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BOUNDED OSCILLATIONS OF HIGHER-ORDER FUNCTIONAL DIFFERENTIAL INEQUALITIES INDUCED BY FORCING FUNCTIONS

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One of important problems in the oscillation theory of nonlinear differential equations and inequalities is the problem of oscillations which are caused by forcing terms and which do not appear in the corresponding unforced equations and inequalities. For some results in this direction we refer the reader in particular to the papers of Graef et al. [2—4], Jones and Rankin [5], Kartsatos [6], Kartsatos and Manougian [7, 8], Kusano [9], Kusano and Naito [10], Lovelady [11], Singh and Kusano [13] and Vencková [14]. Most of these studies have considered the pertubations which represent the unbounded oscillatory functions. However, an obvious example of the equation

$$x''(t) - x(t) = -2\sin t$$

which admits bounded oscillatory solution $x(t) = \sin t$ and for which the corresponding homogeneous equation is nonoscillatory, indicates that the oscillation of at least all bounded solutions can be generated also by bounded forcings.

The purpose of this note is to show that under certain conditions such a situation really occurs. Our results extend the oscillation theorem of Lovelady [11] concerning the linear ordinary differential equation

$$x^{(n)}(t) + (-1)^{n+1}q(t)x(t) = h(t)$$

to the more general nonlinear functional differential inequalities

$$x(t)\{L_nx(t) + f(t, x(g_1(t)), \dots, x(g_m(t))) - h(t)\} \leqq 0 \quad \text{for } n \text{ odd,} \tag{1}$$

and

$$x(t)\{L_nx(t) - f(t, x(g_1(t)), \dots, x(g_m(t))) - h(t)\} \geqq 0 \quad \text{for } n \text{ even,} \tag{2}$$

where $n \geqq 2$ and L_n is the general disconjugate differential operator defined by $L_0 x(t) = x(t)$ and

$$L_k x(t) = a_k(t)(L_{k-1}x(t))', \; k = 1, 2, \ldots, n, \; a_n(t) = 1 .$$

We shall assume that $a_i(t)$, $i = 1, 2, \ldots, n-1$, are positive and continuous functions on $[t_0, \infty)$ and the operator L_n is in the so-called first canonical form in the sense that

$$\int_{t_0}^{\infty} a_i^{-1}(t)\,\mathrm{d}t = \infty \tag{3}$$

for $i = 1, 2, \ldots, n-1$. In what follows we use $\mathscr{D}(L_n)$ to denote the set of all real-valued functions $y(t)$ defined on $[t_y, \infty)$, $t_y \geqq t_0$, and such that $L_i y(t)$, $i = 0, 1, \ldots, n$, exist and are continuous on $[t_y, \infty)$.

We restrict our considerations only to those solutions $x(t)$ of (1) (or (2)) which exist on some ray $[t_x, \infty)$, $t_x \geqq t_0$, and are nontrivial on any neighbourhood of infinity. Such a solution is called oscillatory if it has arbitrarily large zeros in $[t_x, \infty)$, and it is called nonoscillatory otherwise.

We shall use the following notation:

$$y^+(t) = \max\{y(t), 0\}, \; y^-(t) = \max\{-y(t), 0\},$$

$$\langle y, g\rangle^+(t) = (y^+(g_1(t)), \ldots, y^+(g_m(t))),$$

$$\langle y, g\rangle^-(t) = (y^-(g_1(t)), \ldots, y^-(g_m(t))),$$

$$\omega_1(t, T) = \int_T^t \frac{1}{a_1(s)}\,\mathrm{d}s,$$

$$\omega_k(t, T) = \int_T^t \frac{1}{a_k(s)}\,\omega_{k-1}(s, T)\,\mathrm{d}s, \quad k = 2, 3, \ldots, n-1, \quad t \geqq T \geqq t_0 .$$

For the inequalities (1) and (2) the following conditions will be assumed without further mention:

