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MEASURES OF FUZZINESS OF FUZZY PARTITIONS

SLÁVKA BODJANOVÁ, Bratislava

Introduction

Fuzzy clustering algorithms of Ruspini [5], Bezdek [2], Dunn [4] and Backer [1] all yield fuzzy partitions as clustering solutions for partitioning finite data sets. It was shown [2] that fuzzy partitions can be characterized by special cluster validity functionals, e.g. classification entropy, partition coefficient, etc. Backer [1] evaluates fuzzy clusters as follows: if the amount of fuzziness in fuzzy partition is low, it means that the clusters are reasonably separable and that the partition reflects the real data structure reasonably well. On the other hand, if the amount of fuzziness is high, it means that fuzzy set separability is low and that either the partition does not reflect the real structure well or that almost no structure is present in the data. The measure of fuzziness of fuzzy partitions is therefore an important problem in fuzzy clustering. Our purpose in this paper is to develop an axiomatic framework for a measure of fuzziness of fuzzy partitions and to show a more general way of constructing these measures.

Partition spaces are defined in Section 2, where also some cluster validity functionals from [1], [2] are given. In Section 3 we introduce four conditions which we think a measure of fuzziness of fuzzy partitions should satisfy; some examples of measures of fuzziness of fuzzy partitions are given. In Section 4 we propose three ways of a more general constructing of measures of fuzziness of fuzzy partitions. Section 5 contains our definition of min (max) α-combination of fuzzy sets. We suggest some measures of fuzziness of fuzzy clusters. The connection between measures of fuzziness and measures of dissimilarity of fuzzy partitions is shown in Section 6.

1 Partition spaces

Let $X = \{x_1, x_2, \ldots, x_n\} \subset R^p$ be a given finite data set. We fix the integer k, $2 \leq k < n$ and denote by V_{kn} the usual vector space of real $k \times n$ matrices.

Partitions of X are defined in [2] as follows:

$$P_k = \Big\{U \in V_{kn}\colon u_{ij} \in \{0, 1\} \text{ for all } i, j; \quad \sum_i u_{ij} = 1 \text{ for all } j; \\ \sum_j u_{ij} > 0 \text{ for all } i\Big\} \tag{1.1}$$

We call P_k hard k-partition space associated with X.

Here u_{ij} is the value of a characteristic function $u_i\colon X \to \{0, 1\}$; u_{ij} specifies the membership of $x_j \in X$ in a partitioning subset $Y_i \subset X$. Fuzzy k-partition space for X is defined as follows:

$$P_{fk} = \Big\{U \in V_{kn}\colon u_{ij} \in \langle 0, 1\rangle \text{ for all } i, j; \quad \sum_i u_{ij} = 1 \text{ for all } j; \\ \sum_j u_{ij} > 0 \text{ for all } i\Big\} \tag{1.2}$$

Here u_{ij} is the grade of membership of $x_j \in X$ in a fuzzy subset $u_i\colon X \to \langle 0, 1\rangle$.

The superset of P_k defined by

$$P_{k0} = \Big\{U \in V_{kn}\colon u_{ij} \in \{0, 1\} \text{ for all } i, j; \quad \sum_i u_{ij} = 1 \text{ for all } j; \\ \sum_j u_{ij} \geq 0 \text{ for all } i\Big\} \tag{1.3}$$

is a degenerate hard k-partition space for X.

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is a degenerate fuzzy k-partition space for X.

In [2] some partitioning characterizations are given. For example: degree of separation introduced by L. A. Zadeh

for $U \in P_{fk0}$:

$$Z(U, k) = 1 - \max_j (\min_i u_{ij}); \tag{1.5}$$

partition coefficient introduced by J. C. Bezdek

for $U \in P_{fk0}$:

$$F(U, k) = \frac{1}{n} \sum_i \sum_j u_{ij}^2; \tag{1.6}$$

classification entropy introduced by J. C. Bezdek for $U \in P_{fk0}$:

$$H(U, k) = \frac{1}{n} \sum_i \sum_j u_{ij} \cdot \log_a u_{ij}, \tag{1.7}$$

where $a \in (1, \infty)$ and $u_{ij} \cdot \log_a u_{ij} = 0$ for $u_{ij} = 0$;
proportion exponent introduced by M. P. Windham
for $U \in P_{fk0} - P_{k0}$:

$$P(U, k) = -\log_2 \left\{ \prod_{j=1}^{n} \left(\sum_{t=1}^{I_j} (-1)^{t+1} \binom{k}{t} (1 - \max_t u_{tj})^{k-1} \right) \right\} \tag{1.8}$$

where I_j is the greatest integer in $[\max_t u_{tj}]$.

De Luca and Termini [3] defined the measure of fuzziness of fuzzy sets. Backer [1] proposed partitioning characterization in terms of fuzziness of fuzzy sets as follows:
for $U \in P_{fk0}$:

$$\zeta(U, k) = 1 - \frac{2}{k-1} \sum_{r=1}^{k-1} \sum_{s=r+1}^{k} f(u_r \cap u_s), \tag{1.9}$$

where f is a measure of fuzziness of fuzzy set $u_r \cap u_s$.

2 Measures of fuzziness of fuzzy partitions

Let us denote by $L(X)$ the class of all fuzzy sets built on a finite set $X = \{x_1, x_2, \ldots, x_n\} \subset R^p$. $L(X)$ can be partially ordered by relation $\leq_s$ called "sharpned" by De Luca and Termini [3] and defined for all $u, v \in L(X)$ as follows:
u is more fuzzy than v, i.e. $v \leq_s u$ if and only if

$$v(x) \leq u(x) \text{ for } u(x) < \frac{1}{2} \tag{2.1a}$$

and

$$v(x) \geq (x) \text{ for } u(x) > \frac{1}{2}. \tag{2.1b}$$

De Luca and Termini [3) introduced for every fuzzy set $u \in L(X)$ a measure $d(u)$ of its "fuzziness". They impose for measure d to verify the following properties:

(1) $d(u) = 0$ iff for all $x \in X$: $u(x) = 0$ or $u(x) = 1$ (2.2a)

(2) $d(u)$ is maximum iff for all $x \in X$: $u(x) = \frac{1}{2}$ (2.2b)

(3) if $u, v \in L(X)$ are such that $v \leq_s u$, then $d(v) \leq d(u)$ (2.2c)

We try to introduce the relation sharpned on P_{fk0} and to give for every $U \in P_{fk0}$ a measure of its fuzziness.

Remark: Every $U \in P_{fk0}$ is a k-collection of fuzzy sets $\{u_1, u_2, \ldots, u_k\}$. Throughout this paper we denote by $U^{(p)}$ the k-collection $u_{j_1}, \ldots, u_{j_k}$, where $\{j_1, j_2, \ldots, j_k\}$ is any permutation of $\{1, 2, \ldots, k\}$.

