UNIVERSITAS COMENIANA
ACTA MATHEMATICA UNIVERSITATIS COMENIANAE
LII—LIII—1987

A REMARK ON THE THEORY OF SETS OF DISTANCES

ĽUBICA HOLÁ, Bratislava

Let 2^X (or 2^Y) be the set of all subsets of a metric space X (or Y respectively). Let $f: X \to Y$ be a function. Define $F: 2^X \to 2^Y$ by $F(A) = f(A)$ for any $A \in 2^X$.

In this paper we give some conditions under which the function F preserves a topological convergence of sets and the convergence of sets in the Hausdorff metric.

We apply the obtained results for the study of the sets of distances.

Let (Z, d) be a metric space and 2^Z be the set of all subsets of Z. If $E \subset Z$ and $\varepsilon > 0$, let $S_\varepsilon(E]$ denote the union of all open ε-balls whose centres run over E, and let $S_\varepsilon[x)$ denote the open ε-ball about a point x.

If E and F are nonempty subsets of Z, and for some $\varepsilon > 0$ both $S_\varepsilon[F] \supset E$, and $S_\varepsilon[E] \supset F$, then the Hausdorff distance h_d between them is given by: $h_d(E, F) = \inf\{\varepsilon: S_\varepsilon[E] \supset F, S_\varepsilon[F] \supset E\}$. Otherwise we put $h_d(E, F) = \infty$. For $E = \emptyset$ we put $h_d(E, E) = 0$ and $h_d(E, F) = \infty$ for $F \neq \emptyset$.

Then $(2^Z, h_d)$ is a pseudometric space [3].

If F_n is a sequence in 2^Z, the upper and lower closed limits of the sequence are defined as follows [5]: $\mathrm{Ls}\, F_n$ ($\mathrm{Li}\, F_n$)is the set of all x's in Z such that each neighbourhood of x meets infinitely (all but finitely respectively) many sets F_n. It is easy to check that both $\mathrm{Li}\, F_n$ and $\mathrm{Ls}\, F_n$ are closed sets and $\mathrm{Li}\, F_n \subset \mathrm{Ls}\, F_n$. If $F = \mathrm{Li}\, F_n = \mathrm{Ls}\, F_n$, we will write $F = \mathrm{Lim}\, F_n$, and we say that $\{F_n\}$ is topologically convergent to F.

If (Z, d) is an arbitrary metric space, then the convergence of a sequence of sets to a closed set with respect to Hausdorff distance h_d ensures a topological convergence, and if X is compact, then the topological convergence implies the convergence in the Hausdorff distance [1].

Let (X, d_x) and (Y, d_y) be metric spaces. Let $F(X, Y)$ denote the set of all functions from X to Y. We can identify the members of $F(X, Y)$ with their graphs in $X \times Y$ and define the distance between functions to be the Hausdorff distance between their graphs as induced by some metric compatible with the product uniformity. For definiteness and computational simplicity, we take ϱ

defined by:

$$\varrho[(x_1, y_1), (x_2, y_2)] = \max\{d_x(x_1, x_2), d_y(y_1, y_2)\}.$$

For $f, g \in F(X, Y)$ we put $L(f, g) = h_\varrho(G(f), G(g))$, where $G(f)$ (or $G(g)$) is the graph of the function f (or g). Then $(F(X, Y), L)$ is a pseudometric space.

Let R denote the set of real numbers and u denote the usual metric in R.

In what follows let (X, d_x), (Y, d_y) be metric spaces. Let $f\colon X \to Y$ be a function. Define the function $F\colon 2^X \to 2^Y$ by $F(A) = f(A)$ for any $A \in 2^X$.

1 Preservation of convergence of sets in the Hausdorff metric

Proposition 1.1. Let $f\colon X \to Y$ be a uniformly continuous function. Then the function $F\colon (2^X, h_{d_x}) \to, 2^Y, h_{d_y})$ is uniformly continuous.

