UNIVERSITAS COMENIANA
ACTA MATHEMATICA UNIVERSITATIS COMENIANAE
LII—LIII—1987

REMARKS ON STRONG AND SYMMETRIC DIFFERENTIABILITY OF REAL FUNCTIONS

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Introduction

A function f: $\langle a, b\rangle \to R$ is said to be strongly differentiable at the point $x \in \langle a, b\rangle$ if there exists the finite limit

$$\max_{\substack{[y, z] \to [x, x] \\ y \neq z}} \frac{f(z) - f(y)}{z - y} = f^*(x)$$

(cf. [1], [4]).

Let the function f be differentiable on $\langle a, b\rangle$. It is said to be uniformly differentiable at the point $x \in \langle a, b\rangle$ if for each $\varepsilon > 0$ there exists a $\delta > 0$ and a neighbourhood $U(x)$ of the point x such that for each $y \in U(x)$ and each h, $0 < |h| < \delta$, we have $|\Phi(y, h)| < \varepsilon$, where

$$\Phi(y, h) = \frac{f(y + h) - f(y)}{h} - f'(y)$$

(cf. [7]).

Let the function f be symmetrically differentiable on $\langle a, b\rangle$ (for $x < a$ ($x > b$) we put $f(x) = f(a)$ ($f(x) = f(b)$)). The function f is said to be uniformly symmetrically differentiable at the point $x \in \langle a, b\rangle$ if for each $\varepsilon > 0$ there exists a $\delta > 0$ and a neighbourhood $U(x)$ of x such that for each $y \in U(x)$ and each h, $0 < |h| < \delta$, we have $|\Psi(y, h)| < \varepsilon$, where

$$\Psi(y, h) = \frac{f(y + h) - f(y - h)}{2h} - f^s(y)$$

($f^s(y)$ stands for the symmetric derivative of the function f at y) (cf. [10]).

A point x is said to be a point of strong differentiability, a point of uniform differentiability and a point of uniform symmetric differentiability of the function f if f is strongly differentiable, uniformly differentiable and uniformly symmetrically differentiable at x, respectively.

This paper consists of two parts. In the first part we shall give simple proofs of certain results concerning strong differentiability and uniform symmetric differentiability. The second part contains new proofs of two results concerning the sets of points of strong differentiability and points of uniform differentiability of functions.

1 On strong differentiation and uniform symmetric differentiation

At first we shall give simple proofs of the following two theorems. In this part of the paper we suppose that the function f is defined on a certain interval.

Theorem 1.1. Suppose f is a symmetrically differentiable function and that $f^*(x_o)$ exists. Then f^s is continuous at x_0.

Proof. Suppose there is a sequence $\{x_n\}_{n=1}^{\infty}$, $x_n \to x_o$ with $f^s(x_n) \nrightarrow f^s(x_o)$. There is an h_n, $0 < h_n < \frac{1}{n}$ such that

$$\left|\frac{f(x_n + h_n) - f(x_n - h_n)}{2h_n} - f^s(x_n)\right| < \frac{1}{n}$$

$(n = 1, 2, \ldots)$.

Then both $x_n \pm h_n \to x_o$ and

$$\frac{f(x_n + h_n) - f(x_n - h_n)}{2h_n} \nrightarrow f^s(x_o) = f^*(x_o)$$

Theorem 1.2. Suppose f is a continuous symmetrically differentiable function and that f^s is continuous at x_o. Then f is uniformly symmetrically differentiable at x_o.

Proof. Suppose f is not uniformly symmetrically differentiable at x_o. Then there is an $\varepsilon_o > 0$, a sequence $\{x_n\}_{n=1}^{\infty}$, $x_n \to x_o$, and a sequence $\{h_n\}_{n=1}^{\infty}$, $h_n \to 0$ such that either

a) $$\frac{f(x_n + h_n) - f(x_n - h_n)}{2h_n} \geqq f^s(x_o) + \varepsilon_o$$

or

b) $$\frac{f(x_n + h_n) - f(x_n - h_n)}{2h_n} \leqq f^s(x_o) - \varepsilon_o$$

According to the Quasi-Mean Value Theorem (cf. [9], Theorem 5.7, p. 166) there are two sequences $\{c_n\}_{n=1}^{\infty}$, $\{d_n\}_{n=1}^{\infty}$ such that $c_n, d_n \in (x_n - h_n, x_n + h_n)$ $(n = 1, 2, \ldots)$ and

$$f^s(c_n) < \frac{f(x_n + h_n) - f(x_n - h_n)}{2h_n} < f^s(d_n)$$

$(n = 1, 2, \ldots)$.
In the case a) above, $d_n \to x_o$ but $f^s(d_n) \not\to f^s(x_o)$.
In the case b) we have $c_n \to x_o$ but $f^s(c_n) \not\to f^s(x_o)$.
The proof is finished.