(i) $h \in C([t_0, \infty), R)$ and there exists a bounded oscillatory function $p \in \mathscr{D}(L_n)$ such that $L_n(t) = h(t)$ on $[t_0, \infty)$ and

$$a = \liminf_{t\to\infty} p(t) < 0 < \limsup_{t\to\infty} p(t) = b;$$

(ii) $f \in C([t_0, \infty) \times R^m, R)$ has the following properties:

$$f(t, x_1, \ldots, x_m) > 0 \text{ for } (x_1, \ldots, x_m) \in R_+^m, \; t \geqq t_0,$$

$$f(t, x_1, \ldots, x_m) < 0 \text{ for } (x_1, \ldots, x_m) \in R_-^m, \; t \geqq t_0,$$

where $R_+ = (0, \infty)$ and $R_- = (-\infty, 0)$, and, moreover, for any $y \in \mathscr{D}(L_n)$

and any $t_1 \geqq t_0$ such that

$$y(t) \geqq (p(t) - a)^+, \text{ resp. } -y(t) \geqq (p(t) - b)^-,$$

on $[t_1, \infty)$, there exists $t_2 \geqq t_1$ such that

$$f(t, \langle y, g\rangle(t)) \geqq f(t, \langle p - a, g\rangle^+(t)),$$

resp.

$$f(t, \langle -y, g\rangle(t)) \geqq f(t, \langle p - b, g\rangle^-(t))),$$

on $[t_2, \infty)$;

(iii) $g_i \in C([t_0, \infty), R)$, $\lim_{t\to\infty} g_i(t) = \infty$, $i = 1, 2, \dots, m$.

Theorem 1. Suppose that the conditions (i)—(iii) are satisfied. If, moreover,

$$\int_T^\infty \omega_{n-1}(\tau, T) f(\tau, \langle p - a, g\rangle^+(\tau))\,d\tau = \infty \tag{4}$$

and

$$\int_T^\infty \omega_{n-1}(\tau, T) f(\tau, \langle p - b, g\rangle^-(\tau))\,d\tau = \infty \tag{5}$$

for every $T \geqq t_0$, then all bounded solutions of (1) (or (2)) are oscillatory.

Proof. In order to avoid repetition, we consider only the inequality (1).

Let $x(t)$ be a bounded nonoscillatory solution of (1). Assume that this solution is positive for $t \geqq t_1 \geqq t_0$. By (iii), we choose $t_2 \geqq t_1$ such that $x(g_i(t)) > 0$ for $t \geqq t_2$ and $i = 1, 2, \dots, m$. Let $u(t) = x(t) - p(t)$. Then, in view of (1) and (ii), we obtain that

$$L_n u(t) \leqq -f(t, x(g_1(t)), \dots, x(g_m(t) < 0$$

for $t \geqq t_2$. Consequently, by taking into account the fact that the function $L_0 u(t)$ is bounded and eventually positive, from the well-known generalized Kigurad-ze's Lemma (see for example [12]) it follows that there is a $t_3 \geqq t_2$ such that

$$(-1)^k L_k u(t) > 0 \tag{6}$$

for $t \geqq t_3$ and $k = 0, 1, \dots, n$. In particular, $L_1 u(t) < 0$ for $t \geqq t_3$, and so $u(t)$ is decreasing on $[t_3, \infty)$.

We claim that $\lim_{t\to\infty} u(t) \geqq -a$. Indeed, in the opposite case we have

$$\liminf_{t\to\infty} x(t) - a \leqq \limsup_{t\to\infty} (x(t) - p(t)) = \lim_{t\to\infty} u(t) < -a,$$

which contradicts the positivity of $x(t)$. Therefore, $u(t) \geqq -a$ for $t \geqq t_3$ and, consequently,

$$x(t) \geqq p(t) - a, \quad t \geqq t_3.$$

Since $x(t) > 0$ for $t \geqq t_3$, this implies

$$x(t) \geqq (p(t) - a)^+. \tag{7}$$

On the other hand, it is easy to verify that

$$u(t) \leqq u(t_3) + \sum_{j=1}^{n-1} (-1)^{j-1} L_j u(t) \omega_j(t, t_3) -$$

$$- \int_{t_3}^{t} \omega_{n-1}(\tau, t_3) f(\tau, \langle x, g \rangle(\tau))\, \mathrm{d}\tau$$

and so, by (6), (7) and (ii),

$$u(t) \leqq u(t_3) - \int_{t_3}^{t} \omega_{n-1}(\tau, t_3) f(\tau, \langle p - a, g \rangle^+(\tau))\, \mathrm{d}\tau$$

for $t \geqq t_3$.

Finally, letting $t \to \infty$ and taking (4) into account, we get the contradiction to the positivity of $u(t)$.

A similar argument holds for $x(t) < 0$, and this completes the proof.