Proof. Let $\varepsilon > 0$. The uniform continuity of f implies that there exists $\delta > 0$ such that for any $x, y \in X$ with $d_x(y, x) < \delta$ we have $d_y(f(x), f(y)) < \varepsilon$.

Let $A, B \in 2^X$ be such that the following inequality holds: $h_{d_x}(A, B) < \delta$. We show that then $h_{d_y}(F(A), F(B)) < \varepsilon$.

Let $y \in F(A)$. There is $x \in A$ for which $f(x) = y$. Since $A \subset S_\delta(B]$ there exists $b \in B$ such that $d_x(b, x) < \delta$, thus $d_y(f(b), y) < \varepsilon$, i.e. $y \in S_\varepsilon[f(B)] = S_\varepsilon[FB)]$. Hence $F(A) \subset S_\varepsilon[F(B)]$.

The proof of the inclusion $F(B) \subset S_\varepsilon[F(A)]$ is similar.

Proposition 1.2. Let $f\colon X \to Y$ be a continuous function. Then the function $F\colon (2^X, h_{d_x}) \to (2^Y, h_{d_y})$ is continuous at any compact subset of X.

Proof. Let $A, A_n \in 2^X$ $(n = 1, 2, \ldots)$, A be a compact set and $\lim_n h_{d_x}(A, A_n) = 0$. We show that $\lim_n h_{d_y}(F(A), F(A_n)) = 0$.

Let $\varepsilon > 0$. Since $S_\varepsilon[F(A)]$ is an open set in Y, the continuity of f implies that $f^{-1}(S_\varepsilon[f(A)]$ is an open set in X containing A. A is compact, i.e. there exists $\delta > 0$ such that $S_\delta[A] \subset f^{-1}(S_\varepsilon[f(A)])$. There exists $N_1 \in N$ such that for any $n \geqq N_1$ $A_n \subset S_\delta[A]$ and thus $f(A_n) \subset S_\varepsilon[f(A)]$ for any $n \geqq N_1$.

We show that there exists $N_2 \in N$ such that $f(A) \subset S_\varepsilon[f(A_n)]$ for any $n \geqq N_2$.

Let $x \in A$. There exists δ_x such that for any $z \in S_{\delta_x}[x]$ we have $d_y(f(z), f(x)) < \varepsilon/2$. The compactness of A implies that there exist points $x_1, x_2, \ldots x_n$ in A such that $A \subset \bigcup_{i=1}^{n} S_{\delta_{x_i}/2}[x_i]$.

Put $\eta = \min\{\delta_{x_i}\ i = 1, 2, \ldots, n\}$.

There exists $N_2 \in N$ such that $A \subset S_{\eta/2}[A_n]$ for any $n \geqq N_2$. Then $f(A) \subset \subset S_\varepsilon[f(A_n)]$ for any $n \geqq N_2$. (Let $n \geqq N_2$. Let $a \in A$. Let i be such that $a \in S_{\delta_{x_i}/2}[x_i]$. Let $y \in A_n$ be such that $a \in S_{\eta/2}[y)$. Then $y \in S_{\delta_{x_i}}[x_i]$ and thus $d_y(f(y), f(a)) < \varepsilon$).

Put $N_0 = \max\{N_1, N_2\}$. For any $n \geqslant N_0$ we have $h_{d_y}(f(A_n), f(A)) < \varepsilon$, i.e. $h_{d_y}(F(A_n), F(A)) < \varepsilon$ for any $n \geqq N_0$.

Proposition 1.3. Let (X, d_x) be a metric space. Let A be a closed totally bounded set in X, which is not compact. There exists a continuous function f: $X \to R$ such that the function F: $(2^X, h_{d_x}) \to (2^R, h_u)$, defined by $F(B) = f(B)$ for any $B \in 2^X$, is not continuous at A.

Proof. There exists a cauchy sequence $\{x_n\}$, $x_n \in A$ $(n = 1, 2, \ldots)$ which has no cluster point in X and $x_i \neq x_j$ for $i \neq j$.

