Let U_p be a p-median of T and V_p be a p-median of G. Then

$$1 \leqslant \frac{H_G(V_p)}{H_T(U_p)} \leqslant 2.$$

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SÚHRN

O PROBLÉMOCH *p*-CENTRA A *p*-MEDIANA V KAKTUSOVÝCH GRAFOCH

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Práca je venovaná štúdiu problému využitia *p*-centra a *p*-mediánu pre niektoré triedy grafov.

Nech $G = (V, E)$ je vrcholovo-ohodnotený kaktus s cyklom maximálnej dĺžky m a T, T_1 sú stromy, ktoré sme dostali z G podľa určenej konštrukcie. Označme vrcholovo-ohodnotený *p*-polomer (resp. súčet vrcholovo-ohodnotených vzdialeností) grafu G symbolom $F_G(V_p)$ (resp. $H_G(V_p)$).

Nech V_p, U_p, W označujú v prípade 1. *p*-centrá a v prípade 2. *p*-mediány grafov G, T, T_1.

Potom platí:

1. $F_T(U_p) \leqslant F_G(V_p) \leqslant F_{T_1}(W_p)$ a tiež

$$1 \leqslant \frac{F_G(V_p)}{F_T(U_p)} \leqslant 2, \quad \text{keď } m = 3.$$

2. $H_T(U_p) \leqslant H_G(V_p) \leqslant H_{T_1}(W_p)$ a tiež

$$1 \leqslant \frac{H_G(V_p)}{H_T(U_p)} \leqslant 2, \quad \text{keď } m = 3.$$

РЕЗЮМЕ

О ПРОБЛЕМАХ p-ЦЕНТРА И p-МЕДИАНА КАКТУС ГРАФА

Мохамед Гассан, Братислава

В работе изучаются проблемы определения p-центра и p-медиана для некоторых классов графов.

Пусть $G = (V, E)$ вершинно-взвешенный каркас с цыклом максимальной длины m и T, T_1 — деревья получены из G по определённой конструкции. Обозначим вершинно-взвешенный p-радиус (или же сумму вершинно-взвешенных расстояний) графа G символом $F_G(V_p)$ (или же $H_G(V_p)$).

Пусть V_p, U_p, W обозначают в случае 1. p-центра и в случае 2. p-медианы графов G, T, T_1. Потом имеет место

1. $F_T(U_p) \leqslant F_G(V_p) \leqslant F_{T_1}(W_p)$ и тоже

$$1 \leqslant \frac{F_G(V_p)}{F_T(U_p)} \leqslant 2, \qquad \text{если } m = 3.$$

2. $H_T(U_p) \leqslant H_G(V_p) \leqslant H_{T_1}(W_p)$ и тоже

$$1 \leqslant \frac{H_G(V_p)}{H_T(U_p)} \leqslant 2, \qquad \text{если } m = 3.$$