The following theorem shows that the strong differentiability is a consequence of the uniform symmetric differentiability for symmetrically differentiable functions.

Theorem 1.3. If f is a symmetrically differentiable function and it is uniformly symmetrically differentiable at x_o, then f is strongly differentiable at x_o.

Proof. We actually prove that if f is uniformly symmetrically differentiable at x_o, but not strongly differentiable at x_o, then f is approximately discontinuous in a neighbourhood of x_o. This, of course, contradicts the fact that symmetrically differentiable functions are measurable.

Assume $x_o = 0$ and $f^s(0) = 0$. Suppose there are sequences $a_n \to 0$, $b_n \to 0$ such that

$$\left|\frac{f(b_n) - f(a_n)}{b_n - a_n}\right| \to M > 0$$

As f is uniformly symmetrically differentiable at 0 there is a δ, $0 < \delta < 1$ such that if $x \in (-\delta, \delta)$ and $|h| < \delta$, $h \neq 0$, then

$$\left|\frac{f(x+h) - f(x-h)}{2h} - f^s(x)\right| < \frac{M}{8} \tag{1}$$

For $x = c_n = \dfrac{a_n + b_n}{2}$ and $h = h_n = \dfrac{b_n - a_n}{2}$ we find from (1)

$$\left|\frac{f(b_n) - f(a_n)}{b_n - a_n} - f^s(c_n)\right| < \frac{M}{8}$$

Hence, there is an N such that for $n \geqq N$

$$|f^s(c_n)| > \frac{3M}{4}$$

We suppose $f^s(c_n) > \dfrac{3M}{4}$, the other case being analogous.

Suppose now that $y \in (0, \delta)$ is a point of approximate continuity of f. Then, there is an interval $I = (y - \delta_1, y + \delta_1)$, $0 < \delta_1 < \dfrac{3}{11} y$, and a set $E \subset I$ of density exceeding $\dfrac{3}{4}$ in I such that if $x \in E$, then

$$|f(x) - f(y)| < \frac{M|y|}{8} = \frac{My}{8} \tag{2}$$

As $f^s(0) = 0$,

$$\left|\frac{f(x) - f(-x)}{2x}\right| < \frac{M}{8}$$

for each $x \in E$ (see (1)), and so

$$x) - f(-x)| < \frac{M|x|}{4} = \frac{Mx}{4} \tag{3}$$

Then, combining (2) and (3),

$$|f(y) - f(-x)| \leqq |f(y) - f(x)| + |f(x) - f(-x)| < \frac{My}{8} + \frac{Mx}{4} < \frac{3M}{8}(y + \delta_1)$$

and so,

$$f(y) - \frac{3M}{8}(y + \delta_1) < f(-x) \tag{4}$$

Now, by (1) (putting $x = c_n$, $h = c_n + x$)

$$\left|\frac{f(2c_n + x) - f(-x)}{2(c_n + x)} - f^s(c_n)\right| < \frac{M}{8}$$

$$\frac{5M}{8} < f^s(c_n) - \frac{M}{8} < \frac{f(2c_n + x) - f(-x)}{2(c_n + x)} < f^s(c_n) + \frac{M}{8},$$

$$\frac{5M}{4}(c_n + x) + f(-x) < f(2c_n + x)$$

and by (4)

$$\frac{5M}{4}(c_n + x) + f(y) - \frac{3M}{8}(y + \delta_1) < f(2c_n + x),$$

$$f(y) + \frac{M}{4}\left[5c_n + 5x - \frac{3}{2}(y + \delta_1)\right] < f(2c_n + x)$$

For large n,

$$f(y) + \frac{M}{4}\left[4x - \frac{3}{2}(y + \delta_1)\right] < f(2c_n + x),$$

$$f(y) + \frac{My}{4} < f(y) + \frac{M}{4}\left[\frac{5}{2}y - \frac{11}{2}\delta_1\right] < f(2c_n + x)$$

That is, for large n, $E_n^* = \{2c_n + x : x \in E\} \cap I$ is disjoint from E. But as $c_n \to 0$, and as E_n^* is a translate of E, E_n^* also has a large density in I and this entails a contradiction. The proof is finished.

2 On points of uniform differentiability and strong differentiability

In [11] the following results are proved.

(A) If f is a differentiable function on $\langle a, b\rangle$, then a point of the interval $\langle a, b\rangle$ is a point of uniform differentiability of the function f if and only if it is a point of strong differentiability of f.

(B) If f is a continuous and symmetrically differentiable function on $\langle a, b\rangle$, then a point of $\langle a, b\rangle$ is a point of uniform symmetric differentiabili of f if and only if it is a point of strong differentiability of f.

The structure of the set of all points of uniform differentiability of an arbitrary differentiable function is described in [7]; the analogous investigations for uniform symmetric differentiability are made in [10]. in these papers the following theorems are proved.