Choose ε_n for any $n \in N$ such that $0 < \varepsilon_n < 1/n$ and the family $S_{\varepsilon_n}[x_n]$ is pairwise disjoint. Define a function f: $X \to R$ by

$$f(x) = \begin{cases} n(1 - (1/\varepsilon_n)\, d_x(x_n, x)) & \text{if } x \in S_{\varepsilon_n}[x_n] \\ 0 & \text{for other } x. \end{cases}$$

Since the sequence $\{x_n\}$ has no cluster point in X, the function f is continuous.

Define a sequence $\{A_n\}$ in 2^X as follows: $A_n = A \setminus \left(\bigcup_{i=n}^{\infty} S_{\varepsilon_i}[x_i]\right)$ for $n = 1, 2, \ldots$.

Then $\lim_n h_{d_x}(A_n, A) = 0$. Let $\varepsilon > 0$. It is sufficient to prove that $A \subset S_\varepsilon[A_n]$ for n sufficiently large.

Let j be such that $2/j < \varepsilon$. There exists $N_1 \geqq j$ such that $d_x(x_n, x_m) < 1/j$ for any $n, m \geqq N_1$. Then $\bigcup_{i=n}^{\infty} S_{\varepsilon_i}[x_i] \subset S_\varepsilon[x_{N_1}]$ for any $n \geqq (N_1 + 1)$, i.e. $A \subset S_\varepsilon[A_n]$ for any $n \geqq (N_1 + 1)$.

It is easy to check that $h_u(f(A_n), f(A)) = \infty$ for any n.

Proposition 1.4. Let (X, d_x) be a metric space. Let A be a closed connected set in X, which is not compact. There exists a continuous function f: $X \to R$ such that the function F: $(2^X, h_{d_x}) \to (2^R, h_u)$, defined by $F(B) = f(B)$ for any $B \in 2^X$, is not continuous at A.

Proof. There exists a sequence $\{x_n\}$, $x_n \in A$ $(n = 1, 2, \ldots)$ which has no cluster point in X and $x_i \neq x_j$ for $i \neq j$.

Choose ε_n for any $n \in N$ such that $0 < \varepsilon_n < 1/n$ and the family $S_{\varepsilon_n}[x_n]$ is pairwise disjoint. Let f: $X \to R$ and $\{A_n\}$ be as above. Then $\lim_n h_{d_x}(A_n, A) = 0$ but $h_u(F(A_n), F(A)) = \infty$ for any $n \in N$. (The connectedness of A implies that for any $n \in N$ there exists $y_n \in A$ such that $d_x(y_n, x_n) = \varepsilon_n$. Thus $h_{d_x}(A, A_n) < 2/n$ for any $n \in N$.)

Proposition 1.5. Let (X, d_x), (Y, d_y) be metric spaces. Let f: $X \to Y$ be a function. Then f is continuous if and only if the function F: $(2^X, h_{d_x}) \to (2^Y, h_{d_y})$, defined by $F[A] = f(A)$ for any $A \in 2^X$, is continuous at any compact set.

Proof. The proof is evident.

In the following part of the paper we will study behaviour of the function H: $(F(X, Y), L) \times (2^X, h_{d_x}) \to (2^Y, h_{d_y})$ defined by: $H(f, C) = f(C)$.

Let f: $X \to Y$ be a continuous function and C be a compact subset of X. We show that H is continuous at (f, C). First we show the following lemma.

Lemma 1.1. Let X, Y be metric spaces. Let f_n: $X \to Y$ $(n = 1, 2, \dots)$ be functions and f: $X \to Y$ be a continuous function. If the sequence $\{f_n\}$ converges to f in the pseudometric L, then $\{f_n\}$ uniformly converges to f on any compact subset of X.

