UNIVERSITAS COMENIANA
ACTA MATHEMATICA UNIVERSITATIS COMENIANAE
LII—LIII—1987

# RANDOM VARIABLES WITH VALUES IN A VECTOR LATTICE (MEAN VALUE AND CONDITIONAL MEAN VALUE OPERATORS)

PETER MALIČKÝ, Bratislava

This paper develops the integration theory for funcitons $f\colon \Omega \to V$ on an arbitrary probability measure space $(\Omega, \mathscr{S}, P)$ with values in a $\sigma$-complete vector lattice $V$. In other words, we develop the mean value theory for random variables with values in a vector lattice $V$. We also construct the conditional mean value operator for $V$-random variables.

The construction of the mean value and conditional mean value operators is based on an abstract theorem concerning extensions of nonnegative linear operators (Theorem 4.11). We give two variants of the mean value theory — $L^\infty$- and $L^1$-theory.

If a $\sigma$-complete vector lattice satisfies certain conditions, the integration theory based on the pointwise convergence may be developed (see [1], [4], [5], [9]).

We require only $\sigma$-completeness of a vector lattice and our construction is based on the convergence which is uniform a.e. We obtain similar results as [8] using weaker assumptions.

## 1 Preliminaries

We shall work with vector lattices, i.e. real vector spaces that are lattices and satisfy the following identities:

$$a + (b \vee c) = (a + b) \vee (a + c)$$

$$\lambda(b \vee c) = (\lambda b) \vee (\lambda c) \quad \text{for } \lambda > 0.$$

A vector lattice $X$ is called $\sigma$-complete if every contable upper bounded subset $A \subset X$ has the least upper bound, which is denoted by $\sup A$ or $\bigvee\limits_{a \in A} a$.
$\inf A$ or $\bigwedge\limits_{a \in A} a$ denote the grates lower bound of the set $A$. The symbol $|a|$ denotes $(a) \vee (-a)$.

**Definition 1.1.** Let $X$ be a $\sigma$-complete vector lattice. We say that a sequence $\{a_n\}_{n=1}^{\infty}$ decreases to 0, and we write $a_n \searrow 0$, if $\forall n$: $0 \leq a_{n+1} \leq a_n$ and $\bigwedge_{n=1}^{\infty} a_n = 0$.

We say that a sequence $\{x_n\}_{n=1}^{\infty} \subset X$ converges to $x \in X$ and we write $x_n \to x$ (we say that $\{x_n\}_{n=1}^{\infty}$ is fundamental) if $\exists \{a_n\}_{n=1}$: $a_n \searrow 0$ and $\forall n$: $|x - x_n| \leq a_n (\forall n, m$: $n \leq m \Rightarrow |x_n - x_m| \leq a_n)$.

**Proposition 1.2.** For any sequences $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ of a $\sigma$-complete lattice $X$, any $x, y \in X$ and any $c \in R$:

$$a_n \searrow 0, \quad b_n \searrow 0 \Rightarrow (a_n + b_n) \searrow 0$$

$$c \geq 0, \quad a_n \searrow 0 \Rightarrow (c \cdot a_n) \searrow 0$$

$$x_n \to x, \quad x_n \to y \Rightarrow x = y$$

$$x_n \to x, \quad y_n \to y \Rightarrow (x_n + y_n) \to (x + y)$$

$$x_n \to x \Rightarrow (c \cdot x_n) \to c \cdot x$$

$$x_n \to x \Rightarrow \{x_n\}_{n=1}^{\infty} \quad \text{is fundamental}$$

$$\{x_n\}_{n=1}^{\infty} \quad \text{is fundamental} \Rightarrow x_n \to \bigvee_{i=1}^{\infty} \bigwedge_{j=i}^{\infty} x_j = \bigwedge_{i=1}^{\infty} \bigvee_{j=i}^{\infty} x_j$$

$$x_n \to x, \quad \forall n\colon x_n \geq 0 \Rightarrow x \geq 0.$$

## 2 Elementary theory of the integral

Let $\Omega$ be a set and $V$ be a $\sigma$-complete vector lattice. The symbol $F(\Omega, V)$ denotes the set of all function $f\colon \Omega \to V$.

**Proposition 2.1.** $F(\Omega, V)$ is a $\sigma$-complete vector lattice under the natural operations and ordering.

Let $(\Omega, \mathscr{S}, P)$ be a probability measure space. Two functions $f, g \in F(\Omega, V)$ are called equivalent if there exists a set $E \in \mathscr{S}$ such that:

$$P(E) = 0$$

$$f(x) = g(x) \quad \text{for all} \quad x \in \Omega \backslash E.$$

The symbol $[f]$ denotes the equivalence class of the function $f \in F(\Omega, V)$. Put $\mathscr{F}(\Omega, \mathscr{S}, P, V) = \{\xi\colon \xi = [f], f \in F(\Omega, V)\}$.

**Definition 2.2.** We say that a sequence $\{f_n\}_{n=1}^{\infty} \subset F(\Omega, V)$ converges to a function $f \in F(\Omega, V)$ uniformly almost everywhere, if there exist a sequence $\{a_n\}_{n=1}^{\infty} \subset V$ and a set $E \in \mathscr{S}$ such that:

$a_n \searrow 0$, $P(E) = 0$ and $|f_n(x) - f(x)| \leq a_n$ for all natural $n$ and all $x \in \Omega \backslash E$.

We say that a sequence $\{\xi_n\}_{n=1}^{\infty} \subset \mathscr{F}(\Omega, \mathscr{S}, P, V)$ converges to $\xi \in \mathscr{F}(\Omega, \mathscr{S},$

$P, V)$ uniformly almost everywhere if there exist $f_n \in \xi_n$, $f \in \xi$ such that $\{f_n\}_{n=1}^{\infty}$ converges to $f$ uniformly almost everywhere.

**Proposition 2.3.** $\mathscr{F}(\Omega, \mathscr{S}, P, V)$ is a $\sigma$-complete vector lattice under the natural operations and ordering.

A function $f \in F(\Omega, V)$ is called elementary if there exist sequences $\{a_i\}_{i=1}^n \subset$, $\{B_i\}_{i=1}^n \subset \mathscr{S}$ such that:

$$\Omega = \bigcup_{i=1}^{n} B_i, \quad B_i \cap B_j = \emptyset \quad \text{for all } i \neq j,$$

$f(x) = a_i$ whenever $x \in B_i$

$\left(\text{or equivalently } f(x) = \sum_{i=1}^{n} \chi_{B_i}(x)\, a_i\right)$.

For such a function $f$ put:

$$E(f) = \int_{\Omega} f(x)\, dP(x) = \sum_{i=1}^{n} a_i P(B_i). \tag{1}$$

The set of all elementary functions $f \in F(\Omega, V)$ is denoted by the symbol $L_0^{\infty}(\Omega, \mathscr{S}, P, V)$.

**Proposition 2.4.**

(i) $L_0^{\infty}(\Omega, \mathscr{S}, P, V)$ is a vector sublattice of the vector lattice $F(\Omega, V)$.

For any $f, g \in L_0^{\infty}$, $c \in \boldsymbol{R}$ and $B \in \mathscr{S}$ we have:

(ii) $f, g$ are equivalent $\Rightarrow E(f) = E(g)$

(iii) $E(f+g) = E(f) + E(g)$

(iv) $E(c \cdot f) = c \cdot E(f)$

(v) $f(x) = a$ for all $x \Rightarrow E(f) = a$

(vi) $f \geq 0 \Rightarrow E(f) \geq 0$

(vii) $\chi_B \cdot f \in L_0^{\infty}(\Omega, \mathscr{S}, P, V)$.

Put $\mathscr{L}_0^{\infty}(\Omega, \mathscr{S}, P, V) = \{\xi \in \mathscr{F}(\Omega, \mathscr{S}, P, V);\ \xi = [f] \text{ for some } f \in L_0^{\infty}(\Omega, \mathscr{S}, P, V)\}$.

For all $\xi \in \mathscr{L}_0^{\infty}(\Omega, \mathscr{S}, P, V)$ put:

$$E(\xi) = E(f), \quad \text{where } f \in \xi.$$

$E$ is defined correctly.

**Proposition 2.4.** $\mathscr{L}_0^{\infty}(\Omega, \mathscr{S}, P, V)$ is a vector sublattice of the vector lattice $\mathscr{F}(\Omega, \mathscr{S}, P, V)$ and for any $\xi, \eta \in \mathscr{L}_0^{\infty}(\Omega, \mathscr{S}, P, V)$, $c \in \boldsymbol{R}$ and $B \in \mathscr{S}$ we have:

(i) $E(\xi + \eta) = E(\xi) + E(\eta)$

(ii) $E(c\xi) = cE(\xi)$

(iii) $\xi = a \in \mathrm{V}$ a.e. $\Rightarrow E(\xi) = a$

(iv) $\xi \geq 0 \Rightarrow E(\xi) \geq 0$

(v) $[\chi_B] \cdot \xi \in \mathscr{L}_0^{\infty}(\Omega, \mathscr{S}, P, V)$.