Definition 2.1. If U, $W \in P_{fk0}$ are such that there exists $W^{(p)}$ such that $w_{ij}^{(p)}$

$$w_{ij}^{(p)} \leq u_{ij} \text{ for } u_{ij} < \frac{1}{k} \tag{2.3a}$$

and

$$w_{ij}^{(p)} \geq u_{ij} \text{ for } u_{ij} > \frac{1}{k}, \tag{2.3b}$$

we say that W is a sharpned version of U denoted by $W \prec U$.

Theorem 2.1. P_{fk0} is partially ordered by relation $\prec$. The proof is evident.

Definition 2.2. Consider a real nonnegative function φ: $P_{fk0} \to R$. This function is called a measure of fuzziness of partitions form P_{fk0} if the following properties are true:

For all $U \in P_{fk0}$:

(P1) $\varphi(U) = \varphi(U^{(p)})$ (2.4a)

(P2) $\varphi(U) = 0$ iff $U \in P_{k0}$ (2.4b)

(P3) $\varphi(U)$ is maximum iff $U = \left[\frac{1}{k}\right]$ (2.4c)

(P4) if U, $W \in P_{fk0}$ are such that $W \prec U$, then $\varphi(W) \leq \varphi(U)$ (2.4d)

Theorem 2.2. Consider a function φ: $P_{fk0} \to R$, where for all $U \in P_{fk0}$:

$$\varphi(U) = 1 - \frac{k}{n(k-1)} \sum_i \sum_j \left(u_{ij} - \frac{1}{k}\right)^2. \tag{2.5}$$

The function φ is a measure of fuzziness of partitions from P_{fk0}.

Proof:

It is sufficient to verify if the function φ satisfies the properties (P1)—(P4) in Definition 2.2.

(P1) holds obviously.

For all j:

$$0 \leq \sum_i \left(u_{ij} - \frac{1}{k}\right)^2 = \sum_i u_{ij}^2 - \frac{2}{k} \sum_i u_{ij} + \frac{k}{k^2} = \sum_i u_j^2 - \frac{1}{k} \leq 1 - \frac{1}{k} = \sum_i \left(w_{ij} - \frac{1}{k}\right)^2, \tag{1}$$

where $W = [w_{ij}] \in P_{k0}$.

Hence

$$0 \le \sum_i \sum_j \left(u_{ij} - \frac{1}{k}\right)^2 \le \frac{n}{k}(k-1)$$

$$1 - 0 \ge 1 - \frac{k}{n(k-1)} \sum_i \sum_j \left(u_{ij} - \frac{1}{k}\right)^2 \ge 1 - \frac{k}{n(k-1)} \cdot \frac{n}{k}(k-1)$$

and

$$0 \le \varphi(U) \le 1. \tag{2}$$

From (1), (2): $\varphi(U) = 0 \Leftrightarrow$ for all j:

$$\sum_i \left(u_{ij} - \frac{1}{k}\right)^2 = 1 - \frac{1}{k} \Leftrightarrow U \in P_{k0},$$

which proves (P2) in Definition 2.2.

$\varphi(U) = 1 \Leftrightarrow$ for all j: $\sum_i \left(u_{ij} - \frac{1}{k}\right)^2 = 0 \Leftrightarrow$ for all i, j: $u_{ij} = \frac{1}{k} \Leftrightarrow U = \left[\frac{1}{k}\right]$, which proves (P3) in Definition 2.2.

(P4): If $W \prec U$, then

for $u_{ij} > \frac{1}{k}$:

$$w_{ij} \ge u_{ij} > \frac{1}{k} \quad \text{and} \quad \left(w_{ij} - \frac{1}{k}\right)^2 \ge \left(u_{ij} - \frac{1}{k}\right)^2 \tag{3}$$

for $u_{ij} < \frac{1}{k}$:

$$w_{ij} \le u_{ij} < \frac{1}{k} \quad \text{and} \quad \left(w_{ij} - \frac{1}{k}\right)^2 \ge \left(u_{ij} - \frac{1}{k}\right)^2. \tag{4}$$

From (3), (4):

$$\sum_i \sum_j \left(w_{ij} - \frac{1}{k}\right)^2 \ge \sum_i \sum_j \left(u_{ij} - \frac{1}{k}\right)^2, \tag{5}$$

hence $\varphi(W) = 1 - \frac{k}{n(k-1)} \sum_i \sum_j \left(w_{ij} - \frac{1}{k}\right)^2 \le 1 - \frac{k}{n(k-1)} \sum_i \sum_j \left(u_{ij} - \frac{1}{k}\right)^2 = \varphi(U)$, which proves (P4) in Definition 2.2.

Theorem 2.3. Consider a partition $U \in P_{fk0}$ and the partition coefficient $F(U, k)$ (see (1.6.)). Then

$$\varphi(U) = 1 - F(U, k) \tag{2.6}$$

is a measure of fuzziness of U.

Proof:

Bezdek has proved [2] that the partition coefficient $F(U, k)$ satisfies the following properties:

$$\frac{1}{k} \leq F(U, k) \leq 1 \tag{2.7a}$$

$$F(U, k) = \frac{1}{k} \Leftrightarrow U = \left[\frac{1}{k}\right] \tag{2.7b}$$

$$F(U, k) = 1 \Leftrightarrow U \in P_{k0}. \tag{2.7c}$$

It is clear $\varphi(U) = 1 - F(U, k)$ satisfies these conditions:

$$0 \leq \varphi(U) \leq 1 - \frac{1}{k} \tag{2.8a}$$

$$\varphi(U) = 1 - \frac{1}{k} \Leftrightarrow U = \left[\frac{1}{k}\right] \tag{2.8b}$$

$$\varphi(U) = 0 \Leftrightarrow U \in P_{k0}. \tag{2.8c}$$

(2.8b) is (P3) in Definition 2.2., (2.8c) is (P2) in Definition 2.2 φ obviously satisfies (P1).

Let us verify (P4):

Using (5) from the proof of Theorem 2.2. we have:

if $W \prec U$ then for all j: $\sum_i \left(w_{ij} - \frac{1}{k}\right)^2 = \sum_i w_{ij}^2 - \frac{1}{k} \geq \sum_i u_{ij}^2 - \frac{1}{k} = \sum_i \left(u_{ij} - \frac{1}{k}\right)^2$,

hence:

for all j:

$$\sum_i w_{ij}^2 \geq \sum_i u_{ij}^2. \tag{1}$$

From (1): $\varphi(W) = 1 - \frac{1}{n} \sum_i \sum_j w_{ij}^2 \leq 1 - \frac{1}{n} \sum_i \sum_j u_{ij}^2 = \varphi(U)$, which proves (P4).