Proof. Let K be a compact set in X and $\varepsilon > 0$. Let $x \in K$. The continuity of f at x implies that there exists $r_x > 0$ such that for any $z \in S_{r_x}[x]$ we have $d_y(f(x), f(z)) < \varepsilon/4$. There exists set $\{x_1, x_2, \dots, x_n\}$ such that $K \subset \bigcup_{i=1}^{n} S_{r_{x_i}/2}[x_i]$. Put $\delta = \min\{\varepsilon/2, r_{x_i}/2\ i = 1, 2, \dots, n\}$. Let $N_0 \in N$ be such that the following inequality holds: $L(f_n(f) < \delta$. Then for any $n \geqq N_0$ and for any $x \in K$ we have $d_y(f_n(x), f(x)) < \varepsilon$. (Let $x \in K$, $n \geqslant N_0$. Since $(l(f_n, f) < \delta$, there exists y such that $\varrho[(x, f_n(x)), (y, f(y))] < \delta$. Let i be such that $x \in S_{r_{x_i}/2}[x_i]$. Then $d_y(f_n(x), f(x)) \leqslant d_y(f_n(x), f(y)) + d_y(f(y), f(x)) < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

Theorem 1.1. Let (X, d_x), (Y, d_y) be metric spaces. Then the function H: $(F(X, Y), L) \times (2^X, h_{d_x}) \to (2^Y, h_{d_y})$, defined by $H(g, A) = g(A)$ for any $A \in 2^X$, is continuous at any point (f, C) where f is a continuous function and C is a compact set.

Proof. Suppose that there exist a continuous function f: $X \to Y$ and a compact set $C \subset X$ such that H is not continuous at (f, C). Then there are $\varepsilon > 0$ and sequences $\{f_n\}$, $\{C_n\}$ such that $\lim_n L(f_n, f) = 0$, $\lim_n h_{d_x}(C_n, C) = 0$ and $h_{d_y}(f_n(C_n), f(C)) \geqq \varepsilon$ for any n [1).

There is $N_0 \in N$ such that for any $n \geqq N_0$ the following inclusion holds: $f_n(C_n) \subset S_\varepsilon[f(C)]$.

By Proposition 1.2. the function F, defined by $F(A) = f(A)$ for any $A \in 2^X$, is continuous at C. There is $\delta > 0$ such that $F(B) \subset S_{\varepsilon/2}(F(C)]$ for any $B \in 2^X$ for which $h_{d_x}(B, C) < \delta$. Put $\eta = \min\{\varepsilon, \delta\}$. There is $N_0 \in N$ such that for any $n \geqslant N_0$ we have $L(f_n, f) < \eta/2$ and $h_{d_x}(C_n, C) < \eta/2$.Let $n \geqslant N_0$. Then $f_n(C_n) \subset S_{\eta/2}[f(S_{\eta/2}[C_n])] \subset S_{n/2}[f(S_\eta[C])] \subset S_{\eta/2}[f(S_\delta[C])] \subset S_{\eta/2}[S_{\varepsilon/2}[f(C)]] \subset S_\varepsilon[f(C)]$.

Thus by (1) we have $f(C) \not\subset S_\varepsilon[f_n(C_n)]$ for any $n \geqq N_0$, i.e. there is a sequence $\{y_n\}$, $y_n \in f(C)$ $(n = 1, 2, \dots)$ such that $y_n \notin S_\varepsilon[f_n(C_n)]$ for any $n \geqq N_0$.

The compactness of $f(C)$ implies that there is a point $y \in f(C)$ which is a cluster point of the sequence $\{y_n\}$. Let $\{y_{k_n}\}$ be a subsequence of $\{y_n\}$ which is convergent to the point y.

Let $N_1 \in N$ be such that for any $n \geqslant N_1$ we have $y \notin S_{\varepsilon/2}[f_{k_n} C_{k_n})]$ (2).