Let $\mathscr{M}$ be the class of all linear subspaces $X$ of $\mathscr{F}(\Omega, \mathscr{S}, P, V)$ such that:

$$(\xi_n \to \xi \quad \text{uniformly a. e.}, \quad \xi_n \in X) \Rightarrow \xi \in X,$$

$$\mathscr{L}_0^\infty(\Omega, \mathscr{S}, P, V) \subset X.$$

Put $\mathscr{L}^\infty(\Omega, \mathscr{S}, P, V) = \bigcap_{X \in \mathscr{M}} X.$

**Proposition 2.5.**

(i) $\mathscr{L}^\infty(\Omega, \mathscr{S}, P, V)$ is a vector subspace of the vector space $\mathscr{F}(\Omega, \mathscr{S}, P, V)$.

(ii) For any $\{\xi_n\}_{n=1}^\infty \subset \mathscr{L}^\infty(\Omega, \mathscr{S}, P, V)$ and $\xi \in \mathscr{F}(\Omega, \mathscr{S}, P, V)$ we have: $\xi_n \to \xi$ uniformly a.e. $\Rightarrow \xi \in \mathscr{L}^\infty(\Omega, \mathscr{S}, P, V)$.

(iii) $\forall \xi \in \mathscr{L}^\infty(\Omega, \mathscr{S}, P, V) \exists a \in V$ such that: $|\xi| \leq a$ a.e.

**Proof:**

(i) and (ii) follow immediately from the construciton of $\mathscr{L}^\infty(\Omega, \mathscr{S}, P, V)$.

Let $X_0$ be the set of all $\xi \in \mathscr{F}(\Omega, \mathscr{S}, P, V)$ such that there exists $a \in V$ for which $|\xi| \leq a$.

Clearly, $X_0 \in \mathscr{M}$ and this implies: $\mathscr{L}^\infty(\Omega, \mathscr{S}, P, V) \subset X_0$.

In Section 4 we shall construct a positive linear operator $\bar{E}$: $\mathscr{L}^\infty(\Omega, \mathscr{S}, P, V) \to V$, which is an extension of $E$.

There are no problems with the integration of functions with values in a finite-dimensional vector space. Namely, let $(\Omega, \mathscr{S}, P)$ be a probability measure space, $V$ be a finite dimensional real vector space and $\{e_i\}_{n=1}^n$ be its basis.

Every function $f: \Omega \to V$ may be written in the form $f(x) = \sum_{i=1}^n \varphi_i(x)\, e_i$, where all $\varphi_i$ are real functions.

We say that $f$ is integrable, if all $\varphi_i$ are integrable, and in this case we put

$$E(f) = \int_\Omega f(x)\, dP(x) \underset{\text{def}}{=} \sum_{i=1}^n \left( \int_\Omega \varphi_i(x)\, dP(x) \right) e_i. \tag{2}$$

Obviously, the integrability and the integral of the function $f: \Omega \to V$ do not depend on the basis $\{e_i\}_{i=1}^n$ of $V$.

**Definition 2.6.** Let $(\Omega, \mathscr{S}, P)$ be a probability measure space and $V$ be a $\sigma$-complete vector lattice. A function $f \in F(\Omega, V)$ is called simple integrable if there exists a finite dimensional vector subspace $V_0 \subset V$ such that:

$f(x) \in V_0$ for all $x \in \Omega$, and $f$ is integrable in the above sense.

The set of all simple integrable functions is denoted by $L_0^1(\Omega, \mathscr{S}, P, V)$.

**Proposition 2.7.**

(i) $L_0^1(\Omega, \mathscr{S}, P, V)$ is a vector subspace of the vector space $F(\Omega, V)$ and $L_0^\infty(\Omega, \mathscr{S}, P, V) \subset L_0^1(\Omega, \mathscr{S}, P, V)$.

(ii) If $f \in L_0^\infty(\Omega, \mathscr{S}, P, V)$, then the integrals of $f$ by the formulas (1) and (2) are the same.

For any $f, g \in L_0^1(\Omega, \mathscr{S}, P, V)$, $c \in \boldsymbol{R}$, $a \in V$ and $B \in \mathscr{S}$ we have:

(iii) $E(f + g) = E(f) + E(g)$

(iv) $E(c \cdot f) = c \cdot E(f)$

(v) $(\forall x: f(x) = a) \Rightarrow E(f) = a$
(vi) $f \geq 0 \Rightarrow E(f) \geq 0$
(vii) $\chi_B \cdot f \in L_0^1(\Omega, \mathscr{S}, P, V)$.

**Proof:**

All parts except (vi) are obvious. We shall prove (vi).

Let $f$: $\Omega \to V$ be simple integrable and nonnegative. It means there exists a finite dimensional vector subspace $V_0 \subset V$ such that for all $x \in \Omega$

$$f(x) \in V_0, \quad f(x) \geq 0 \quad \text{and } f \text{ is integrable.}$$

Denote $C = \{x \in V: x \geq 0\}$.

Clearly, $f(\Omega) \subset V_0 \cap C$, and $V_0 \cap C$ is a convex set in $V_0$.

The proof is reduced to the following lemma.

**Lemma 2.9.** Let $(\Omega, \mathscr{S}, P)$ be a probability measure space, $V$ be a finite dimensional real vector space, and $K$ be a convex subset of $V$.

If $f$: $\Omega \to V$ is integrable and $f(\Omega) \subset K$, then $\int_\Omega f(x)\,dP(x) \in K$.

**Proof:**

The lemma holds when $\dim V = 1$. Suppose that the lemma is true for all $(n-1)$-dimensional vector spaces. We are going to prove it for all $n$-dimensional vector spaces.

We are to prove that for all $a \bar{\in} K$: $\int_\Omega f(x)\,dP(x) \neq$ a.

Put $g(x) = f(x) - a$ and $K' = K - a$.

It suffices to prove $\int_\Omega g(x)\,dP(x) \neq 0$.

We have: $0 \bar{\in} K'$, $g(\Omega) \subset K'$, and $K'$ is convex.

There exists linear functional $\Phi$: $V \to \boldsymbol{R}$ such that $\forall y \in K'$: $\Phi(y) \geq 0$ and $\exists y \in K'$: $\Phi(y) > 0$. (See [10], p. 113.)

We have: $\Phi(g(x)) \geq 0$ for all $x \in \Omega$.

If $P(\{x: \Phi(g(x)) > 0\}) > 0$, then $\Phi\left(\int_\Omega g(x)\,dP(x)\right) = \int_\Omega \Phi(g(x))\,dP(x) > 0$, which means $\int_\Omega g(x)\,dP(x) \neq 0$.

If $\Phi(g(x)) = 0$ a.e. on $\Omega$, we may assume that $\Phi(g(x) = 0$ for all $x \in \Omega$.

Put $V_0 = \{y \in V: \Phi(y) = 0\}$. $V_0$ is an $(n-1)$-dimensional vector space and $g(\Omega) \subset V_0 \cap K'$ and $0 \bar{\in} K' \cap V_0$.

In this case we can use the inductive assumption.

Put $\mathscr{L}_0^1(\Omega, \mathscr{S}, P, V) = \{[f]: f \in L_0^1(\Omega, \mathscr{S}, P, V)\}$ and $E([f]) = E(f)$ for $f \in L_0^1(\Omega, \mathscr{S}, P, V)$.

**Proposition 2.10.** $\mathscr{L}_0^1(\Omega, \mathscr{S}, P, V)$ is a vector subspace of the vector space

$\mathscr{F}(\Omega, \mathscr{S}, P, V)$. For any $\xi$, $\eta \in \mathscr{L}_0^1(\Omega, \mathscr{S}, P, V)$, $c \in \boldsymbol{R}$, $a \in V$ and $B \in \mathscr{S}$ we have:

(i) $[\chi_B] \cdot \xi \in \mathscr{L}_0^1(\Omega, \mathscr{S}, P, V)$

(ii) $E(\xi + \eta) = E(\xi) + E(\eta)$

(iii) $E(c \cdot \xi) = c \cdot E(\xi)$

(iv) $\xi \geq 0 \Rightarrow E(\xi) \geq 0$

(v) $\xi = a$ almost everywhere $\Rightarrow E(\xi) = a$.

Let $\mathscr{L}^1(\Omega, \mathscr{S}, P, V)$ be the minimal vector subspace of $\mathscr{F}(\Omega, \mathscr{S}, P, V)$, which contains $\mathscr{L}_0^1(\Omega, \mathscr{S}, P, V)$ and is closed with respect to the almost everywhere uniform convergence. We shall describe an extension $\bar{E}$ of $E$ onto $\mathscr{L}^1(\Omega, \mathscr{S}, P, V)$ in Section 4.

