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ON THE BOUNDARY VALUE PROBLEM WITH MATRIX PARAMETER

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Consider the real boundary value problem

$$Y''(x) + \Lambda Y(x) = 0 \quad (0 \leqslant x \leqslant 1), \tag{1}$$

$$Y(0) = Y(1) = 0, \tag{2}$$

where $Y = \operatorname{col}(y_1, y_2) \in C^2(\langle 0, 1\rangle) \times C^2(\langle 0, 1\rangle)$ and parameter Λ is a real 2×2 matrix $\Lambda = (\lambda_{ij})$ $i, j = 1, 2$.

In this paper the necessary and sufficient condition for the existence of a nontrivial solution of the problem (1), (2) is given.

Theorem. The boundary value problem (1), (2) has a nontrivial solution in $C^2(\langle 0, 1\rangle) \times C^2(\langle 0, 1\rangle)$ iff

$$\det\Lambda = k^2\pi^2(\operatorname{tr}\Lambda - k^2\pi^2), \tag{3}$$

where $k \neq 0$ is an integer, i.e. iff at least one eigenvalue of Λ is $\lambda = k^2\pi^2$. This solution is of the form

$$Y_k(x, \Lambda) = C_k\begin{pmatrix} \lambda_{12} \\ k^2\pi^2 - \lambda_{11}\end{pmatrix}\sin k\pi x, \tag{4}$$

$C_k \in R$ being a constant. If both eigenvalues of Λ are $\lambda_1 = k^2\pi^2$, $\lambda_2 = l^2\lambda^2$, then the solution of (1), (2) is

$$Y_{k,l}(x, \Lambda) = C_k\begin{pmatrix} \lambda_{12}k^2\pi^2 - \lambda_{11}\end{pmatrix}\sin k\pi x + C_l\begin{pmatrix} \lambda_{12} \\ l^2\pi^2 - \lambda_{11}\end{pmatrix}\sin l\pi x, \tag{5}$$

C_k, $C_l \in R$ being arbitrary constants.

Remark. (4) is equivalent to

$$Y_k(x, \Lambda) = C_k\begin{pmatrix} k^2\pi^2 - \lambda_{22} \\ \lambda_{21}\end{pmatrix}\sin k\pi x$$

and (5) is equivalent to

$$Y_{k,l}(x,\Lambda) = C_k\left(\frac{k^2\pi^2 - \lambda_{22}}{\lambda_{21}}\right)\sin k\pi x + C_l\left(\frac{l^2\pi^2 - \lambda_{22}}{\lambda_{21}}\right)\sin l\pi x \tag{5'}$$

as can be obtained from (3). The form (4′) of the solution is more general when $\lambda_{12} = 0$ and inversely, (4) is more general when $\lambda_{21} = 0$. The same holds for (5′) and (5).

Proof of the theorem. Equation (1) is equivalent to the system

$$\begin{aligned} y_1'' &= -\lambda_{11}y_1 - \lambda_{12}y_2 \\ y_2'' &= -\lambda_{21}y_1 - \lambda_{22}y_2 \end{aligned} \tag{1'}$$

the solutions of which can be chosen in the form

$$y_1(x) = c_1\,e^{rx},\ y_2(x) = c_2\,e^{rx}. \tag{S}$$

Here, r is a root of the characteristic equation

$$r^4 + (\operatorname{tr}\Lambda)r^2 + \det\Lambda = 0. \tag{6}$$

Considering (1′) and (S) we obtaine

$$\begin{aligned} c_1(\lambda_{11} + r^2) + c_2\lambda_{12} &= 0 \\ c_1\lambda_{21} + c_2(\lambda_{22} + r^2) &= 0. \end{aligned}$$

Solution $(c_1,\ c_2)$ of this system is nontrivial iff r is a root of (6). Hence $(c_1, c_2) = (-\lambda_{12},\ \lambda_{11} + r^2) = (\lambda_{22} + r^2,\ -\lambda_{21})$.