Choose $x \in C$ which $f(x) = y$. Since $\lim_n h_{d_x}(C_{k_n}, C) = 0$ there is a sequence $\{x_{k_n}\}$, $x_{k_n} \in C_{k_n}$ $(n = 1, 2, \ldots)$ which is convergent to x. Put $K = \{x_{k_1}, \ldots, x_{k_n}, \ldots\} \cup \cup \{x\}$. By Lemma 1.1. the sequence $\{f_{k_n}\}$ uniformly converges to f on the set K. Since f is continous at x there is $\delta_1 > 0$ such that the following inclusion holds: $f(S_{\delta_1}[x]) \subset S_{\varepsilon/4}[f(x)]$. The convergence of the sequence $\{x_{k_n}\}$ to the point x and the uniform convergence of the sequence $\{f_{k_n}\}$ to f on the set K implies that there is $N_2 \in N$ such that for any $n \geqslant N_2$ we have $d_x(x_{k_n}, x) < \delta_1$ and $d_y(f_{k_n}(u), f(u)) < \varepsilon/4$ for any $u \in K$. Let $n \geqslant \max\{N_1, N_2\}$. Then $d_y(f_{k_n}(x_{k_n}), f(x)) \leq d_y(f_{k_n}(x_{k_n}), f(x_{k_n})) + d_y(f(x_{k_n}), f(x)) < \varepsilon/4 + \varepsilon/4 = \varepsilon/2$. Thus $y = f(x)$ is a point of the set $S_{\varepsilon/2}[f_{k_n} C_{k_n})]$, which contradicts (2).

Let $C(X, Y)$ be the set of all continuous functions from X to Y and $\mathscr{Z}(X)$ (or $(\mathscr{Z}(Y))$ be the set of all compact subsets of X (or Y) respectively).

Corollary 1.1. Let (X, d_x), (Y, d_y) be metric spaces. Then the function $H: (C(X, Y), L) \times (\mathscr{Z}(X), h_{d_x}) \to (\mathscr{Z}(Y), h_{d_y})$, defined by $H(g, A) = g(A)$ for any $A \in \mathscr{Z}(X)$, is continuous.

For a compact metric space X Corollary 1.1 can be obtained from [2] 3.12.26 and from Lemma 1.1.

2 Preservation of topological convergence of sets

Proposition 2.1. Let X, Y be metric spaces. Let f: $X \to Y$ be a continuous function. Let $A_n \in 2^X$ $(n = 1, 2, \ldots)$. Then $F(\mathrm{Li}\, A_n) \subset \mathrm{Li}\, F(A_n)$, where: F: $2^X \to 2^Y$ is the function defined by $F(B) = f(B)$ for any $B \in 2^X$.

Proof. Let $y \in F(\mathrm{Li}\, A_n)$. There is $x \in Li\, A_n$ such that $y = f(x)$. There is a sequence $\{x_n\}$, $x_n \in A_n$ $(n = 1, 2, \ldots)$ which is convergent to x.

The continuity of f at x implies that $\{f(x_n)\}$ converges to $f(x)$, i.e. $y = = f(x) \in \mathrm{Li}\, f(A_n) = \mathrm{Li}\ F(A_n)$.

Proposition 2.2. Let X, Y be metric spaces. Let f: $X \to Y$ be a function with a closed graph. Let $A_n \in 2^X$ $(n = 1, 2, \ldots)$. If X is compact, then $\mathrm{Ls}\, F(A_n) \subset F(\mathrm{Ls}\, A_n)$, where F: $2^X \to 2^Y$ is the function defined by $F(B) = f(B)$ for any $B \in 2^X$.

Proof. Let $y \in \mathrm{Ls}\, F(A_n)$. There is an increasing sequence of positive integers $\{n_k\}$ and a sequence $\{y_{n_k}\}$, $y_{n_k} \in F(A_{n_k})$ such that $\{y_{n_k}\}$ converges to y. Let $\{x_{n_k}\}$ be a sequence of points of X such that $f(x_{n_k}) = y_{n_k}$ for any K. Let x be a cluster point of the sequence $\{x_{n_k}\}$. Then x is an element of $\mathrm{Ls}\, A_n$. The point (x, y) is a cluster point of the sequence $\{(x_{n_k}, f(x_{n_k}))\}$, i.e. $(x, y) \in \overline{G(f)}$. Since $\overline{G(f)} = G(f)$, $y = f(x)$, and thus $y \in f(\mathrm{Ls}\, A_n) = F(\mathrm{Ls}\, A_n)$.