Theorem 2.4.

Consider a function φ: $P_{fk0} \to R$, for all $U \in P_{fk0}$:

$$\varphi(U) = n - \sum_j \max_i u_{ij}. \tag{2.9}$$

The function φ is a measure of fuzziness of partitions from P_{fk0}.

Proof:

(P1) is evident.

(P2), (P3):

for all j:

$$\frac{1}{k} \leq \max_i u_{ij} \leq 1 \tag{1}$$

$$\frac{n}{k} \leq \sum_j \max_i u_{ij} \leq n,$$

hence

$$0 \leq \varphi(U) \leq n\left(1 - \frac{1}{k}\right). \tag{2}$$

From (1), (2): $\varphi(U) = 0 \Leftrightarrow$ for all j: $\max u_{ij} = 1 \Leftrightarrow U \in P_{k0}$, which proves (P2).
$\varphi(U) = n\left(1 - \frac{1}{k}\right) \Leftrightarrow$ for all j: $\max\limits_i u_{ij} = \frac{1}{k} \Leftrightarrow U = \left[\frac{1}{k}\right]$, which proves (P3).

(P4): If $W \prec U$, then for all j: $\max\limits_i w_{ij} \geq \max\limits_i u_{ij}$ so

$$\varphi(W) = n - \sum_j \max_i w_{ij} \leq n - \sum_j \max_i u_{ij} = \varphi(U),$$

which proves (P4).

Theorem 2.5. Classification entropy defined for all $U \in P_{fk0}$ by (1.7) satisfies properties (P1)—(P4) in Definition 2.2.

Proof:

(P1) holds obviously.

Let us consider function h: $\langle 0, 1\rangle \to R$ defined by $h(x) = -x \cdot \log_a x$, for $x \in (0, 1\rangle$, $a \in (1, \infty)$, $h(0) = 0$.

Then $h'(x) = -\log_a x - \dfrac{1}{\log a}$ and $h''(x) \leq 0$.

Hence $h(x + \delta) \leq h(x) + \delta \cdot h'(x)$ for all $\delta \in (0, 1)$.

Let us consider $u(x_j) \in U \in P_{fk0}$ such that $u_{ij} \leq \frac{1}{k}$ for $i \leq r$, $u_{ij} > \frac{1}{k}$ for $i > r$. If $V \prec U$, then $v_{ij} = u_{ij} + \delta_i$, where $\delta_i \leq 0$ for $i \leq r$ and $\delta_i \geq 0$ for $i > r$.

Obviously, $\sum\limits_{i=1}^{r} -\delta_i = \sum\limits_{i=r+1}^{k} \delta_i$.

$$\begin{aligned}
h(v(x_j)) &= \sum_i^k - v_{ij} \cdot \log_a v_{ij} = \sum_i^k (u_{ij} + \delta_i) \cdot \log_a (u_{ij} + \delta_i) \leq -\sum_i^k u_{ij} \cdot \log_a u_{ij} + \\
&+ \sum_i^k \delta_i\left(-\log_a u_{ij} - \frac{1}{\log a}\right) = h(u(x_j)) + \sum_{i=1}^{r} -\delta_i\left(\log_a u_{ij} + \frac{1}{\log a}\right) + \\
&+ \sum_{i=r+1}^{k} \delta_i\left(-\log_a u_{ij} - \frac{1}{\log a}\right) \leq h(u(x_j)) + \left(\log_a \frac{1}{k} + \frac{1}{\log a}\right) \cdot \sum_{i=1}^{r} -\delta_i + \\
&+ \sum_{i=r+1}^{k} \delta_i\left(-\log_a u_{ij} - \frac{1}{\log a}\right) = h(u(x_j)) + \sum_{i=r+1}^{k} \delta_i\left(\log_a \frac{1}{k} - \log_a u_{ij}\right) \leq h(u(x_j))
\end{aligned}$$

Hence $H(V) = \sum\limits_j h(v(x_j)) \leq \sum\limits_j h(u(x_j)) = H(U)$, which proves (P4) in Definition 2.2. It is easy to prove that function H satisfies also the properties (P2) and (P3) in Definition 2.2.

3 Construction of measures of fuzziness of fuzzy partitions

Throughout this and the next section we denote by J_1 the set $\{1, 2, \ldots, k\}$ and by J_2 the set $\{1, 2, \ldots, n\}$. Every partition $U \in P_{fk0}$ can be considered as a collection of

a) elements $u_{ij} \in \langle 0, 1 \rangle$ $(i, j) \in J_1 \times J_2$,

b) fuzzy sets $u_i \in L(X)$,

$$u_i = \{u_{i1}, u_{i2}, \ldots, u_{in}\} = \{u_i(x_1), u_i(x_2), \ldots, u_i(x_n)\}, \; i \in J_1,$$

c) k-dimensional vectors $u^T(x_j) = [u_1(x_j), u_2(x_j), \ldots, u_k(x_j)], j \in J_2$.

Taking into consideration these collections we suggest three different ways of constructing a measure of fuzziness of fuzzy partitions.

Theorem 3.1. Let $f: \langle 0, 1 \rangle \to R$ such that

(i) f is increasing on $\left\langle 0, \frac{1}{k} \right\rangle$ and decreasing on $\left\langle \frac{1}{k}, 1 \right\rangle$,

(ii) $f(0) = f(1)$.

Then the function φ: $P_{fk0} \to R$ defined for all $U \in P_{fk0}$ by

$$\varphi(U) = K \cdot \sum_i \sum_j f(u_{ij}) + A, \tag{3.1}$$

where $0 \neq K$, $A \in R$ are appropriate constants, satisfies properties (P1)—(P4) in Definition 3.2.

Proof:

Let us denote by h the value of $f(0) = f(1)$ and by m the value of $f\left(\frac{1}{k}\right)$. Obviously, $h < m$.

Let us define for all $U \in P_{fk0}$:

$$\varphi(U) = \frac{1}{n \cdot k} \sum_i \sum_j f(u_{ij}) - h. \tag{3.2}$$

We shall prove that (3.2) statisfies (P1)—(P4) in Def. 2.2.

(P1) holds obviously.