## 3 Elementary theory of the conditional mean value operator

Let $(\Omega, \mathscr{S}, P)$ be a probability measure space and $\mathscr{S}_0 \subset \mathscr{S}$ be a $\sigma$-subalgebra. Let $\mathscr{L}^1(\Omega, \mathscr{S}, P, \boldsymbol{R})$ and $\mathscr{L}^1(\Omega, \mathscr{S}_0, P, \boldsymbol{R})$ be the sets of all equivalence classes of integrable real functions. The Radon—Nikodym theorem guarantees the existence (and the uniqueness) of the operator:

$$E(.|\mathscr{S}_0)\colon \mathscr{L}^1(\Omega, \mathscr{S}, P, \boldsymbol{R}) \to \mathscr{L}^1(\Omega, \mathscr{S}_0, P, \boldsymbol{R})$$

such that for any $\xi$, $\eta \in \mathscr{L}^1(\Omega, \mathscr{S}, P, \boldsymbol{R})$, $c \in \boldsymbol{R}$ and $B \in \mathscr{S}_0$

(3) $$E(\xi + \eta|\mathscr{S}_0) = E(\xi|\mathscr{S}_0) + E(\eta|\mathscr{S}_0)$$

(4) $$E(c\xi|\mathscr{S}_0) = cE(\xi/\mathscr{S}_0)$$

(5) $$\xi \geq 0 \Rightarrow E(\xi|\mathscr{S}_0) \geq 0$$

(6) $$\xi = c \quad \text{almost everywhere} \Rightarrow E(\xi|\mathscr{S}_0) = c \quad \text{a.e.}$$

(7) $$E([\chi_B]\,\xi|\mathscr{S}_0) = [\chi_B] \cdot E(\xi|\mathscr{S}_0)$$

(8) $$E(E(\xi|\mathscr{S}_0)) = E(\xi).$$

Now we are going to construct analogous operators on the spaces $\mathscr{L}_0^\infty(\Omega, \mathscr{S}, P, V)$ and $\mathscr{S}_0^1(\Omega, \mathscr{S}, P, V)$.

Take $\xi \in \mathscr{L}_0^\infty(\Omega, \mathscr{S}, P, V)$. There is $f \in L_0^\infty(\Omega, \mathscr{S}, P, V)$ such that $f \in \xi$. For some sequences $\{a_i\}_{i=1}^n \subset V$ and $\{B_i\}_{i=1}^n \subset \mathscr{S}$ we have: $f(x) = \sum_{i=1}^n \chi_{B_i}(x)\, a_i$. Put

(9) $$E(\xi|\mathscr{S}_0) = \sum_{i=1}^n E([\chi_{B_i}])|\mathscr{S}_0)\, a_i.$$

**Proposition 3.1.**

(i) For any $\xi$, $\eta \in \mathscr{L}_0^\infty(\Omega, \mathscr{S}, P, V)$, $c \in \boldsymbol{R}$, $a \in V$ and $B \in \mathscr{S}_0$ (3)—(7) hold, where in (6) $c$ is replaced by $a$.

Moreover,

(ii) $E(\xi|\mathscr{S}_0) \in \mathscr{L}^{\infty}(\Omega, \mathscr{S}_0, P, V)$ for any $\xi \in \mathscr{L}_0^{\infty}(\Omega, \mathscr{S}, P, V)$.

Remark.

Since we do not know how to integrate $\xi \in \mathscr{L}^{\infty}(\Omega, \mathscr{S}, P, V)$, we cannot formulate (8) in Proposition 3.1, but see Proposition 3.2.

**Proof:**

Part (i) follows from the construction and properties of the real conditional mean value operator.

(ii) It suffices to prove that $E([a \cdot \chi_B]|\mathscr{S}_0) \in \mathscr{L}^{\infty}(\Omega, \mathscr{S}_0, P, V)$ for any $a \in V$ and $B \in \mathscr{S}$.

We have $E([a \cdot \chi_B]|\mathscr{S}_0) = aE([\chi_B]|\mathscr{S}_0)$.

From the properties (5) and (6) it follows that $E([\chi_B]|\mathscr{S}_0) \in \mathscr{L}^{\infty}(\Omega, \mathscr{S}_0, P, \boldsymbol{R})$. Therefore, there is a sequence $\{\varphi_n\}_{n=1}$ of real elementary functions which converges to $E([\chi_B]|\mathscr{S}_0)$ uniformly almost everywhere.

We have $[a \cdot \varphi_n] \in \mathscr{L}_0^{\infty}(\Omega, \mathscr{S}_0, P, V) \subset \mathscr{L}^{\infty}(\Omega, \mathscr{S}_0, P, V)$ and $[a \cdot \varphi_n] \to$ $\to a \cdot E([\chi_B]|\mathscr{S}_0)$ uniformly almost everywhere. From Proposition 2.5 it follows that $E([a\chi_B]|\mathscr{S}_0) \in \mathscr{L}^{\infty}(\Omega, \mathscr{S}_0, P, V)$.

Now we are going to construct $E(.|\mathscr{S}_0)$ on $\mathscr{L}_0^1(\Omega, \mathscr{S}, P, V)$. Let $\xi \in \mathscr{L}_0^1(\Omega, \mathscr{S}, P, V)$. There exists $f \in L_0^1(\Omega, \mathscr{S}, P, V)$, $f \in \xi$. For $f$ there exist a finite-dimensional vector subspace $V_0 \subset V$, its basis $\{e_i\}_{i=1}^n$ and real integrable functions $\{\varphi_i\}_{i=1}^n$, $\varphi_i$: $\Omega \to \boldsymbol{R}$ such that:

$f(x) = \sum_{i=1}^{n} \varphi_i(x) e_i$. We put:

$$E(\xi|\mathscr{S}_0) = \sum_{i=1}^{n} E([\varphi_i]|\mathscr{S}_0) \cdot e_i. \tag{10}$$

**Proposition 3.2.**

(i) For any $\xi$, $\eta \in \mathscr{L}_0^1(\Omega, \mathscr{S}, P, V)$, $c \in \boldsymbol{R}$, $a \in V$ and $B \in \mathscr{S}_0$ (3)—(8) hold, where in (6) $a$ is instead of $c$.

(ii) $E(\xi|\mathscr{S}_0) \in \mathscr{L}_0^1(\Omega, \mathscr{S}_0, P, V)$ for any $\xi \in \mathscr{L}_0^1(\Omega, \mathscr{S}, P, V)$.

(iii) For any $\xi \in \mathscr{L}_0^{\infty}(\Omega, \mathscr{S}, P, V)$ the values $E(\xi|\mathscr{S}_0)$ by the formulas (9) and (10) are the same.

**Proof:**

The property $\xi \geq 0 \Rightarrow E(\xi|\mathscr{S}_0) \geq 0$ for any $\xi \in \mathscr{L}_0^1(\Omega, \mathscr{S}, P, V)$ may be proved in an analogous way as Proposition 2.9.

The other properties are obvious.

## 4 Extension theory

In this section we shall prove an abstract theorem on the extensions of nonnegative linear operators, and then we shall apply it to the operators $E$ and $E(.|\mathscr{S}_0)$.

**Definition 4.1.** Let $X$ be a $\sigma$-complete vector lattice and $A$ be its vector sublattice. $A$ is called a $\sigma$-sublattice, if for every countable set $B \subset A$:

$$\sup B \in A \text{ whenever } \sup B \text{ exists in } X.$$

**Definition 4.2.** Let $X$ be a $\sigma$-complete lattice and $A$ be its vector $\sigma$-sublattice. We say that a sequence $\{x_n\}_{n=1}^{\infty} \subset X$ converges to $x \in X$ by $A$ (is fundamental by $A$) if there exists $\{a_n\}_{n=1}^{\infty} \subset A$, $a_n \searrow 0$ such that: $\forall n$: $|x_n - x| \leq a_n$ ($\forall n, m$: $m \geq n \Rightarrow |x_n - x_m| \leq a_n$).

**Example 4.3.** Let $(\Omega, \mathscr{S}, P)$ be a probability measure space and $V$ be a $\sigma$-complete vector lattice. Put $X = \mathscr{F}(\Omega, \mathscr{S}, P, V)$ and $A = \{\xi \in \mathscr{F}(\Omega, \mathscr{S}, P, V)$: $\xi = [\mathrm{const}]\}$.

In Section 2 it was said that $X$ is a $\sigma$-complete vector lattice. Obviously, $A$ is a vector $\sigma$-sublattice of $X$ and for any sequence $\{\xi_n\}_{n=1}^{\infty} \subset X$ we have:

$\xi_n \to \xi$ by $A \Leftrightarrow \xi_n \to \xi$ uniformly almost everywhere.

**Example 4.4.** Let $X$ be a $\sigma$-complete lattice and $A = X$. Then for any sequence $\{x_n\}_{n=1}^{\infty} \subset X$ we have: $x_n \to x$ by $A \Leftrightarrow x_n \to x$.