The roots of (6) are

$$\begin{aligned} r_{1,2} &= \pm\frac{1}{2}\left[\left(-\operatorname{tr}\Lambda + \sqrt{(\operatorname{tr}\Lambda)^2 - 4\det\Lambda}\right)\right]^{1/2}, \\ r_{3,4} &= \pm\frac{1}{2}\left[\left(-\operatorname{tr}\Lambda - \sqrt{(\operatorname{tr}\Lambda)^2 - 4\det\Lambda}\right)\right]^{1/2}. \end{aligned} \tag{7}$$

It is obvious that the numbers $-r_1^2,\ -r_3^2$ are the eigenvalues of the matrix Λ.

Let us now consider the two following cases:

(A) off-diagonal elements of Λ are both nonzero, i.e. $\lambda_{21} \neq 0$ as well as $\lambda_{12} \neq 0$;

(B) at least one of off-diagonal elements of Λ is equal to zero, i.e. $\lambda_{12} = 0$ or $\lambda_{21} = 0$.

(A)

As to the multiplicity of roots of (6), we shall distinct four cases:

(A_1)

$$r_1 = r_2 = r_3 = r_4.$$

Clearly, $r_i = 0$ for $i = 1, 2, 3, 4$ and there is no nontrivial solution of (1), (2).

(A_2)

There exist four distinct roots of (6).

In this case, the general solution of equation (1) is

$$Y(x) = D_1\begin{pmatrix} -\lambda_{12} \\ \lambda_{11} + r_1^2 \end{pmatrix}\exp(r_1 x) + D_2\begin{pmatrix} -\lambda_{12} \\ \lambda_{11} + r_1^2 \end{pmatrix}\exp(-r_1 x) +$$

$$+ D_3\begin{pmatrix} -\lambda_{12} \\ \lambda_{11} + r_3^2 \end{pmatrix}\exp(r_3 x) + D_4\begin{pmatrix} -\lambda_{12} \\ \lambda_{11} + r_3^2 \end{pmatrix}\exp(-r_3 x),$$

D_i, $i = 1, 2, 3, 4$ being constants. Taking (2) into account we obtain $D_1 + D_2 = 0$, $D_3 + D_4 = 0$, $D_1(\exp r_1 - \exp(-r_1)) = 0$, $D_3(\exp r_3 - \exp(-r_3)) = 0$. It follows from these relations that $Y(x)$ is a nontrivial solution of (1), (2) iff $r_1 = ik\pi$ or $r_3 = il\pi$, $k, l = \pm 1, \pm 2, \ldots$. A real form of this nontrivial solution is

$$Y_{k,l}(x) = C_k\begin{pmatrix} -\lambda_{12} \\ \lambda_{11} - k^2\pi^2 \end{pmatrix}\sin k\pi x + C_l\begin{pmatrix} -\lambda_{12} \\ \lambda_{11} - l^2\pi^2 \end{pmatrix}\sin l\pi x.$$

(A_3)

Equation (6) has two double roots.

We have $r_1 = r_3$. As in (A_2) we obtain $r_1 = ik\pi$, $k = \pm 1, \pm 2, \ldots$, and a general form of real nontrivial solution of (1), (2) is

$$Y_k(x) = C_k\begin{pmatrix} -\lambda_{12} \\ \lambda_{11} - k^2\pi^2 \end{pmatrix}\sin k\pi x.$$

(A_4)

There exists one double root of (6).

In this case, no other possibility than $r_1 = 0$ and $r_3 = il\pi$ or $r_1 = ik\pi$ and $r_3 = 0$, $k, l = \pm 1, \pm 2, \ldots$, can occur. A real nontrivial solution of (1), (2) is the same as in (A_3).

(B)

Let $\lambda_{12} = 0$. (The case $\lambda_{21} = 0$ is analogous.)