Example 1. For an illustration consider the equation

$$\begin{aligned} &(e^{-t} x'(t))' - e^{-\frac{\pi}{2}} x\left(t - \frac{\pi}{2}\right) - e^{-\pi} x(t - \pi) = \\ &= e^{-2t}(3 \cos t - \sin t) - e^{-t}(\cos t + \sin t), \quad t \geqq \pi. \end{aligned} \tag{8}$$

All the conditions of Theorem 1 are satisfied with $p(t) = (1 - e^{-t}) \sin t$, $a = -1$ and $b = 1$. Consequently, all bounded solutions of (8) are oscillatory. One such solution is $x(t) = -e^{-t} \sin t$.

Example 2. All assumptions of Theorem 1 hold also in the case of the nonlinear equation

$$x''(t) - \frac{4}{3} x^3(t - \pi) = -\frac{1}{3} \sin 3t, \tag{9}$$

where we have $p(t) = \sin 3t/27$. Thus every bounded solution oscillates. For example, $x(t) = \sin t$ is one such solution.

Following the results of Grace and Lalli in [1], the oscillation criterion given in Theorem 1 can be easily extended to the set of solutions $x(t)$ of (1) (or (2)) with the property $x(t)/\alpha_1(t) \to 0$ as $t \to \infty$, where

$$\alpha_1(t) = \int_{t_0}^{t} \frac{1}{a_1(s)}\, \mathrm{d}s.$$

This extension is based on the following lemma which can be proved analogously as Theorem 1 in [1].

Lemma. Let the conditions (i)—(iii) be satisfied. If $x(t)$ is a nonoscillatory solution of (1) (or (2)) such that

$$x(t)/\alpha_1(t) \to 0 \quad \text{as} \quad t \to \infty,$$

then there exists a $t_1 \geqq t_0$ such that the function $u(t) = x(t) - p(t)$ satisfies

$$(-1)^k u(t) L_k u(t) > 0 \tag{10}$$

for $t \geqq t_1$ and $k = 1, 2, \ldots, n$.

Theorem 2. Suppose that the conditions of Theorem 1 hold. Then every solution $x(t)$ of (1) (or (2)) such that

$$x(t)/\alpha_1(t) \to 0 \quad \text{as} \quad t \to \infty$$

is oscillatory.

The proof of this theorem follows along the lines of that of Theorem 1, and so we omit it.

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SÚHRN

OHRANIČENÉ OSCILÁCIE FUNKCIONÁLNYCH DIFERENCIÁLNYCH NEROVNÍC VYŠŠÍCH RÁDOV INDUKOVANÉ NÚTIACIMI FUNKCIAMI

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V práci je dokázané kritérium oscilatoričnosti všetkých ohraničených riešení nelineárnych funkcionálnych diferenciálnych nerovníc

$$x(t)\{L_n x(t) + f(t, x(g_1(t)), \ldots, x(g_m(t))) - h(t)\} \leqq 0, \; n \text{ nepárne},$$

$$x(t)\{L_n x(t) - f(t, x(g_1(t)), \ldots, x(g_m(t))) - h(t)\} \geqq 0, \; n \text{ párne},$$

kde $n \geqq 2$, L_n je zovšeobecnený diskonjugovaný diferenciálny operátor a $h(t)$ reprezentuje ohraničenú oscilatorickú nútiacu silu.

РЕЗЮМЕ

ОГРАНИЧЕННЫЕ КОЛЕБАНИЯ ДИФФЕРЕНЦИАЛЬНО-ФУНКЦИОНАЛЬНЫХ НЕРАВЕНСТВ ВЫСШИХ ПОРЯДКОВ ИНДУЦИРОВАННЫЕ ВЫНУЖДАЮЩИМИ ФУНКЦИЯМИ

Ярослав Ярош, Братислава

В работе доказан признак колеблемости всех ограниченных решений нелинейных дифференциально-функциональных неравенств

$$x(t)\{L_n x(t) + f(t, x(g_1(t)), \ldots, x(g_m(t))) - h(t)\} \leqq 0, \; n \text{ нечетное},$$

$$x(t)\{L_n x(t) - f(t, x(g_1(t)), \ldots, xg_m(t))) - h(t)\} \geqq 0, \; n \text{ четное},$$

где $n \geqq 2$, L_n обобщенный осцилляционный дифференциальный оператор и $h(t)$ представляет ограниченную колеблющуюся вынуждающую силу.