Theorem I. Let $f: \langle a, b\rangle \to R$ be a differentiable function on $\langle a, b\rangle$. Then the set of all points of uniform differentiability of the function f is residual in $\langle a, b\rangle$.

Theorem II. Let $f: \langle a, b\rangle \to R$ be a continuous and symmetrically differentiable function on $\langle a, b\rangle$. Then the set of all points of uniform symmetric differentiability of the function f is residual in $\langle a, b\rangle$.

From Theorem II and (B) we get the following result which is equivalent to (B) and is a little weaker than Theorem 1 from [1].

Theorem 2.1. Let $f: \langle a, b\rangle \to R$ be a continuous and symmetrically differentiable function on $\langle a, b\rangle$. Then the set of all points of strong differentiability of the function f is residual in $\langle a, b\rangle$.

The proofs of Theorem I and Theorem II are in [7] and [10] based on some properties of a certain function $\alpha(f, x)$. We shall give new proofs of Theorem I and Theorem 2.1 based on some knowledges about sets of points of absolute continuity of real function (cf. [2], [3], [6]).

Proof of Theorem I. Denote by Δ_f^* the set of all points of strong differentiability of the function f in $\langle a, b\rangle$. From Theorem 3 of [4] we get

$$\Delta_f^* = G_f \cap L \tag{5}$$

where G_f denotes the set of all points of absolute continuity of the function f (cf. [2], [3], [12]) and L is the set of all $x \in \langle a, b\rangle$ at which there exists a finite

$$\lim_{\substack{t \to x \\ t \in \Delta}} f'(t)$$

$\Delta = \Delta_f$ being the set of all points of differentiability of the function f.

It is proved in [2], [12] that the set G_f has the form $\langle a, b\rangle \setminus H$, where H is a nowhere dense set in $\langle a, b\rangle$. Hence, G_f is a residual set in $\langle a, b\rangle$.

Since f' belongs to the first Baire class, the set $C(f')$ of all continuity points of f' is residual in $\langle a, b\rangle$. Evidently we have $L \supset C(f')$, thus L is residual in $\langle a, b\rangle$, too. But then the assertion follows at once from (5).

Proof of Theorem 2.1. Using the notation from the proof of Theorem I we get (5).

On account of Theorem HS from [6] the set G_f is residual in $\langle a, b\rangle$. It suffices therefore to show that L is a residual set.

Since f is continuous on $\langle a, b\rangle$ and symmetrically differentiable, it follows from Theorem 11 of the paper [8] that the complement of the set $\Delta(=\Delta_f)$ of points of differentiability of f is a σ-porous set. Hence it is a set of the first category (cf. [5], [13]). Thus we have

$$\Delta = \langle a, b\rangle \setminus A \tag{6}$$

A being a set of the first category in $\langle a, b\rangle$.

Denote by $C_1(f')$ the set of all points $x \in \Delta$ at which f' is continuous (relative to the set Δ). Since f' is a function of the first Baire class on Δ, the set $C_1(f')$ is residual in Δ. Hence

$$C_1(f') = \Delta \setminus B \tag{7}$$

B being a subset of Δ of the first category in Δ (and therefore at of the first category in $\langle a, b\rangle$, too).

We get from (6), (7) the equality

$$C_1(f') = \langle a, b\rangle \setminus (A \cup B)$$

Hence $C_1(f')$ is a residual set in $\langle a, b\rangle$. The definition of the set L yields $L \supset C_1(f')$. Hence the set L is residual in $\langle a, b\rangle$. This ends the proof.

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Received: 1. 9. 1986

SÚHRN

POZNÁMKY O SYMETRICKEJ A SILNEJ DIFERENCOVATEĽNOSTI REÁLNYCH FUNKCIÍ

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V práci sú podané jednoduché dôkazy niektorých výsledkov o silnej diferencovateľnosti a rovnomernej symetrickej diferencovateľnosti reálnych funkcií. Ďalej práca obsahuje nové dôkazy istých známych výsledkov o bodoch silnej diferencovateľnosti a bodoch rovnomernej symetrickej diferencovateľnosti. Tieto dôkazy sa zakladajú na istých vlastnostiach množín bodov absolútnej spojitosti funkcií.

РЕЗЮМЕ

ЗАМЕЧАНИЯ О СИММЕТРИЧЕСКОЙ И СИЛЬНОЙ ДИФФЕРЕНЦИРУЕМОСТИ ДЕЙСТВИТЕЛЬНЫХ ФУНКЦИЙ

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В работе показаны простые доказательства некоторых результатов относительно сильной дифференцируемости и равномерной симметрической дифференцируемости действительных функций. Далее работа содержит новые доказательства некоторых известных результатов, касающихся пунктов сильной дифференцируемости и пунктов равномерной симметрической дифференцируемости. Эти доказательства основаны на некоторых свойствах множеств пунктов абсолютной непрерывности функций.