Proposition 2.3. If X is a noncompact metric space, then there are a continuous function f: $X \to R$ and a sequence $\{A_n\}$, $A_n \in 2^X$ $(n = 1, 2, \dots)$ such that $\mathrm{Ls} f(A_n) \not\subset f(\mathrm{Ls}\, A_n)$.

Proof. There is a sequence $\{x_n\}$ of distinct points of X which has no cluster point in X. Choose ε_n for any $n \in N$ such that $0 < \varepsilon_n < 1/n$ and the family $S_{\varepsilon_n}[x_n]$ is pairwise disjoint and define a function f: $X \to R$ by: $f(x) = 1 - (1/\varepsilon_{2k}) d_x(x_{2k}, x)$ for $x \in S_{\varepsilon_{2k}}[x_{2k}]$ $k = 1, 2, \dots$ and $f(x) = 0$ for other x.

Define a sequence $\{A_n\}$ in 2^X as follows: $A_n = \{x_1, x_n\}$ for $n = 1, 2, \dots$. Then $\mathrm{Ls}\, A_n = \mathrm{Lim}\, A_n = \{x_1\}$ and $\mathrm{Ls} f(A_n) = \{0, 1\}$. Thus $\mathrm{Ls} f(A_n) \not\subset f(\mathrm{Ls}\, A_n)$.

Let A_n $(n = 1, 2, \dots)$ be subsets of X, and f: $X \to Y$ be a continuous function. Suppose that there exists $\mathrm{Lim}\, A_n$. The proof of Proposition 2.3. shows that Lim-$f(A_n)$ need not exist for a noncompact metric space X.

Proposition 2.4. Let X be a compact metric space, Y be a metric space. Let A, $A_n \in 2^X$ $(n = 1, 2, \dots)$ and $A = \mathrm{Lim}\, A_n$. Let f: $X \to Y$ be a continous function. Then $F(A) = \mathrm{Lim}\, F(A_n)$, where the function F: $2^X \to 2^Y$ is defined by $F(B) = f(B)$ for any $B \in 2^X$.

Proof. Since X is compact, by Theorem in [7] we have $\lim\limits_n h_{d_x}(A, A_n) = 0$. The compactness of A and Proposition 1.2. imply that $\lim\limits_n h_{d_y}(F(A), F(A_n)) = 0$, i.e. $F(A) = \mathrm{Lim}\, F(A_n)$.

For a sequence of connected subsets of X, Proposition 2.4. remains valid also for a locally compact metric space X.

In the following assertions we suppose that topological limits of sequences exist and that they are nonempty sets.

Proposition 2.5. Let (X, d_x) be a metric space. Let F_n $(n = 1, 2, \dots)$ be connected subset of X and let $\mathrm{Lim}\, F_n$ exist. Then for any open set U for which $U \supset \mathrm{Lim}\, F_n$ and $\bar{U}$ is compact, there exists $N_U \in N$ such that $F_n \subset U$ for any $n \geqslant N_U$.

Proof. The compactness of $\mathrm{Lim}\, F_n$ implies that there exists $\varepsilon > 0$ such that $S_{2\varepsilon}[\mathrm{Lim}\, F_n] \subset U$. Suppose that for any $n \in N$ there exists $k_n \geqslant n$ such that $F_{k_n} \cap (X \setminus U) \neq \emptyset$. Put $F = \mathrm{Lim}\, F_n$.

There are two possibilities:

(a) $\{x \in X\colon d_x(x, F) = \varepsilon\} = \emptyset$

(b) $\{x \in X\colon d_x(x, F) = \varepsilon\} \neq \emptyset$.

(a) If $\{x\colon d_x(x, F) = \varepsilon\} = \emptyset$, then $S_\varepsilon[F] = \{x\colon d_x(x, F) < \varepsilon\}$ is an open and closed subset of X.