**Proposition 4.5.** Let $X$ be a $\sigma$-complete vector lattice and $A$ be its vector $\sigma$-sublattice. For any sequences $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty} \subset X$, any $x, y \in X$ and $c \in \boldsymbol{R}$ we have:

$$x_n \to x \text{ by } A,\ y_n \to y \text{ by } A \Rightarrow (x_n + y_n) \to (x + y) \text{ by } A \tag{11}$$

$$x_n \to x \text{ by } A \Rightarrow cx_n \to cx \text{ by } A \tag{12}$$

$$x_n \to x \text{ by } A,\ x_n \to y \text{ by } A \Rightarrow x = y \tag{13}$$

$$(x_n \to x \text{ by } A,\ \forall n\colon x_n \geq 0) \Rightarrow x \geq 0 \tag{14}$$

$$x_n \to x \text{ by } A \Rightarrow |x_n| \to |x| \text{ by } A \tag{15}$$

$$x_n \to x \text{ by } A \Rightarrow \{x_n\}_{n=1}^{\infty} \text{ is fundamental by } A \tag{16}$$

$$\{x_n\}_{n=1}^{\infty} \text{ is fundamental by } A \Rightarrow x_n \to \bigwedge_{i=1}^{\infty}\left(\bigvee_{j=i}^{\infty} x_j\right) = \bigvee_{i=1}^{\infty}\left(\bigwedge_{j=i}^{\infty} x_j\right) \text{ by } A. \tag{17}$$

**Definition 4.6.** Let $X$ be a $\sigma$-complete vector lattice and $A$ its vector $\sigma$-sublattice. Let $X_0$ be a vector subspace of $X$. $\bar{X}_0$ denotes the minimal vector subspace of $X$ which contains $X_0$ and is closed with respect to the convergence by $A$.

Now we are going to describe $\bar{X}_0$ by the transfinite induction over the sets of all countable ordinals. Put $X_\alpha = X_0$ for $\alpha = 0$. Let $0 < \alpha < \omega_1$ where $\omega_1$ is the first uncountable ordinal. Suppose that $X_\beta$ have already been for all ordinals $\beta < \alpha$. Put $X_\alpha = \left\{x \in X\colon \exists \{x_n\}_{n=1}^{\infty} \subset \bigcup_{\beta < \alpha} X_\beta,\ x_n \to x \text{ by } A\right\}$.

**Proposition 4.7.**

(i) For all ordinals $\alpha < \omega_1$ $X_\alpha$ are vector subspaces of $X$ such that: $X_\beta \subset X_\alpha$ whenever $\beta < \alpha < \omega_1$

(ii) $\bar{X}_0 = \bigcup_{\alpha < \omega_1} X_\alpha$

(iii) If $X_0$ is a vector sublattice of $X$, then $\bar{X}_0$ is a vector sublattice of $X$ as well.

**Proof.**

Part (i) may be easily proved using the transfinite induction.

(ii) The inclusion $X_\alpha \subset \bar{X}_0$ for all $\alpha < \omega_1$ may be proved by the transfinite induction. We have: $\bigcup_{\alpha < \omega_1} X_\alpha \subset \bar{X}_0$. Obviously, $\bigcup_{\alpha < \omega_1} X_\alpha$ is a vector subspace of $X$.

If we show that $\bigcup_{\alpha < \omega_1} X_\alpha$ is closed with respect to the convergence by A, we shall have: $\bar{X}_0 \bigcup_{\alpha < \omega_1} X_\alpha$.

Let $\{x_n\}_{n=1}^\infty \subset \bigcup_{\alpha < \omega_1} X_\alpha$, $x_n \to x$ by $A$.

There is a sequence of ordinals $\{\alpha_n\}_{n=1}^\infty$ such that $\alpha_n < \omega_1$, $x_n \in X_{\alpha_n}$ for all natural $n$. There is some $\alpha_0 < \omega_1$ such that $\alpha_n < \alpha_0$ for all natural $n$.

We have $\{x_n\}_{n=1}^\infty \subset \bigcup_{\beta < \alpha_0} X_\beta$ and $x_n \to x$ by $A$, which means $x \in X_{\alpha_0} \subset \bigcup_{\alpha < \omega_1} X_\alpha$.

(iii) If $X_0$ is a sublattice of $X$, then using the transfinite induction and the property (15) it is easy to prove that all $X_\alpha$ are sublattices of $X$. Since $\alpha < \beta < \omega_1$ implies $X_\alpha \subset X_\beta$, it follows that $\bar{X}_0 = \bigcup_{\alpha < \omega_1} X_\alpha$ is a sublattice of $X$.

**Definition 4.8.** Let $X$ and $Y$ be vector lattices and $X_0$ be a vector subspace of $X$. A linear opeator $T$: $X_0 \to Y$ is called nonnegative, if $\forall x \in X_0$: $x \geq 0 \Rightarrow$ $\Rightarrow T(x) \geq 0$.

A nonnegative linear operator $T$: $X_0 \to Y$ is called $o$-continuous, if $\forall \{x_n\}_{n=1}^\infty \subset X_0$: $x_n \searrow 0 \Rightarrow T(x_n) \searrow 0$.

**Definition 4.9.** Let $X$ and $Y$ be $\sigma$-complete vector lattices, $A$ and $B$ be vector $\sigma$-sublattices of $X$ and $Y$ respectively, and $X_0$ be a vector subspace of $X$. A linear operator $T$: $X_0 \to Y$ is called $A$—$B$-continuous, if

$$\forall \{x_n\}_{n=1}^\infty \subset X_0 \colon x_n \to 0 \text{ by } A \Rightarrow T(x_n) \to 0 \text{ by } B.$$

**Proposition 4.10.** Let $X$ and $Y$ be $\sigma$-complete vector lattices, $A$ and $B$ be vector $\sigma$-sublattices of $X$ and $Y$ respectively, $X_0$ be a vector subspace of $X$, and $T$: $X_0 \to Y$ be a nonnegative linear operator.

(i) If $A \subset X_0$, then $T$ is $A$—$B$-continuous if and only if the restriction of $T$ onto $A$ is $A$—$B$-continuous.

(ii) If $A \subset X_0$, $T(A) \subset B$ and the restriction of $T$ onto $A$ is $o$-continuous, then $T$ is $A$—$B$-continuous.

(iii) If $T$ is $A$—$B$-continuous, then $\forall \{x_n\}_{n=1}^\infty \subset X_0$: $x_n \to x \in X_0$ by $A \Rightarrow$ $\Rightarrow T(x_n) \to T(x)$ by $B$.

**Proof:**

(i) Suppose $A \subset X_0$ and the restriction of $T$ onto $A$ is $A$—$B$-continuous. Let $\{x_n\}_{n=1}^{\infty} \subset X_0$, $x_n \to 0$ by $A$.

There is a sequence $\{a_n\}_{n=1}^{\infty} \subset A$ such that: $a_n \searrow 0$ and $|x_n| \leq a_n$ for all natural $n$.

We have $a_n \to 0$ by $A$. Since the restriction of $T$ onto $A$ is $A$—$B$-continuous, there is some $\{b_n\}_{n=1}^{\infty} \subset B$ such that $b_n \searrow 0$ and $|T(a_n)| \leq b_n$ for all natural $n$.

We have: $-a_n \leq x_n \leq a_n$ and $-T(a_n) \leq T(x_n) \leq T(a_n)$, because $T$ is nonnegative and linear.

Finally, $|T(x_n)| \leq T(a_n) \leq b_n$, which means that $T(x_n) \to 0$ by $B$, and $T$ is $A$—$B$-continuous. The opposite implication is obvious.

(ii) Let $A \subset X_0$, $T(A) \subset B$, and the restriction of $T$ onto $A$ be $o$-continuous. Let $\{x_n\}_{n=1}^{\infty} \subset X_0$, $x_n \to 0$ by $A$. Then $|x_n| \leq a_n$ for some sequence $\{a_n\}_{n=1}^{\infty}$, $a_n \searrow 0$.

We have $|T(x_n)| \leq T(a_n) \searrow 0$ and $\{T(a_n)\}_{n=1}^{\infty} \subset B$. $T(x_n) \to 0$ by $B$, and $T$ is $A$—$B$-continuous.

Part (iii) is obvious.

**Theorem 4.11.** Let $X$ and $Y$ be $\sigma$-complete vector lattices, $A$ and $B$ vector $\sigma$-sublattices of $X$ and $Y$ respectively, $X_0$ and $Y_0$ be vector subspaces of $X$ and $Y$ respectively, and $T$: $X_0 \to Y$ be a linear nonnegative $A$—$B$-continuous operator.

(i) If $A \subset X_0$, then there exists a unique linear nonnegative operator $\bar{T}$: $\bar{X}_0 \to Y$ such that $\bar{T}(x) = T(x)$ for all $x \in X_0$.

(ii) $\bar{T}$ is $A$—$B$-continuous and $T(X_0) \subset Y_0$ implies $\bar{T}(\bar{X}_0) \subset \bar{Y}_0$.

**Proof:**

We shall use the description of $\bar{X}_0$ from Proposition 4.7. Let $P$ be the set of all ordinals $\alpha < \omega_1$ such that there exists a unique linear nonnegative operator $T_\alpha$: $X_\alpha \to Y$, which is the extension of $T$.

Suppose $P \neq \{\alpha\colon \alpha < \omega_1\}$. Then the set $P' = \{\alpha\colon \alpha < \omega_1\} \backslash P$ is nonempty. Let $\alpha_0 = \min P'$. We are going to prove $\alpha_0 \in P$, which will be a contradiction.

Obviously, $0 \in P$ and $0 < \alpha_0$.