(P2), (P3): from (i), (ii):

$$h \leq f(u_{ij}) \leq \mathrm{m} \quad \text{for all } i, j \tag{1}$$

$$\frac{1}{n \cdot k} \sum_i \sum_j h - h \leq \frac{1}{n \cdot k} \sum_i \sum_j f(u_{ij}) - h \leq \frac{1}{n \cdot k} \sum_i \sum_j m - h$$

$$0 \leq \varphi(U) \leq m - h. \tag{2}$$

From (1), (2), (i):

$$\varphi(U) = 0 \Leftrightarrow \text{for all } i, j\colon f(u_{ij}) = h \Leftrightarrow u_{ij} \in \{0, 1\} \Leftrightarrow U \in P_{k0},$$

which proves (P2).

$$\varphi(U) = m - h \Leftrightarrow \text{ for all } i, j\text{: } f(u_{ij}) = m \Leftrightarrow u_{ij} = \frac{1}{k} \Leftrightarrow U = \left[\frac{1}{k}\right],$$

which proves (P3).

(P4): If $W \prec U$, then from (i):

for $u_{ij} \in \left\langle 0, \frac{1}{k}\right\rangle$:

$$w_{ij} \leq u_{ij} \Rightarrow f(w_{ij}) \leq f(u_{ij}), \tag{3}$$

for $u_{ij} \in \left\langle \frac{1}{k}, 1\right\rangle$:

$$w_{ij} \geq u_{ij} \Rightarrow f(w_{ij}) \leq f(u_{ij}). \tag{4}$$

From (3), (4):

$$\varphi(W) = \frac{1}{n \cdot k} \sum_i \sum_j f(w_{ij}) - h \leq \frac{1}{n \cdot k} \sum_i \sum_j f(u_{ij}) - h = \varphi(U),$$

which proves (P4).

We establish an example of function f satisfying conditions (i), (ii) in Theorem 3.1.

Example 3.1. Let f: $\langle 0, 1 \rangle \to R$ such that

$$f(t) = t \quad \text{for } t \in \left\langle 0, \frac{1}{k}\right)$$

$$f(t) = 1 - t \quad \text{for } t \in \left\langle \frac{1}{k}, 1\right\rangle.$$

Function f satisfies conditions (i), (ii) in Theorem 3.1. The proof is evident.

We suggest generalization of the relation sharpned by De Luca and Termini (see (2.1a), (2.1b)) as follows:

Definition 3.1. If $\alpha \in (0, 1)$ and $u, v \in L(X)$ such that

$$v(x) \leq u(x) \quad \text{for } u(x) < \alpha \tag{3.3a}$$

and

$$v(x) \geq u(x) \quad \text{for } u(x) > \alpha, \tag{3.3b}$$

we say that v is an α- sharpned version of u denoted by $v \underset{\alpha}{\prec} u$.

Remarks:

(1) for $\alpha = \frac{1}{2}$ we obtain the relation sharpned by De Luca and Termini

(2) for $U, W \in P_{fk0}$:

$$W \prec U \Leftrightarrow \text{ for all } i\text{: } w_i \underset{\frac{1}{k}}{\prec} u_i. \tag{3.4}$$

α — sharpned for $\alpha = 1/k$ will be useful in the next theorem.

Theorem 3.2. Denote by $\mathscr{P}(X)$ the set of ordinary (hard) subsets of X and denote by $u_{(1/k)}$ the fuzzy set from $L(X)$ for which $u_{(1/k)}(x) = \frac{1}{k}$ for all $x \in X$. Let τ: $L(X) \to R$ such that
(i) for w_1, $w_2 \in \mathscr{P}(X)$, $u \in L(X) - \mathscr{P}(X) \cap \{u_{(1/k)}\}$:

$$\tau(w_1) = \tau(w_2) < \tau(u) < \tau(u_{(1/k)});$$

(ii) if u, $w \in L(X)$ are such that $w \underset{\frac{1}{k}}{\prec} u$, then $\tau(w) \leq \tau(u)$.

The function φ: $P_{fk0} \to R$ defined for all $U \in P_{fk0}$ by

$$\varphi(U) = K \cdot \sum_i \tau(u_i) + A, \tag{3.5}$$

where $0 \neq K$, $A \in R$ are appropriate constants, satisfies (P1)—(P4) in Def. 2.2.

Proof:

Let us denote by h the value of $\tau(v)$, where $v \in \mathscr{P}(X)$, and by m the value of $\tau(u_{(1/k)})$. Obviously, $h < m$. Let us define for all $U \in P_{fk0}$:

$$\varphi(U) = \frac{1}{k} \sum_i \tau(u_i) - h. \tag{3.6}$$

We shall prove that (3.6) satisfies (P1)—(P4) in Def. 2.2.

(P1) holds obviously.

(P2), (P3):

From (i): for all i:

$$h \leq \tau(u_i) \leq m, \tag{1}$$

hence

$$0 \leq \varphi(u) \leq m - h. \tag{2}$$

From (1), (2), (i):

$$\varphi(U) = 0 \Leftrightarrow \text{for all } i\text{: } \tau(u_i) = h \Leftrightarrow u_i \in \mathscr{P}(X) \Leftrightarrow U \in P_{k0},$$

$$\varphi(U) = m - h \Leftrightarrow \text{for all } i\text{: } \tau(u_i) = m \Leftrightarrow u_i = u_{(1/k)} \Leftrightarrow U = \left[\frac{1}{k}\right]$$

(P4): $W \prec U \Leftrightarrow$ for all i: $w_i \underset{\frac{1}{k}}{\prec} u_i$. Then from (ii):

for all i:

$$\tau(w_i) \leq \tau(u_i),$$

hence

$$\varphi(W) = \frac{1}{k} \sum_i \tau(w_i) \leq \frac{1}{k} \sum_i \tau(u_i = \varphi(U).$$

Remarks:

(1) Function τ: $L(X) \to R$ defined for all $u \in L(X)$ by

$$\tau(u) = \sum_j f(u^T(x_j)), \tag{3.7}$$

where f satisfies conditions (i), (ii) in Theorem 3.1., satisfies properties (i), (ii) in Theorem 3.2.

(2) Function τ in Theorem 3.2. for $k = 2$ and $\tau(u) = 0$ for $u \in \mathscr{P}(X)$ is the measure of fuzziness of fuzzy sets by De Luca and Termini.

We establish an example of function τ satisfying conditions (i), (ii) in Theorem 3.2.

Example 3.2.

For $u \in L(X)$ denote by

$$I_1(u) = \left\{x \in X;\ 0 < u(x) < \frac{1}{k}\right\}$$

$$I_2(u) = \left\{x \in X;\ 1 > u(x) > \frac{1}{k}\right\}$$

$$I_3(u) \triangleq \left\{x \in X;\ u(x) = \frac{1}{k}\right\}.$$

Then $X - I_1(u) \cup I_2(u) \cup I_3(u) = \{x \in X;\ u(x) \in \{0, 1\}\}$.