Since $\mathrm{Lim}\, F_n \neq \emptyset$, there exists $z \in \mathrm{Lim}\, F_n$. There exists $N_1 \in N$ such that $F_n \cap S_\varepsilon[z] \neq \emptyset$ for any $n \leq N_1$. By assumption there exists $k_{N_1} \geqslant N_1$ such that $F_{k_{N_1}} \cap (X \setminus U) \neq \emptyset$,

Then $F_{k_{N_1}} = (F_{k_{N_1}} \cap S_\varepsilon[F]) \cup (F_{k_{N_1}} \cap (X \setminus S_\varepsilon[F]))$ and that is contradictory to the

connectedness of $F_{k_{N_1}}$, sine $F_{k_{N_1}} \cap S_\varepsilon[F]$ and $F_{k_{N_1}} \cap (X \setminus S_\varepsilon[F])$ are both open in $F_{k_{N_1}}$.
(b) Suppose $\{x: d_x(x, F) = \varepsilon\} \neq \emptyset$.

Choose $z \in \operatorname{Lim} F_n$. There exists $N_2 \in N$ such that for any $n \geqslant N_2$ $F_n \cap S_\varepsilon[z] \neq \emptyset$. By assumption there exists $k_{N_2} \geqq N_2$ such that

$$F_{k_{N_2}} \cap (X \setminus U) \neq \emptyset, \text{ i.e. } F_{k_{N_2}} \cap (X \setminus \{x: d_x(x, F) \leqslant \varepsilon\}) \neq \emptyset.$$

The connectedness of $F_{k_{N_2}}$ implies that there exists $y \in F_{k_{N_2}}$ for which $d_x(y, F) = \varepsilon$. Let $\{k_n\}$ be an increasing sequence of positive integers such that

$$F_{k_n} \cap \{y: d_x(y, F) = \varepsilon\} \neq \emptyset.$$

Since the set $\{y: d_x(y, F) = \varepsilon\}$ is compact, there is a point $y_0 \in \operatorname{Ls} F_n \cap \{y: d_x(y, F) = \varepsilon\}$ and that is a contradiction.

Corollary 2.1. Let (X, d_x) be a locally compact metric space. Let F_n $(n = 1, 2, \ldots)$ be connected subsets of X and let $\operatorname{Lim} F_n$ exist. If $\operatorname{Lim} F_n$ is a compact set, then $\lim_n h_{d_x}(F_n, \operatorname{Lim} F_n) = 0$.

Corollary 2.2. Let (X, d_x) be a locally compact metric space. Let (Y, d_y) be metric space. Let F_n $(n = 1, 2, \ldots)$ be connected subsets of X and let $\operatorname{Lim} F_n$ exist. If $\operatorname{Lim} F_n$ is a compact set and $f: X \to Y$ is a continuous function, then $\operatorname{Lim} f(F_n) = f(\operatorname{Lim} F_n)$.

3 A remark on the sets of distances

Let (X, d) be a metric space. If A and B are nonempty subsets of X, let $D(A, B)$ denote the set of all numbers $d(x, y)$ $x \in A$, $y \in B$. If $A = B$ we put $D(A, B) = D(A)$ and for $A = \emptyset$ we put $D(A) = \emptyset$. The set $D(A)$ is called the set of distances of A.

We can consider D as a function on the space of all subsets of the set X to the space of all subsets of R, which assings to any subset A of X the subset $D(A)$ of R.

Proposition 3.1. The function D: $(2^X, h_d) \to (2^R, h_u)$ is uniformly continuous.

Proof. The metric d: $X \times X \to R$ is uniformly continuous with respect to the product metric ϱ_1, ϱ_1 $[(x, y), (u, v)] = \max\{d(x, u), d(y, v)\}$.

Define F: $(2^{X \times X}, h_{\varrho_1}) \to (2^R, h_u)$ by: $F(C) = \{d(x, y): (x, y) \in C\}$. By Proposition 1.1. F is uniformly continuous.

Let $B \in 2^X$. Then $D(B) = F(B \times B)$. We show that the function D: $(2^X, h_d) \to$ $\to (2^R, h_u)$ is also uniformly continuous. Let $\varepsilon > 0$. There exists $\delta > 0$ such that for any $I, J \in 2^{X \times X}$ such that $h_{\varrho_1}(I, J) < \delta$ we have $h_u(F(I), F(J)) < \varepsilon$.