Let $\beta \leq \alpha < \alpha_0$. Then $\alpha, \beta \in P$.

The restriction of $T_\alpha$ onto $X_\beta$ is a nonnegative linear operator on $X_\beta$, which is the extension of $T$. Since $\beta \in P$, we have $T_\alpha(x) = T_\beta(x)$ for all $x \in X_\beta$.

Let $T'$: $\bigcup_{\alpha < \alpha_0} X_\alpha \to Y$ be the operator defined by the formula $T'(x) = T_\alpha(x)$ whenewer $x \in X_\alpha$, $\alpha < \alpha_0$.

$T'$ is defined correctly; it is linear and nonnegative extension of $T$ onto $\bigcup_{\alpha < \alpha_0} X_\alpha$.

By Proposition 4.10, $T'$ is $A$—$B$-continuous.

Take $x \in X_{\alpha_0}$. There exists $\{x_n\}_{n=1}^{\infty} \subset \bigcup_{\alpha < \alpha_0} X_\alpha$ such that $x_n \to x$ by $A$. We shall show that the sequence $\{T'(x_n)\}_{n=1}^{\infty}$ converges by $B$. There is some sequence $\{a_n\}_{n=1}^{\infty} \subset A$ such that $|x_n - x| \leq a_n$ for all natural $n$ and $a_n \searrow 0$.

We have:

(18) $|x_n - x_m| \leq |x_n - x| + |x_m - x| \leq a_n + a_m \leq 2a_n$ whenever $m \geq n$.

Since $a_n \to 0$ by $A$, and $T'$ is $A$—$B$-continuous,

(19) $T'(a_n) \leq b_n$ for some sequence $\{b_n\}_{n=1}^{\infty} \subset B$, $b_n \searrow 0$.

The properties of $T'$ and the inequalities (18) and (19) give $|T'(x_n) - T'(x_m)| = |T'(x_n - x_m)| \leq 2T(a_n) \leq b_n$ whenever $m \geq n$. The last inequality shows that the sequence $\{T'(x_n)\}_{n=1}^{\infty}$ is fundamental by $B$, and according to Proposition 4.5 it converges to some $y \in Y$ by $B$.

If $\{x_n'\}_{n=1}^{\infty}$ is another sequence converging to $x$ by $A$, then $\{(x_n - x_n')\}_{n=1}^{\infty}$ converges to 0 by $A$ and $\{T'(x_n) - T'(x_n')\}_{n=1}^{\infty}$ converges to 0 y $B$ because $T'$ is $A$—$B$-continuous. It shows that $\{T'(x_n)\}_{n=1}^{\infty}$ and $\{T'(x_n')\}_{n=1}^{\infty}$ have the same limit by $B$.

Put $T_{\alpha_0}(x) = \lim T'(x_n)$.

We obtain a linear operator $T_{\alpha_0}\colon X_{\alpha_0} \to Y$, which is an extension of $T$.

We are going to show that $T_{\alpha_0}$ is nonnegative.

Let $x \in X_{\alpha_0}$, $x \geq 0$. There exist $\{x_n\}_{n=1}^{\infty} \subset \bigcup\limits_{\alpha < \alpha_0} X_\alpha$ and $\{a_n\}_{n=1}^{\infty} \subset A$ such that: $|x_n - x| \leq a_n$ for all natural $n$ and $a_n \searrow 0$.

We have:

$|x_n + a_n - x| \leq |x_n - x| + a_n \leq 2a_n$ and $-a_n \leq x_n - x \leq a_n$ for all natural $n$. The first inequality shows that $(x_n + a_n) \to x$ by $A$ and the left side of the second inequality gives $x_n + a_n \geq x \geq 0$.

We have $T'(x_n + a_n) \to T_{\alpha_0}(x)$ and $T_{\alpha_0}(x) \geq 0$ by (14) from Proposition 4.5.

If $S\colon X_{\alpha_0} \to Y$ is another linear nonnegative extension of $T$, then the restriction of $S$ onto $X_\alpha$ for any $\alpha < \alpha_0$ is a nonnegative linear extension of $T$ onto $X_\alpha$. Since $\alpha < \alpha_0$ implies $\alpha \in P$, we have $S(x) = T_\alpha(x)$ for all $x \in X_\alpha$.

It means that $S(x) = T'(x)$ for all $x \in \bigcup\limits_{\alpha < \alpha_0} X_\alpha$.

If $x \in X_{\alpha_0}$, then $x_n \to x$ by $A$ for some sequence $\{x_n\}_{n=1}^{\infty} \subset \bigcup\limits_{\alpha < \alpha_0} X_\alpha$. We have $S(x_n) = T'(x_n) \to T_{\alpha_0}(x)$.

Since $S$ is $A$—$B$-continuous, by Proposition 4.10 $S(x) = T_{\alpha_0}(x)$. We have just proved that $\alpha_0 \in P$, which contradicts $\alpha_0 \in P'$. Therefore $P' = \emptyset$ and $P = \{\alpha\colon \alpha < \omega_1\}$. It means that for every $\alpha < \omega_1$ there exists a unique nonnegative linear extension $T_\alpha$ of $T$ onto $X_\alpha$.

If we put $\bar{T}(x) = T_\alpha(x)$ for $x \in X_\alpha$, we obtain a linear nonnegative extension of $T$ onto $\bar{X}_0$.

If $S$ is another such extension, it must coincide with $T_\alpha$ on $X_\alpha$. Therefore $\bar{T}(x) = S(x)$ for all $x \in \bar{X}_0$.

The proof of part (i) is complete.

(ii) $A$—$B$-continuity of $T$ follows from Proposition 4.10. The implication $T(X_0) \subset Y_0 \Rightarrow \bar{T}(\bar{X}_0) \subset \bar{Y}_0$ may be easily proved using the transfinite induction.

**Proposition 4.12.** Let $X$, $Y$ and $Z$ be $\sigma$-complete vector lattices, $A$, $B$ and $C$ be vector $\sigma$-sublattices of $X$, $Y$ and $Z$ respectively, $X_0 \subset X$ and $Y_0 \subset Y$ be vector subspaces, $T\colon X_0 \to Y$ and $S\colon Y_0 \to Z$ be linear nonnegative operators, which are $A$—$B$-continuous and $B$—$C$-continuous respectively.

If $T(X_0) \subset Y_0 = \bar{Y}_0$, $A \subset X_0$ and $B \subset Y_0$, then $\overline{S \circ T} = S \circ \bar{T}$, where $\overline{S \circ T}$ and $\bar{T}$ are extensions of $S \circ T$ and $T$ respectively onto $\bar{X}_0$.

Let $(\Omega, \mathscr{S}, P)$ be a probability measure space and $V$ be a $\sigma$-complete vector lattice. Put $X = \mathscr{F}(\Omega, \mathscr{S}, P, V)$, $A = \{[f]\colon f \text{ is constant}\}$, $B = Y = V$, $X_0 = \mathscr{L}_0^\infty(\Omega, \mathscr{S}, P, V)$ and $T = E$. Since $E([\text{const}]) =$ the same constant, the restriction of $T$ onto $A$ is $o$-continuous, and $T$ is $A$—$B$-continuous by Proposition 4.10.

We may apply Theorem 4.11 and we obtain:

**Theorem 4.13.**

(i) There exists a unique linear nonnegative extension

$$\bar{E}\colon \mathscr{L}^\infty(\Omega, \mathscr{S}, P, V) \to V \quad \text{of } E.$$

(ii) For any sequence $\{\xi_n\}_{n=1}^\infty \subset \mathscr{L}^\infty(\Omega, \mathscr{S}, P, V)$ and $\xi \in \mathscr{L}^\infty(\Omega, \mathscr{S}, P, V)$: $\xi_n \to \xi$ uniformly a.e. $\Rightarrow E(\xi_n) \to E(\xi)$.

The structure of $\mathscr{L}^\infty(\Omega, \mathscr{S}, P, V)$ is described in Proposition 2.5 and in the following one.

**Proposition 4.14.**

(i) $\mathscr{L}^\infty(\Omega, \mathscr{S}, P, V)$ is a vector sublattice of $\mathscr{F}(\Omega, \mathscr{S}, P, V)$

(ii) $\forall \xi \in \mathscr{L}^\infty(\Omega, \mathscr{S}, P, V), \forall B \in \mathscr{S}\colon [\chi_B] \cdot \xi \in \mathscr{L}^\infty(\Omega, \mathscr{S}, P, V)$.

**Proof:**

Part (i) follows from Propositions 2.4 and 4.7.

(ii) Using Proposition 4.7 we have a family $\{\mathscr{L}_\alpha\}_{\alpha < \omega_1}$ of linear subspaces of $\mathscr{F}(\Omega, \mathscr{S}, P, V)$ such that:

$$\mathscr{L}_0 = \mathscr{L}^\infty(\Omega, \mathscr{S}, P, V)$$

$\mathscr{L}_\alpha = \{\xi\colon \exists \{\xi_n\}_{n=1}^\infty \subset \bigcup_{\beta < \alpha} \mathscr{L}_\beta,\ \xi_n \to \xi \text{ almost everywhere}\}$ $\mathscr{L}^\infty(\Omega, \mathscr{S}, P, V) = \bigcup_{\alpha < \omega_1} \mathscr{L}_\alpha$.