Let τ: $L(X) \to R$ such that

$$\tau(u) = \min\{\min_{x \in I_1(u)} u(x),\ \min_{x \in I_2(u)} (1 - \mathrm{u(x)})\} \quad \text{if } I_1(u) \cup I_2(u) \neq \emptyset$$

$$= \left(1 - \frac{1}{k}\right) \cdot \operatorname{card} I_3(u) \quad \text{otherwise.}$$

Function τ satisfies properties (i), (ii) in Theorem 3.2.

Proof:

(i): if $u \in \mathscr{P}(X)$, then $I_1(u) \cup I_2(u) = \emptyset$, card $I_3(u) = 0$, hence

$$\tau(u) = 0. \tag{1}$$

If $u = u_{(1/k)}$, then $I_1(u) \cup I_2(u) = \emptyset$, card $I_3(u) = n$, hence

$$\tau(u_{(1/k)}) = \left(1 - \frac{1}{k}\right) \cdot n \tag{2}$$

If $u \in L(X) - \mathscr{P}(X) \cup \{u_{(1/k)}\}$, then

a) if $I_1(u) \cup I_2(u) = \emptyset$, then $I_3(u) \neq \emptyset$ and $0 <$ card $I_3(u) = p < n$

$$\tau(u) = \left(1 - \frac{1}{k}\right) \cdot p < \left(1 - \frac{1}{k}\right) \cdot n \tag{3}$$

b) if $I_1(u) \cup I_2(u) \neq \emptyset$, then

$$\tau(u) = \min\{\min_{x \in I_1(u)} u(x), \quad \min_{x \in I_2(u)} (1 - u(x))\} < 1 - \frac{1}{k}$$

$$0 < \tau(u) < \left(1 - \frac{1}{k}\right) \cdot n \tag{4}$$

From (1), (2), (3), (4) we obtain:
for $w_1, w_2 \in \mathscr{P}(X)$, $u \in L(X) - \mathscr{P}(X) \cup \{u_{(1/k)}\}$:

$$\tau(w_1) = \tau(w_2) < \tau(u) < \tau(u_{(1/k)})$$

which is (i) in Theorem 3.2.

(ii): if $w \underset{\frac{1}{k}}{\prec} u$, then

$$w(x) \leq u(x) \quad \text{for } u(x) < \frac{1}{k} \tag{5}$$

$$w(x) \geq u(x) \quad \text{for } u(x) > \frac{1}{k}. \tag{6}$$

It is clear that

$$\min_{x \in I_1(w)} w(x) \leq \min_{x \in I_1(u)} u(x) \tag{7}$$

and

$$\min_{x \in I_2(w)} (1 - w(x)) \leq \min_{x \in I_2(u)} (1 - u(x)). \tag{8}$$

If $I_1(u) \cup I_2(u) = \emptyset$, then $I_1(w) \cup I_2(w) \neq \emptyset$ and from (7), (8) we have:

$$\tau(w) \leq \tau(u). \tag{9}$$

If $I_1(u) \cup I_2(u) = \emptyset$, then

a) if $I_3(u) = \emptyset$, then $u \in \mathscr{P}(X)$ and from (5), (6): $w = u$, hence

$$\tau(w) = \tau(u); \tag{10}$$

b) if $I_3(u) \neq \emptyset$, card $I_3(u) > 0$, then

1. if $\emptyset \neq I_1(w) \cup I_2(w) \subset I_3(u)$ we obtain

$$\tau(w) = \min\{\min_{x \in I_1(w)} (w(x), \quad \min_{x \in I_2(w)} (1 - w(x))\} < 1 - \frac{1}{k} \leq \left(1 - \frac{1}{k}\right) \cdot \operatorname{card} I_3(u) = \tau(u); \tag{11}$$

2. if $I_1(w) \cup I_2(w) = \emptyset$, then $\emptyset \neq I_3(w) \subset I_3(u)$ we obtain

$$\operatorname{card} I_3(w) \leq \operatorname{card} I_3(u),$$

hence

$$\tau(w) = \left(1 - \frac{1}{k}\right) \cdot \operatorname{card} I_3(w) \leq \left(1 - \frac{1}{k}\right) \cdot \operatorname{card} I_3(u) = \tau(u); \tag{12}$$

3. if $I_1(w) \cup I_2(w) \cup I_3(w) = \emptyset$, then $w \in \mathscr{P}(X)$ and from (1):

$$\tau(w) = 0, \quad \text{hence} \quad \tau(w) \leq \tau(u). \tag{13}$$

From (9), (10), (11), (12), (13):
if $w \prec_{\frac{1}{k}} u$, then $\tau(w) \leq \tau(u)$, which is (ii) in Theorem 3.2.

Theorem 3.3. Denote by $\mathscr{V}$ the set of all k-dimensional vectors v, $v^T = [v_1, v_2, \ldots, v_k]$, where for all $i \in \{1, \ldots, k\}$: $v_i \in \langle 0, 1 \rangle$ and $\sum_i v_i = 1$. Denote by $\mathscr{V}_I$ the set $\{v \in \mathscr{V}$; such that $v_i \in \{0, 1\}$ for all $i\}$ and by $v_{(1/k)}$ the vector from $\mathscr{V}$ for which $v_{(1/k)i} = \frac{1}{k}$ for all i.

Let $\tau: \mathscr{V} \to R$ such that
(i) for t, $w \in \mathscr{V}_I$, $v \in \mathscr{V} - \mathscr{V}_I \cup \{v_{(1/k)}\}$:

$$\tau(t) = \tau(w) < \tau(v) < \tau(v_{(1/k)});$$

(ii) if w, $v \in \mathscr{V}$ are such that

$$w_r \leq v_r \quad \text{for } v_r < \frac{1}{k} \, r \in \{1, \ldots, k\}$$

and

$$w_s \geq v_s \quad \text{for } v_s > \frac{1}{k} \, s \in \{1, \ldots, k\},$$

then $\tau(w) \leq \tau(v)$.

Then the function φ: $P_{fk0} \to R$ defined for all $U \in P_{fk0}$ by

$$\varphi(U) = K \cdot \sum_j u^T(x_j) + A, \tag{3.8}$$

where $0 \neq K$, $A \in R$ are appropriate constants, satisfies (P1)—(P4) in Definition 2.2.

The proof is analogous to that in Theorem 3.2.

Remark: Function τ: $\mathscr{V} \to R$ defined for all $v \in \mathscr{V}$ by

$$\tau(v) = \sum_i f(v_i), \tag{3.9}$$

where f satisfies conditions in Theorem 3.1., satisfies properties (i), (ii) in Theorem 3.3.