The system (1′) will have the form

$$\begin{aligned} y_1'' &= -\lambda_{11}y_1 \\ y_2'' &= -\lambda_{21}y_1 - \lambda_{22}y_2. \end{aligned} \qquad (1'')$$

It is known that there exists a nontrivial solution of the problem

$$y_1'' = -\lambda_{11}y_1$$

$$y_1(0) = y_1(1) = 0$$

iff $\lambda_{11} = k^2\pi^2$, $k = \pm 1, \pm 2, \ldots$, i.e. $r_1^2 = -\lambda_{11} = -k^2\pi^2$.

(B_1)

Let $\lambda_{11} \neq k^2\pi^2$, $k = \pm 1, \pm 2, \ldots$.

Clearly, $y_1(x) \equiv 0$ on $\langle 0, 1\rangle$ and the problem (1), (2) will have nontrivial solution just when $-r_3^2 = \lambda_{22} = l^2\pi^2$, $l = \pm 1, \pm 2, \dots$.

This solution will have the form

$$Y_l(x) = C_l \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin l\pi x .$$

(B_2)

Let $\lambda_{11} = k^2\pi^2$, $k = \pm 1, \pm 2, \dots$

We have $y_1(x) = \sin k\pi x$ and $y_2(x)$ is the solution of the problem

$$y_2''(x) = -\lambda_{22} y_2(x) - \lambda_{21} \sin k\pi x$$

$$y_2(0) = y_2(1) = 0 .$$

Using the method of variation of the constants we obtain

$$y_2(x) = \frac{\lambda_{21}}{\lambda_{11} - \lambda_{22}} \sin k\pi x = \frac{\lambda_{21}}{k^2\pi^2 - \lambda_{22}} \sin k\pi x, \ \lambda_{22} \neq \lambda_{11} .$$

Hence

$$Y_k(x) = C_k \begin{pmatrix} k^2\pi^2 - \lambda_{22} \\ \lambda_{21} \end{pmatrix} \sin k\pi x .$$

Specially, for $\lambda_{22} = \lambda_{11} = -r_1^2 = -r_3^2 = k^2\pi^2$ the solution will be

$$Y_k(x) = C_k \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin k\pi x .$$

We have proved that (1), (2) has a nontrivial solution iff at least one eigenvalue of Λ is equal to $k^2\pi^2$, $k = \pm 1, \pm 2, \dots$, and that this solution has the form (4), (4′), (5) or (5′). From (7), putting $r_1 = \mathrm{i}k\pi$ ($r_3 = \mathrm{i}l\pi$) we shall obtain (3). Inversely, from (3) and (7) we shall have $r_1 = \mathrm{i}k\pi$ or $r_3 = \mathrm{i}l\pi$. The proof of the theorem is complete.

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SÚHRN

O OKRAJOVEJ ÚLOHE S MATICOVÝM PARAMETROM

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V práci je vyslovená a dokázaná nutná a postačujúca podmienka existencie netriviálneho riešenia $Y(x) \in C^2(\langle 0, 1\rangle) \times C^2(\langle 0, 1\rangle)$ dvojrozmernej okrajovej úlohy s maticovým parametrom

$$Y''(x) + \Lambda Y(x) = 0, \qquad x \in \langle 0, 1\rangle$$
$$Y(0) = Y(1) = 0.$$

Uvádzame tiež všeobecný tvar príslušného riešenia.

РЕЗЮМЕ

О КРАЕВОЙ ЗАДАЧЕ С МАТРИЧНЫМ ПАРАМЕТРОМ

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В работе приведено и доказано необходимое и достаточное условие для существования нетривиального решения $Y(x) \in C^2(\langle 0, 1\rangle) \times C^2(\langle 0, 1\rangle)$ двухмерной краевой задачи с матричным параметром

$$Y''(x) + \Lambda Y(x) = 0, \qquad x \in \langle 0, 1\rangle$$
$$Y(0) = Y(1) = 0.$$

Приведена также общая форма соответствующего решения.