If $\xi \in \mathscr{L}_0$ and $B \in \mathscr{S}$, then $[\chi_B]\,\xi \in \mathscr{L}_0$ by Proposition 2.4.

Let $0 < \alpha < \omega_1$. Suppose that

$$\forall \beta < \alpha \forall \xi \in \mathscr{L}_\beta \forall B \in \mathscr{S}\colon [\chi_B]\,\xi \in \mathscr{L}_\beta.$$

Take $\xi \in \mathscr{L}_\alpha$ and $B \in \mathscr{S}$.

There is some sequence $\{\xi_n\}_{n=1}^{\infty} \subset \bigcup_{\beta<\alpha} \mathscr{L}_\beta$, which converges uniformly a.e. to $\xi$.

We have $[\chi_B]\,\xi_n \in \bigcup_{\beta<\alpha} \mathscr{L}_\beta$ by the inductive assumption.

Obviously, $[\chi_B]\cdot\xi_n \to [\chi_B]\cdot\xi$ uniformly a.e.

Therefore $[\chi_B]\,\xi \in \mathscr{L}_\alpha$.

Let $(\Omega, \mathscr{S}\ P)$ be a probability measure space, $\mathscr{S}_0$ be a $\sigma$-subalgebra of $\mathscr{S}$, and $V$ be a $\sigma$-complete vector lattice. Put $X = \mathscr{F}(\Omega, \mathscr{S}, P, V)$, $Y = \mathscr{F}(\Omega, \mathscr{S}_0, P, V)$,

$$A = \{\xi\colon \xi \in \mathscr{F}(\Omega, \mathscr{S}, P, V),\ \xi = [\text{const}]\}$$

$$B = \{\eta\colon \eta \in \mathscr{F}(\Omega, \mathscr{S}_0, P, V),\ \eta = [\text{const}]\}$$

$$X_0 = \mathscr{L}_0^{\infty}(\Omega, \mathscr{S}, P, V)$$

$$T = E(\,.\,|\mathscr{S}_0).$$

Using Theorem 4.11 we obtain:

**Theorem 4.15.**

(i) There exists a unique linear nonnegative extension $\bar{E}(\,.\,|\mathscr{S}_0)\colon \mathscr{L}^{\infty}(\Omega, \mathscr{S}, P, V) \to \mathscr{F}(\Omega, \mathscr{S}_0, P, V)$ of $E(\,.\,|\mathscr{S}_0)\colon \mathscr{L}_0^{\infty}(\Omega, \mathscr{S}, P, V) \to \mathscr{F}(\Omega, \mathscr{S}_0, P, V)$.

(ii) $\forall \xi \in \mathscr{L}^{\infty}(\Omega, \mathscr{S}, P, V)\colon E(\xi|\mathscr{S}_0) \in \mathscr{L}^{\infty}(\Omega, \mathscr{S}_0, P, V)$

(iii) $\forall \{\xi_n\}_{n=1}^{\infty} \subset \mathscr{L}^{\infty}(\Omega, \mathscr{S}, P, V)\ \forall \xi \in \mathscr{L}^{\infty}(\Omega, \mathscr{S}, P, V)\colon \xi_n \to \xi$ uniformly a.e. $\Rightarrow \bar{E}(\xi_n\,|\,\mathscr{S}_0) \to \bar{E}(\xi|\mathscr{S}_0)$ uniformly a.e.

(iv) $\forall \xi \in \mathscr{L}^{\infty}(\Omega, \mathrm{S}, P, V)\ \forall B \in \mathscr{S}_0\colon \bar{E}([\chi_B]\,\xi|\mathscr{S}_0) = [\chi_B]\cdot\bar{E}(\xi|\mathscr{S}_0)$

(v) $\forall \xi \in \mathscr{L}^{\infty}(\Omega, \mathscr{S}, P, V)\colon \bar{E}(\xi) = \bar{E}(\bar{E}(\xi|\mathscr{S}_0))$.

**Proof:**

Parts (i)—(iii) follow from Theorem 4.11.

(iv) The equality $E[\chi_B]\,\xi|\mathscr{S}_0) = [\chi_B]\,E(\xi|\mathscr{S}_0)$ is true for $B \in \mathscr{S}_0$ and $\xi \in \mathscr{L}_0^{\infty}(\Omega, \mathscr{S}, P, V)$ by Proposition 3.1. Using the transfinite induction it may be proved for arbitrary $\xi \in \mathscr{L}^{\infty}(\Omega, \mathscr{S}, P, V)$.

(v) Let $B \in \mathscr{S}$ and $a \in V$. We have $E(a[\chi_B]) = aP(B)$.

As it was said in the proof of Proposition 3.1 there exists a sequence $\{f_n\}_{n=1}^{\infty}$ of real elementary functions which coverges to $E([\chi_B]|\mathscr{S}_0)$ uniformly a.e. We have:

$$af_n \to aE([\chi_B]|\mathscr{S}_0) = E(a[\chi_B]|\mathscr{S}_0) \quad \text{uniformly a.e.}$$

Therefore $E(a\cdot f_n) \to E(E(a[\chi_B]|\mathscr{S}_0))$, but $E(a\cdot f_n) = aE(f_n) \to aE(E([\chi_B]|\mathscr{S}_0)) = aP(B)$, because $f_n \to E([\chi_B]|\mathscr{S}_0)$ uniformly a.e. and $E(E([\chi_B]|\mathscr{S}_0)) = E([\chi_B])$ by the property (8) of the real conditional mean value operator.

We have just obtained the equality $\bar{E}(\bar{E}(\xi|\mathscr{S}_0)) = \bar{E}(\xi)$ for $\xi = a\cdot[\chi_B]$, where $a \in V$ and $B \in \mathscr{S}$.

By linearity, the last equality holds for all $\xi \in \mathscr{L}_0^\infty(\Omega, \mathscr{S}, P, V)$. In the general case we may apply either Proposition 4.12 or the transfinite induction.

**Theorem 4.16.** If $\mathscr{L}_0^1(\Omega, \mathscr{S}, P, V)$, $\mathscr{L}^1(\Omega, \mathscr{S}, P, V)$, $\mathscr{L}^1(\Omega, \mathscr{S}_0, P, V)$ are replaced by $\mathscr{L}_0^1(\Omega, \mathscr{S}, P, V)$, $\mathscr{L}^1(\Omega, \mathscr{S}, P, V)$ and $\mathscr{L}^\infty(\Omega, \mathscr{S}_0, P, V)$ respectively, then Theorems 4.13 and 4.15 remain true.

The advantage of $\mathscr{L}^1$ is that it is greater than $\mathscr{L}^\infty$, the advantage of $\mathscr{L}^\infty$ is that it is a sublattice of $\mathscr{F}(\Omega, \mathscr{S}, P, V)$.

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*Author's adress*:
Peter Maličký
Katedra teórie pravdepodobnosti
a matematickej štatistiky MFF UK,
Mlynská dolina, 842 15 Bratislava



Received: 28. 8. 1985


## SÚHRN

### NÁHODNÉ VELIČINY S HODNOTAMI VO VEKTOROVOM ZVÄZE (STREDNÁ HODNOTA A PODMIENENÁ STREDNÁ HODNOTA)

Peter MALIČKÝ, Bratislava


V práci je dokázaná jedna veta o rozšírení nezáporných operátorov. S jej pomocou je vybudovaná $L^\infty$ a $L^1$ — teória strednej hodnoty a podmienenej strednej hodnoty náhodných veličín nadobúdajúcich hodnoty v ľubovoľnom $\sigma$-úplnom vektorovom zväze.

## РЕЗЮМЕ

## СЛУЧАЙНЫЕ ВЕЛИЧИНЫ СО ЗНАЧЕНЯМИ В ВЕКТОРНОЙ РЕШЕТКЕ. (МАТЕМАТИЧЕСКОЕ ОЖИДАНИЕ И УСЛОВНОЕ МАТЕМАТИЧЕСКОЕ ОЖИДАНИЕ)

Петер МАЛИЧКИ, Братислава

В работе доказана одна теорема о расширении неотрицательных операторов. С ее помощью построены $L^\infty$ и $L^1$ — теории математического ожидания и условного математического ожидания для случайных величин, принимающих значения в произвольной $\sigma$-полной векторной решетке.

UNIVERSITAS COMENIANA
ACTA MATHEMATICA UNIVERSITATIS COMENIANAE
LII—LIII—1987

# THE CONSTRUCTION OF EXPLICIT ONE-STEP HYBRID METHODS

ANTON HUŤA—JOZEF DANČO, Bratislava

The paper deals with the numerical solution of an ordinary differential equation of the first order by means of hybrid methods. The parameters of the methods are rational numbers and the weights are the Newton—Cotes numbers. The method for finding the rational coefficients of the explicit one-step hybrid methods and the local truncation errors are given. Examples of the fomulas of the 2nd to 10th order and a numerical solution of a simple initial value problem by constructed methods are shown.