Example 3.3. Let τ: $\mathscr{V} \to R$ such that for all $v \in \mathscr{V}$:

$$\tau(v) = 1 - \max_i v_i \qquad i \in \{1, \ldots, k\}.$$

Function τ satisfies properties (i), (ii) in Theorem 3.3.

Proof:

(i) if $v \in \mathscr{V}_I$: $\max_i v_i = 1$, hence $\tau(v) = 0$;

if $v = v_{(1/k)}$: $\max_i v_i = \frac{1}{k}$, hence $\tau(v_{(1/k)}) = 1 - \frac{1}{k}$;

if $v \in \mathscr{V} - \mathscr{V}_I \cup \{v_{(1/k)}\}$: $\frac{1}{k} < \max_i v_i < 1$, hence $0 < \tau(v) < 1 - \frac{1}{k}$, which proves (i) in Theorem 3.3.

(ii) For all $v \in \mathscr{V}$: $\max_i v_i \geq \frac{1}{k}$.

If w, $v \in \mathscr{V}$ are such that $w_r \leq v_r < \frac{1}{k}$ and $w_s \geq v_s > \frac{1}{k}$, $r, s \in \{1, \ldots, k\}$, then

$\max_i w_i \geq \max_i v_i$ and $\tau(w) \leq \tau(v)$, which proves (ii) in Theorem 3.3.

4 Measures of fuzziness of fuzzy partitions based on min (max) α-combination of fuzzy sets

In this section we propose a min (max) α-combination of fuzzy sets as follows:

Definition 4.1. Let u_r, $u_s \in L(X)$ and $\alpha \in (0, 1)$. Minimal α-combination of u_r, u_s is the fuzzy set $u_p \in L(X)$ defined by:
for all $x \in X$:

$$\begin{aligned} u_p(x) &= \min\{u_r(x), u_s(x)\} \quad \text{if } \min\{u_r(x), u_s(x)\} \leq \alpha \\ &= 0 \text{ otherwise.} \end{aligned} \tag{4.1}$$

We denote $u_p = u_{r\alpha} u_s$.

Definition 4.2. Let u_r, $u_s \in L(X)$ and $\alpha \in (0, 1)$. Maximal α-combination of u_r, u_s is the fuzzy set $u_q \in L(X)$ defined by:
for all $x \in X$:

$$\begin{aligned} u_q(x) &= \max\{u_r(x), u_s(x)\} \quad \text{if } \max\{u_r(x), u_s(x)\} \geq \alpha \\ &= 1 \text{ otherwise.} \end{aligned} \tag{4.2}$$

We denote $u_q = u_r^{\alpha} u_s$.

Theorem 4.1. Consider $U \in P_{k0}$. For all $(r, s) \in J_1 \times J_1$:

$$u_p(x) = (u_{r\,1/k} u_s)(x) = 0 \quad \text{for all } x \in X, \tag{4.3a}$$

$$u_q(x) = (u_r^{1/k} u_s)(x) = 1 \quad \text{for all } x \in X. \tag{4.3b}$$

The proof is evident.

Theorem 4.2. Consider $U = \left[\frac{1}{k}\right] \in P_{fk0}$. Then for all $(r, s) \in J_1 \times J_1$:

$$u_p(x) = (u_{r1/k}u_s)(x) = (u_r^{1/k}u_s)(x) = u_q(x) = \frac{1}{k} \quad \text{for all } x \in X. \tag{4.4}$$

The proof is evident.

Theorem 4.3. Consider U, $W \in P_{fk0}$ such that $U \prec W$. Then for all $(r, s) \in J_1 \times J_1$: for all $x \in X$:

$$0 \leq u_p(x) = (u_{r\,1/k}u_s)(x) \leq (w_{r\,1/k}w_s)(x) = w_p(x) \leq \frac{1}{k}, \tag{4.5a}$$

$$\frac{1}{k} \leq w_q(x) = (w_r^{1/k}w_s)(x) \leq (u_r^{1/k}u_s)(x) = u_q(x) \leq 1. \tag{4.5b}$$

Proof:

We must analyze three cases for a fixed $x \in X$:

I. $w_r(x) \geq \frac{1}{k}$ and $w_s(x) \geq \frac{1}{k}$

II. $w_r(x) \leq \frac{1}{k}$ and $w_s(x) \leq \frac{1}{k}$

III. $w_r(x) \geq \frac{1}{k}$ and $w_s(x) \leq \frac{1}{k}$.

Then

I. $u_r(x) \geq w_r(x) \geq \frac{1}{k}$ and $u_s(x) \geq w_s(x) \geq \frac{1}{k}$

$$1 \geq \min\{u_r(x), u_s(x)\} \geq \min\{w_r(x), w_s(x)\} \geq \frac{1}{k}$$

if

$$\min\{w_r(x), w_s(x)\} > \frac{1}{k}: (w_{r\,1/k}w_s)(x) = (u_{r\,1/k}u_s)(x) = 0$$

if

$$\min\{w_r(x), w_s(x)\} = \frac{1}{k}: (u_{r\,1/k}u_s)(x) \leq (w_{r\,1/k}w_s)(x) = \frac{1}{k}.$$

Hence

$$0 \leq u_p(x) \leq w_p(x) \leq \frac{1}{k}. \tag{1a}$$

$$1 \geq \max\{u_r(x), u_s(x)\} \geq \max\{w_r(x), w_s(x)\} \geq \frac{1}{k},$$

hence

$$\frac{1}{k} \leq w_q(x) \leq u_q(x) \leq 1. \tag{1b}$$

II. $u_r(x) \leq w_r(x) \leq \frac{1}{k}$ and $u_s(x) \leq w_s(x) \leq \frac{1}{k}$,

hence

$$0 \leq u_p(x) \leq w_p(x) \leq \frac{1}{k}. \tag{2a}$$

$$0 \leq \max\{u_r(x), u_s(x)\} \leq \max\{w_r(x), w_s(x)\} \leq \frac{1}{k}$$

if

$$\max\{w_r(x), w_s(x)\} < \frac{1}{k}: (w_r^{1/k} w_s)(x) = (u_r^{1/k} u_s)(x) = 1.$$

if

$$\max\{w_r(x), w_s(x)\} = \frac{1}{k}: (w_r^{1/k} w_s)(x) \leqq (u_r^{1/k} u_s)(x) \leq 1.$$

Hence

$$\frac{1}{k} \leq w_q(x) \leq u_q(x) \leq 1. \tag{2b}$$

III. $u_r(x) \geq w_r(x) \geq \frac{1}{k}$ and $u_s(x) \leq w_s(x) \leq \frac{1}{k}$

$$0 \leq \min\{u_r(x), u_s(x)\} = u_s(x) \leq \min\{w_r(x), w_s(x)\} = w_s(x) \leq \frac{1}{k},$$

hence

$$0 \leq u_p(x) \leq w_p(x) \leq \frac{1}{k}. \tag{3a}$$

$$\frac{1}{k} \leq \max\{w_r(x), w_s(x)\} = w_r(x) \leq \max\{u_r(x), u_s(x)\} = u_r(x) \leq 1,$$

hence

$$\frac{1}{k} \leq w_q(x) \leq u_q(x) \leq 1. \tag{3b}$$

(1a), (2a), (3a) prove (4.5a).
(1b), (2b), (3b) prove (4.5b).