Let $A, B \in 2^X$ be such that $h_d(A, B) < \delta$. Since $h_{\varrho_1}(A \times A, B \times B) < \delta$, we have $h_u(D(A), D(B)) < \varepsilon$.

Proposition 3.2. Let X be a compact metric space, let $A_n \in 2^X$ $(n = 1, 2, \ldots)$ and let $\operatorname{Lim} A_n$ exist. Then $D(\operatorname{Lim} A_n) = \operatorname{Lim} D(A_n)$.

Proof. By Theorem in [7] we have $\lim_n h_d(\mathrm{Lim}\, A_n, A_n) = 0$ and by Proposition 3.1. we have $\lim_n h_u(D(\mathrm{Lim}\, A_n), D(A_n)) = 0$, i.e. $\mathrm{Lim}\, D(A_n) = D(\mathrm{Lim}\, A_n)$.

Proposition 3.3. Let X be a locally compact metric space. Let F_n $(n = 1, 2, \ldots)$ be connected subsets of X and let $\mathrm{Lim}\, F_n$ exist. If $\mathrm{Lim}\, F_n$ is a compact set, then $\mathrm{Lim}\, D(F_n) = D(\mathrm{Lim}\, F_n)$.

Proof. By Corollary 2.1. $\lim_n h_d(F_n, \mathrm{Lim}\, F_n) = 0$ and by Proposition 3.1. $\lim_n h_u(D(F_n), D(\mathrm{Lim}\, F_n)) = 0$. Thus $\mathrm{Lim}\, D(F_n) = D(\mathrm{Lim}\, F_n)$.

Remark. The certain type of continuity of the function D is also considered in [6].

REFERENCES

1. Beer, G.: More on convergence of continuous functions and topological convergence of sets, Canad. Math. Bull. Vol. 28 (1), 1985 52—59.
2. Engelking, R.: General Topology, Warszawa 1977.
3. Castaing, C.—Valadier, M.: Convex analysis and measurable multifunctions, Lecture notes in Math. No. 580, Springer Verlag, 1977.
4. Kelley, J.: Generaľ Topology. New York 1057.
5. Kuratowski, K.: Topology, Academic Press, New York, 1966.
6. Neubrunn, T.—Šalát, T.: O množinách vzdialeností množín metrického priestoru, Mat. Fyz. Čas. 9, 1959, 222—235.
7. Waterhouse, V.C.: Uniform convergence and graph convergence, Amer. Math. Monthly 83 (1976) 641—643.

Author's address:
Ľubica Holá
Katedra teórie pravdepodobnosti
a matematickej štatistiky
Matematický pavilón
Mlynská dolina
842 15 Bratislava

Received: 4. 9. 1986

SÚHRN

POZNÁMKA K TEÓRII MNOŽÍN VZDIALENOSTÍ

Ľubica Holá, Bratislava

Nech 2^X, 2^Y sú systémy všetkých podmnožín metrických priestorov X, Y. Nech $f\colon X \to Y$ je funkcia. Pre $A \in 2^X$ položme $F(A) = f(A)$. Potom F je funkcia z 2^X do 2^Y. V článku sa uvádzajú podmienky, pri ktorých funkcia F zachováva topologickú konvergenciu a kovergenciu množín v Hausdorffovskej metrike. Výsledky sú aplikované na množiny vzdialeností.

РЕЗЮМЕ

ПРИМЕЧАНИЯ К ТЕОРИИ МНОЖЕСТВ РАССТОЯНИЙ

Любица Гола, Братислава

Пусть 2^X, 2^Y — системы всех подмножеств метрических пространств X, Y. Пусть f — отображение X в Y. Для $A \in 2^X$ положим $F(A) = f(A)$. Потом F — отображение 2^X в 2^Y. В статье приводятся условия при которых отображение F сохраняет топологическую сходимость и сходимость множеств в метрике Хаусдорффа. Результаты применены к множествам расстояний.