## 1 Introduction

Let us consider the solution of an initial value problem of the first order

$$\begin{aligned} y' &= f(x, y) \\ y(x_0) &= y_0 \end{aligned} \tag{1}$$

by one-step hybrid methods. They were introduced in the papers of Butcher [1] and [2], as a generalization of the Runge—Kutta methods with more stages.

The one-step hybrid methods are defined as a system of difference schemes:

$$y_{n+r_1} = y_n \tag{2}$$

$$y_{n+r_i} = y_n + h \sum_{j=1}^{s} a_{ij} f_{n+r_j} \qquad \text{for } i = 2, 3, \ldots, k \tag{3}$$

$$y_{n+1} = y_n + h \sum_{j=1}^{k} w_j f_{n+r_j}, \tag{4}$$

where

$f_{n+r_i} = f(x_n + r_i h, y_{n+r_i}) \qquad$ for $i = 1, 2, \ldots, k,$

$h$ is the stepsize,

$a_{ij}$ are the unknown coefficients of the method which have to be determined,

$w_j$ are the weights of the formula for finding the next value of the solution $y(x_n + h)$,

whereby for $s = i - 1, i, k$ the method is explicit, semiimplicit, implicit and $k$ is the number of stages.

This paper concentrates on explicit one-step hybrid methods. All the coefficients $a_{ij}$ are rational numbers and all $w_i$ are the Newton—Cotes numbers. They determine the conditions for choosing the unknown numbers $r_i$ for $i = 1, 2, \ldots, k$. These conditions determine $r_i$ as rational numbers as well as the in the interval $\langle 0, 1 \rangle$, so

$$r_i = \frac{i-1}{k-1} \tag{5}$$

for $i = 1, 2, \ldots, k$. This special case of the methods is a consequence of Huťa's paper [3].

## 2 The determination of unknown coefficients

To determine the coefficients $a_{ij}$ the polynomial approximation of the solution $y(x)$

$$y(x) = \sum_{i=0}^{m} b_i x^i$$

will be used. To get the method of the $m$-th order we shall successively put into relation (3) the polynoms of the $k$-th order for $k = 0, 1, \ldots, m - 1$. By comparing the coefficients of the same power of stepsize $h$ with those in the derivatives of $y(x)$ we get the linear equation system

$$\begin{aligned} \sum_{j=1}^{i-1} a_{ij} &= r_i \\ \sum_{j=2}^{i=1} r_i^m a_{ij} &= \frac{r_i^{m+1}}{m+1} \end{aligned} \tag{6}$$

for $m = 1, 2, \ldots, i - 1$ and $i = 2, 3, \ldots, k$.

All coefficients of the linear equation system are rational numbers, thus the solution will also be rational numbers. A special computer program was prepared for solving the linear equation systems in rational numbers. The source listing and some details of the computer program are given in [4]. By means of this program the coefficient $a_{ij}$ of the explicit one-step methods from 2nd to the 10th order were determined. The weights $w_i$ from (4) are well known Newton—Cotes quadrature coefficients. By putting the polynomials of the $k$-th order for $k = 0, 1, \ldots, m - 1$ into the formula (4) we obtain a special type of linear equation systems (6). The linear equation system has the form:

$$\sum_{i=1}^{k} w_i = 1$$

$$\sum_{i=2}^{k} r_i^{m-1} w_i = \frac{1}{m} \tag{7}$$

for $m = 2, 3, \ldots, k$. As we can see, the matrix of the linear equation system is the Van der Monde's matrix and its solution are the Newton—Cotes numbers.

Our one-step hybrid methods, using the Butcher's notation, have the form:

$$\begin{array}{c|ccccc}
r_1 & & & & & \\
r_2 & a_{21} & & & & \\
r_3 & a_{31} & a_{32} & & & \\
\vdots & \vdots & & & & \\
r_k & a_{k1} & a_{k2} & \cdots & a_{k,k-1} & \\
\hline
 & w_1 & w_2 & \cdots & w_{k-1} & w_k
\end{array}$$

### 3 Local truncation error of the methods

Suppose that we have a general form of the one-step methods

$$y_{n+1} = \varphi_f(x_n, y_n, y_{n+1}; h). \tag{8}$$

Then the local truncation error in the point $x_{n+1}$ is given by the formula

$$t_{n+1} = y(x_n + h) - \varphi_f(x_n, y(x_n), y(x_n + h); h), \tag{9}$$

where $y(x)$ is the exact solution of the differential equation (1). The local truncation error for our one-step hybrid methods is (the result transfer from well-known Newton—Cotes quadrature formulas):

$$t_{n+1} = c_k \frac{h^{k+1}}{k!} y^{(k+1)}(x_n) + 0(h^{k+2}) \qquad \text{for } k \text{ even}, \tag{10}$$

where

$$c_k = \frac{1}{k+1} - \sum_{i=2}^{k} w_i r_i^k \tag{11}$$

$$t_{n+1} = c_k \frac{h^{k+2}}{(k+1)!} y^{(k+2)}(x_n) + 0(h^{k+3}) \qquad \text{for } k \text{ odd}, \tag{12}$$

where

$$c_k = \frac{1}{k+2} - \sum_{i=2}^{k} w_i r_i^{k+1}. \tag{13}$$

From the formulas (10)—(13) we can see that we have not obtained the formulas of the odd order. It is the consequence of the Newton—Cotes numbers in the last formula.

## 4 Particular methods

We introduce particular explicit one-step hybrid methods for $k = 2$ up to $k = 10$.

$$
\begin{array}{c|cc}
0 & & \\
1 & 1 & \\
\hline
 & \frac{1}{2} & \frac{1}{2}
\end{array}
$$

$$
\begin{array}{c|ccc}
0 & & & \\
\frac{1}{2} & \frac{1}{2} & & \\
1 & 0 & 1 & \\
\hline
 & \frac{1}{6} & \frac{4}{6} & \frac{1}{6}
\end{array}
$$

$$
\begin{array}{c|cccc}
0 & & & & \\
\frac{1}{3} & \frac{1}{3} & & & \\
\frac{2}{3} & 0 & \frac{2}{3} & & \\
1 & \frac{1}{4} & 0 & \frac{3}{4} & \\
\hline
 & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8}
\end{array}
$$

$$
\begin{array}{c|ccccc}
0 & & & & & \\
\frac{1}{4} & \frac{1}{4} & & & & \\
\frac{1}{2} & 0 & \frac{1}{2} & & & \\
\frac{3}{4} & \frac{3}{16} & 0 & \frac{9}{16} & & \\
1 & 0 & \frac{2}{3} & \frac{-1}{3} & \frac{2}{3} & \\
\hline
 & \frac{7}{90} & \frac{32}{90} & \frac{12}{90} & \frac{32}{90} & \frac{7}{90}
\end{array}
$$