Theorem 4.4. Let $f: \left\langle 0, \frac{1}{k} \right\rangle \to R$ such that

(i) f is increasing on $\left\langle 0, \frac{1}{k} \right\rangle$

(ii) $f(0) = 0$.

Then function φ: $P_{fk0} \to R$ defined for all $U \in P_{fk0}$ by

$$\varphi(U) = \sum_{r=1}^{k-1} \sum_{s=r+1}^{k} \sum_{x \in X} f((u_{r\,1/k} u_s)(x)) \tag{4.6}$$

satisfies properties (P1)—(P4) in Def. 2.2.

Proof:
(P1) holds obviously.

(P2), (P3): for all (r, s) $\in J_1 \times J_1$: $0 \leq (u_{r\,1/k} u_s)(x) \leq \frac{1}{k}$ for all $x \in X$.

f is increasing on $\left\langle 0, \frac{1}{k} \right\rangle$ and $f(0) = 0$, hence

$$f(0) \leq f((u_{r\,1/k} u_s)(x)) \leq f\left(\frac{1}{k}\right). \tag{1}$$

Let us denote by m the value of $f\left(\frac{1}{k}\right)$. Then

$$0 \leq \varphi(U) \leq \frac{1}{2} \cdot k(k-1) \cdot n \cdot m \tag{2}$$

From (1), (2), (i) we obtain:
$\varphi(U) = 0 \Leftrightarrow$ for all $(r, s) \in J_1 \times J_1$:

$$f((u_{r\,1/k} u_s)(x) = 0 \text{ for all } x \in X \Leftrightarrow (u_{r\,1/k} u_s)(x) = 0 \Leftrightarrow U \in P_{k0}.$$

$\varphi(U) = \frac{1}{2} \cdot k(k-1) n \cdot m \Leftrightarrow$ for all $(r, s) \in J_1 \times J_1$:

$$f((u_{r\,1/k} u_s)(x)) = m \text{ for all } x \in X \Leftrightarrow (u_{r\,1/k} u_s)(x) = \frac{1}{k} \Leftrightarrow U = \left[\frac{1}{k}\right].$$

(P4): If $U \prec W$, then $0 \leq (u_{r\,1/k} u_s)(x) \leq (w_{r\,1/k} w_s)(x)$ for all (r, s) $\in J_1 \times J_1$ and for all $x \in X$. From (i) we obtain:
$f((u_{r\,1/k} u_s)(x) \leq f((w_{r\,1/k} w_s)(x))$ for all (r, s) $\in J_1 \times J_1$ and for all $x \in X$. Hence

$$\varphi(U) \leq \varphi(W).$$

Theorem 4.5. Let $g: \left\langle \frac{1}{k}, 1 \right\rangle \to R$ such that

(i) g is decreasing on $\left\langle \frac{1}{k}, 1 \right\rangle$,

(ii) $g(1) = 0$.

Then function φ: $P_{fk0} \to R$ defined for all $U \in P_{fk0}$ by

$$\varphi(U) = \sum_{r=1}^{k-1} \sum_{s=r+1}^{k} \sum_{x \in X} g((u_r^{1/k} u_s)(x)) \tag{4.7}$$

satisfies properties (P1)—(P4) in Def. 2.2.
The proof is analogous to that in Theorem 4.4.

5. Measures of fuzziness and measures of dissimilarity of fuzzy partitions

Definition 5.1. Function D: $P_{fk0} \times P_{fk0} \to R$ is a measure of dissimilarity between two partitions from P_{fk0} if it satisfies the following properties: for U, $W \in P_{fk0}$:

(i) $$D(U, U) = 0, \tag{5.1a}$$

(ii) $$D(U, W) = D(W, U). \tag{5.1b}$$

We can show that a measure of dissimilarity satisfying some further conditions can determine the measure of fuzziness of fuzzy partitions from P_{fk0}.

Definition 5.2. Consider a measure of dissimilarilarity D defined on $P_{fk0} \times P_{fk0}$. D is called entropic measure of dissimilarity if the following properties are true:

(i) if V, $W \in P_{fk0}$ are such that $V \prec W$ and $U = \left[\frac{1}{k}\right]$, then

$$D(V, U) \geq D(W, U); \tag{5.2a}$$

(ii) if V_1, $V_2 \in P_{fk0}$, $U = \left[\frac{1}{k}\right]$ and $W \in P_{fk0} - P_{k0}$, $W \neq U$, then

$$DV_1, U) = D(V_2, U) > D(W, U) > 0. \tag{5.2b}$$

For illustration we establish two following examples:

Example 5.1. Let D: $P_{fk0} \times P_{fk0} \to R$ such that for U, $W \in P_{fk0}$:

$$D(U, W) = \sum_j \left| \sum_i u_{ij}^2 - \sum_i w_{ij}^2 \right|. \tag{5.3}$$

D is an entropic measure of dissimilarity.

Proof:

(i) Obviously, $D(U, U) = 0$,

(ii) obviously, $D(U, W) = D(W, U)$.

Hence D is a measure of dissimilarity.

Consider $U = \left[\frac{1}{k}\right]$, $V \prec W$.

$$D(U, V) = \sum_j \left| \sum_i v_{ij}^2 - \frac{1}{k} \right| \quad \text{and} \quad D(U, W) = \sum_j \left| \sum_i w_{ij}^2 - \frac{1}{k} \right|.$$

It was shown in the proof of Theorem 2.2 that

$$0 \leq \sum_j \left(\sum_i v_{ij}^2 - \frac{1}{k} \right) \leq \frac{n}{k}(k - 1) \tag{1}$$

$$\sum_j\left(\sum_i v_{ij}^2 - \frac{1}{k}\right) = 0 \Leftrightarrow V = \left[\frac{1}{k}\right] \tag{2}$$

$$\sum_j\left(\sum_i v_{ij}^2 - \frac{1}{k}\right) = \frac{n}{k}(k-1) \Leftrightarrow V \in P_{k0}. \tag{3}$$

For $V \prec W$:

$$\sum_j\left(\sum_i v_{ij}^2 - \frac{1}{k}\right) \geq \sum_j\left(\sum_i w_{ij}^2 - \frac{1}{k}\right). \tag{4}$$

From (4): if $V \prec W$, then $D(V, U) \geq D(W, U)$, which proves condition (i) in Definition 5.2.