| | | | | | | |
|---|---|---|---|---|---|---|
| $0$ | | | | | | |
| $\frac{1}{5}$ | $\frac{1}{5}$ | | | | | |
| $\frac{2}{5}$ | $0$ | $\frac{2}{5}$ | | | | |
| $\frac{3}{5}$ | $\frac{3}{20}$ | $0$ | $\frac{9}{20}$ | | | |
| $\frac{4}{5}$ | $0$ | $\frac{8}{15}$ | $\frac{-4}{15}$ | $\frac{8}{15}$ | | |
| $1$ | $\frac{19}{144}$ | $\frac{-10}{144}$ | $\frac{120}{144}$ | $\frac{-70}{144}$ | $\frac{85}{144}$ | |
| | $\frac{19}{288}$ | $\frac{75}{288}$ | $\frac{50}{288}$ | $\frac{50}{288}$ | $\frac{75}{288}$ | $\frac{19}{288}$ |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $0$ | | | | | | | |
| $\frac{1}{6}$ | $\frac{1}{6}$ | | | | | | |
| $\frac{1}{3}$ | $0$ | $\frac{1}{3}$ | | | | | |
| $\frac{1}{2}$ | $\frac{1}{8}$ | $0$ | $\frac{3}{8}$ | | | | |
| $\frac{2}{3}$ | $0$ | $\frac{4}{9}$ | $\frac{-2}{9}$ | $\frac{4}{9}$ | | | |
| $\frac{5}{6}$ | $\frac{95}{864}$ | $\frac{-50}{864}$ | $\frac{600}{864}$ | $\frac{-350}{864}$ | $\frac{425}{864}$ | | |
| $1$ | $0$ | $\frac{11}{20}$ | $\frac{-14}{20}$ | $\frac{26}{20}$ | $\frac{-14}{20}$ | $\frac{11}{20}$ | |
| | $\frac{41}{840}$ | $\frac{216}{840}$ | $\frac{27}{840}$ | $\frac{272}{840}$ | $\frac{27}{840}$ | $\frac{216}{840}$ | $\frac{41}{840}$ |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $0$ | | | | | | | | |
| $\frac{1}{7}$ | $\frac{1}{7}$ | | | | | | | |
| $\frac{2}{7}$ | $0$ | $\frac{2}{7}$ | | | | | | |
| $\frac{3}{7}$ | $\frac{3}{28}$ | $0$ | $\frac{9}{28}$ | | | | | |
| $\frac{4}{7}$ | $0$ | $\frac{8}{21}$ | $\frac{-4}{21}$ | $\frac{8}{21}$ | | | | |
| $\frac{5}{7}$ | $\frac{95}{1\,008}$ | $\frac{-50}{1\,008}$ | $\frac{600}{1\,008}$ | $\frac{-350}{1\,008}$ | $\frac{425}{1\,008}$ | | | |
| $\frac{6}{7}$ | $0$ | $\frac{33}{70}$ | $\frac{-42}{70}$ | $\frac{78}{70}$ | $\frac{-42}{70}$ | $\frac{33}{70}$ | | |
| $1$ | $\frac{751}{8\,640}$ | $\frac{-840}{8\,640}$ | $\frac{8\,547}{8\,640}$ | $\frac{-11\,648}{8\,640}$ | $\frac{14\,637}{8\,640}$ | $\frac{-7\,224}{8\,640}$ | $\frac{4\,417}{8\,640}$ | |
| | $\frac{751}{17\,280}$ | $\frac{3\,577}{17\,280}$ | $\frac{1\,323}{17\,280}$ | $\frac{2\,989}{17\,280}$ | $\frac{2\,989}{17\,280}$ | $\frac{1\,323}{17\,280}$ | $\frac{3\,577}{17\,280}$ | $\frac{751}{17\,280}$ |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $0$ | | | | | | | | | |
| $\frac{1}{8}$ | $\frac{1}{8}$ | | | | | | | | |
| $\frac{1}{4}$ | $0$ | $\frac{1}{4}$ | | | | | | | |
| $\frac{3}{4}$ | $\frac{3}{32}$ | $0$ | $\frac{9}{32}$ | | | | | | |
| $\frac{1}{2}$ | $0$ | $\frac{2}{6}$ | $\frac{-1}{6}$ | $\frac{2}{6}$ | | | | | |
| $\frac{5}{8}$ | $\frac{95}{1\,152}$ | $\frac{-50}{1\,152}$ | $\frac{600}{1\,152}$ | $\frac{-350}{1\,152}$ | $\frac{425}{1\,152}$ | | | | |
| $\frac{3}{4}$ | $0$ | $\frac{33}{80}$ | $\frac{-42}{80}$ | $\frac{78}{80}$ | $\frac{-42}{80}$ | $\frac{33}{80}$ | | | |
| $\frac{7}{8}$ | $\frac{5\,257}{69\,120}$ | $\frac{-5\,880}{69\,120}$ | $\frac{59\,829}{69\,120}$ | $\frac{-81\,536}{69\,120}$ | $\frac{102\,459}{69\,120}$ | $\frac{-505\,668}{69\,120}$ | $\frac{30\,919}{69\,120}$ | | |
| $1$ | $0$ | $\frac{460}{945}$ | $\frac{-954}{945}$ | $\frac{2\,196}{945}$ | $\frac{-2\,459}{945}$ | $\frac{2\,196}{945}$ | $\frac{-954}{945}$ | $\frac{460}{945}$ | |
| | $\frac{984}{28\,350}$ | $\frac{5\,888}{28\,350}$ | $\frac{-928}{28\,350}$ | $\frac{10\,496}{28\,350}$ | $\frac{-4\,540}{28\,350}$ | $\frac{10\,496}{28\,350}$ | $\frac{-928}{28\,350}$ | $\frac{5\,888}{28\,350}$ | $\frac{984}{28\,350}$ |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $0$ | $0$ | | | | | | | | | |
| $\frac{1}{9}$ | $\frac{1}{9}$ | | | | | | | | | |
| $\frac{2}{9}$ | $0$ | $\frac{2}{9}$ | | | | | | | | |
| $\frac{1}{3}$ | $\frac{1}{12}$ | $0$ | $\frac{3}{12}$ | | | | | | | |
| $\frac{4}{9}$ | $0$ | $\frac{8}{27}$ | $\frac{-4}{27}$ | $\frac{8}{27}$ | | | | | | |
| $\frac{5}{9}$ | $\frac{95}{1\,296}$ | $\frac{-50}{1\,296}$ | $\frac{600}{1\,296}$ | $\frac{-350}{1\,296}$ | $\frac{425}{1\,296}$ | | | | | |
| $\frac{2}{3}$ | $0$ | $\frac{11}{30}$ | $\frac{-14}{30}$ | $\frac{26}{30}$ | $\frac{-14}{30}$ | $\frac{11}{30}$ | | | | |
| $\frac{7}{9}$ | $\frac{5\,257}{77\,760}$ | $\frac{-5\,880}{77\,760}$ | $\frac{59\,829}{77\,760}$ | $\frac{-81\,536}{77\,760}$ | $\frac{102\,459}{77\,760}$ | $\frac{-50\,568}{77\,760}$ | $\frac{30\,919}{77\,760}$ | | | |
| $\frac{8}{9}$ | $0$ | $\frac{3\,680}{8\,505}$ | $\frac{-7\,632}{8\,505}$ | $\frac{17\,568}{8\,505}$ | $\frac{-19\,672}{8\,505}$ | $\frac{17\,568}{8\,505}$ | $\frac{-7\,632}{8\,505}$ | $\frac{3\,680}{8\,505}$ | | |
| $1$ | $\frac{231\,407}{3\,628\,800}$ | $\frac{403\,786}{3\,628\,800}$ | $\frac{4\,208\,966}{3\,628\,800}$ | $\frac{8\,935\,522}{3\,628\,800}$ | $\frac{14\,812\,580}{3\,628\,800}$ | $\frac{14\,344\,702}{3\,628\,800}$ | $\frac{10\,502\,666}{3\,628\,800}$ | $\frac{4\,121\,686}{3\,628\,800}$ | $\frac{1\,678\,877}{3\,628\,800}$ | |
| | $\frac{2\,857}{89\,600}$ | $\frac{15\,741}{89\,600}$ | $\frac{1\,080}{89\,600}$ | $\frac{19\,344}{89\,600}$ | $\frac{5\,778}{89\,600}$ | $\frac{5\,778}{89\,600}$ | $\frac{19\,344}{89\,600}$ | $\frac{1\,080}{89\,600}$ | $\frac{15\,741}{89\,600}$ | $\frac{2\,857}{89\,600}$ |

## 5 Numerical results

As an example of using our methods we begin with a very simple initial value problem

$$y' = -y$$
$$y(0) = 1$$

with exact solution $y(x) = \exp(-x)$. On the interval $\langle 0, 1 \rangle$ and the stepsize is $h = 0.1$. Solving the problem one can get the following results:

| Number of stages | Order | The absolute error in $x = 1$. |
|---|---|---|
| 2 | 2 | $0.661\ 10^{-3}$ |
| 3 | 4 | $0.322\ 10^{-3}$ |
| 4 | 4 | $0.766\ 10^{-4}$ |
| 5 | 6 | $0.432\ 10^{-4}$ |
| 6 | 6 | $0.193\ 10^{-4}$ |
| 7 | 8 | $0.143\ 10^{-4}$ |
| 8 | 8 | $0.778\ 10^{-5}$ |
| 9 | 10 | $0.697\ 10^{-5}$ |
| 10 | 10 | $0.392\ 10^{-5}$ |

The numerical results show that the proposed methods can give a good result for the solution of the Cauchy problems. The computations procedure was programmed in double precision in FORTRAN for the SMEP with operation system DOS RV.

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*Author's addresses*:
Anton Huťa
Jozef Dančo
Katedra numerických a optimalizačných metód
Matematicko-fyzikálna fakulta
Univerzity Komenského
Mlynská dolina
842 15 Bratislava
Czechoslovakia

Received: 16. 4. 1986

## SÚHRN

### KONŠTRUKCIA EXPLICITNÝCH JEDNOKROKOVÝCH HYBRIDNÝCH VZORCOV

Anton Huťa—Jozef Dančo, Bratislava

Článok sa zaoberá numerickým riešenim začiatočnej úlohy 1. rádu špeciálnou triedou jednokrokových hybridných metód, ktorých koeficienty sú racionálne čísla s Newtonovými—Cotesovými váhami. V článku je ukázaný postup pre odvedenie racionálnych koeficientov explicitných jednokrokových hybridných vzorcov a formuly pre lokálnu chybu aproximácie. Sú uvedené príklady vzorcov od druhého po desiaty rád presnosti. V práci je riešený jednoduchý typový príklad pomocou odvodených metód.



## РЕЗЮМЕ

### КОНСТРУКЦИЯ НЕЯВНЫХ ОДНОШАГОВЫХ ГИБРИДНЫХ МЕТОДОВ

Антон Хутя—Йосеф Данчо, Братислава

Авторы занимаются численными методами решеня задаши Коши вида одношаговых гибридных методов, коэффициенты которых рациональные числа с весами Невтон—Котеса. В статье разработан алгоритм нахождения коэффициентов неявных методов и погрешности апроксимации. Вводятся примеры методов от второго до десятого порядков точности, с помощью которых решилась простая задача Коши.