From (1), (2), (3): if V_1, $V_2 \in P_{k0}$, $U = \left[\frac{1}{k}\right]$ and $W \in P_{fk0} - P_{k0}$, $W \neq U$, then $D(V_1, U) = D(V_2, U) > D(W, U) > 0$, which proves condition (ii) in Definition 5,2.

So D is an entropic measure of dissimilarity.

Example 5.2. Let D: $P_{fk0} \times P_{fk0} \to R$ such that for U, $W \in P_{fk0}$:

$$D(U, W) = \sum_j \frac{|\max_i u_{ij} - \max_i w_{ij}|}{\sum_i u_{ij}^2 \cdot \sum_i w_{ij}^2}. \tag{5.4}$$

D is a measure of dissimilarity but not an entropic measure of dissimilarity.

Proof:

(i) $D(U, U) = 0$ is evident,

(ii) $D(U, W) = D(W, U)$ is evident,

so D is a measure of dissimilarity.

Let us consider V, $W \in P_{f40}$ as follows:

$$V = \begin{pmatrix} 0{,}25 & 1 & 0 & 0 & 0 \\ 0{,}40 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0{,}35 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad W = \begin{pmatrix} 0{,}20 & 1 & 0 & 0 & 0 \\ 0{,}40 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0{,}40 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Evidently, $W \prec V$.

For $U = \left[\frac{1}{4}\right]$ we have:

$$D(V, U) = \sum_j \frac{|\max_i v_{ij} - 0.25|}{0.25 \cdot \sum_i v_{ij}^2}$$

$$D(V, U) = \frac{0{,}40 - 0{,}25}{0{,}25 \cdot 0{,}345} + 4 \cdot \frac{1 - 0{,}25}{1 \cdot 0{,}25} = 13{,}739$$

$$D(W, U) = \frac{0{,}40 - 0{,}25}{0{,}25 \cdot 0{,}36} + 4 \cdot \frac{1 - 0{,}25}{1 \cdot 0{,}25} = 13{,}667.$$

$D(W, U) < D(V, U)$, hence D does not satisfy condition (i) in Def. 5.1., so D is not an entropic measure of dissimilarity. Now we can establish the connection between entropic measures of dissimilarity and measures of fuzziness of fuzzy partitions from P_{fk0}.

Theorem 5.1. Let D be an entropic measure of dissimilarity defined on $P_{fk0} \times P_{fk0}$, $U = \left[\frac{1}{k}\right]$ and $W \in P_{k0}$. Function H: $P_{fk0} \to R$ defined for all $V \in P_{fk0}$ by

$$H(V) = D(W, U) - D(V, U) \tag{5.5}$$

satisfies conditions (P1)—(P4) in Definition 2.2.

Proof:

(P1) $H(V^{(p)}) = H(V)$ because $(U^{(p)}) = U$.

(P2) $H(V) = D(W, U) - D(V, U) = 0 \Leftrightarrow D(W, U) \Leftrightarrow V \in P_{k0}$.
From (ii) in Def. 5.2.: $D(W, U) = D(V, U) \Leftrightarrow V \in P_{k0}$.

(P3) $H(V) = D(W, U) - D(V, U) \leq D(W, U)$
$H(V) = D(W, U) \Leftrightarrow D(V, U) = 0$.
From (ii) in Def. 5.2: $D(V, U) = 0 \Leftrightarrow V = U = \left[\frac{1}{k}\right]$.

(P4) If $V_1 \prec V_2$ from (i) in Def. 5.2.: $D(V_1, U) \geq D(V_2, U)$, hence

$$H(V_1) = D(W, U) - D(V_1, U) \leq D(W, U) - D(V_2, U) = H(V_2).$$

On the other hand, we can prove also the following theorem:

Theorem 5.2. Let H be a measure of fuzziness defined on P_{fk0}.
Function D: $P_{fk0} \times P_{fk0} \to R$ defined for all $U, W \in P_{fk0}$ by

$$D(U, W) = |H(U) - H(W)| \tag{5.6}$$

is an entropic measure of dissimilarity.

Proof:

(i) $D(U, U) = |H(U) - H(U)| = 0$,

(ii) $D(U, W) = |H(U) - H(W)| = |H(W) - H(U)| = D(W, U)$.
Hence D is a measure of dissimilarity.

(iii) if $W \prec V$, then from (P4): $H(W) \leq H(V)$.
For $U = \left[\frac{1}{k}\right]$ we have:

$$D(W, U) = |H(W) - H(U)| \geq |H(V) - H(U)| = D(V, U),$$

which proves condition (i) in Def. 5.2.

(iv) From (P2): $H(V) = 0 \Leftrightarrow V \in P_{k0}$, from (P3): $H(V) = \max \Leftrightarrow V = \left[\frac{1}{k}\right]$.

If $V_1, V_2 \in P_{k0}$, $U = \left[\frac{1}{k}\right]$ and $U \neq W \in P_{fk0} - P_{k0}$, then

$$D(V_1, U) = |H(V_1) - H(U)| = |H(V_2) - H(U)| = D(V_2, U) =$$
$$= H(U) > |H(U) - H(W)| = D(W, U) > 0,$$

which proves condition (ii) in Def. 5.2. Hence D is an entropic measure of dissimilarity.

Conclusion

We have built an axiomatic generalization of measures of fuzziness of fuzzy partitions. We think that these measures can be useful as cluster validity functionals in fuzzy clustering algorithms.

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Author's adress:
Slávka Bodjanová
Katedra štatistiky VŠE
Odbojárov 10
832 20 Bratislava

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SÚHRN

MIERY NEURČITOSTI FUZZY ROZKLADOV

S. Bodjanová, Bratislava

Výsledkom fuzzy zhlukovacích algoritmov sú fuzzy rozklady konečných množín objektov. Cieľom tohto článku je podať matematickú formalizáciu mier neurčitosti fuzzy rozkladov a navrhnúť niektoré všeobecnejšie postupy pre konštrukciu týchto mier.

РЕЗЮМЕ

МЕРЫ НЕЧЁТКОСТИ НЕЧЁТКИХ РАЗБИЕНИЙ

С. Бодйанова, Братислава

Резуљтатом нечётких алгорифмов кластерного анализа являются нечёткие разбиения конечных множеств объектов. В статье приводится математическая формализация мер нечёткости этих разбиений и некоторые методы конструкции этих мер.