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SOME CONDITIONS THAT IMPLY CONTINUITY OF ALMOST CONTINUOUS MULTIFUNCTIONS

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This paper presents a generalization of some results of [1] for multifunctions. The main theorem states that if $F\colon X \to Y$ is an almost continuous compact-valued multifunction, the set of points of quasicontinuity is dense in X, and Y is a regular space, then F is continuous.

We show the situation in the case of Theorem 3.3. [1] is different for multifunctions. A generalization of this theorem for spaces complete in the sense of Čech is given in [4].

In what follows X, Y denote topological spaces.

A multifunction $F\colon X \to Y$ is a mapping defined on X with values in the power set of Y. A single-valued mapping $f\colon X \to Y$ may be interpreted as a multifunction assigning to $x \in C$ a one-point set $\{f(x)\}$.

For a multifunction $F\colon X \to Y$ we suppose $F(x) \neq \emptyset$ for any $x \in X$. If F is a multifunction from X into Y, then for any $A \subset Y$ we denote $F^-(A) = \{x \in X\colon F(x) \cap A \neq \emptyset\}$, $F^+(A) = \{x \in X\colon F(x) \subset A\}$. For a subset A of topological space, $\bar{A}$ and $\operatorname{Int} A$ denote the closure and the interior of A respectively.

A multifunction $F\colon X \to Y$ is called upper (lower) almost continuous at a point $x_0 \in X$ if for any open set $V \subset Y$ such that $x_0 \in F^+(V)$ $(x_0 \in F^-(V))$, $x_0 \in \operatorname{Int} \overline{F^+(V)}$ $(x_0 \in \operatorname{Int} \overline{F^-(V)})$.

In the case of single-valued function the upper and the lower almost continuity coincides with the notion of almost continuity as defined in [8].

A multifunction $F\colon X \to Y$ is called upper (lower) quasicontinuous at a point $x_0 \in X$ if for any open set V such that $x_0 \in F^+(V)$ $(x_0 \in F^-(V))$, $x_0 \in \overline{\operatorname{Int} F^+(V)}$ $(x_0 \in \overline{\operatorname{Int} F^-(V)})$.

If F is upper and lower almost continuous (upper and lower quasicontinuous) at x_0, then it is said to be almost continuous (quasicontinuous) at x_0.

If F is upper almost continuous (lower almost continuous, upper quasicontinuous, lower quasicontinuous, almost continuous, quasicontinuous) at any $x \in X$, then it is said to be upper almost continuous (lower almost continuous, upper quasicontinuous, lower quasicontinuous, almost continuous, quasicontinuous).

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If F: $X \to Y$ is a multifunction from X into Y, let $K(F) = \{x \in X$: F is quasicontinuous at $x\}$ and $C(F) = \{x \in X$: F is continuous at $x\}$.

Theorem 1.1. Let Y be a regular space and F: $X \to Y$ be an almost continuous multifunction. Let $\overline{K(F)} = X$. Then F is lower semicontinuous. Moreover, if F is a compact-valued multifunction, then F is upper semicontinuous and thus continuous.

Proof: First we prove the case of the lower semicontinuity of F. Suppose there exists $x_0 \in X$ such that F is not lower semicontinuous at x_0. Then there exists an open set V in Y such that $x_0 \in F^-(V)$ but $F^-(V)$ is not a neighbourhood of x_0 (1).

Let $y_0 \in F(x_0) \subset V$. Since Y is a regular space, there exist two open sets V_1, V_2 in Y such that $y_0 \in V_1$, $y_0 \in V_2$ and $V_1 \subset \bar{V}_1 \subset V_2 \subset \bar{V}_2 \subset V$ (2).

Lower almost continuity of F at x_0 and $V_1 \cap F(x_0) \neq \emptyset$ imply that there exists an open set U in X such that $x_0 \in U$ and $U \subset \overline{F^-(V_1)}$ (3). If $s \in U \cap K(F)$, then $s \in F^-(\bar{V}_1)$ (4).

Suppose (4) does not hold. Then there exists $s \in U \cap K(F)$ such that $s \in F^+(Y - \bar{V}_1)$. Upper quasicontinuity of F at s implies that there exists a non-empty open set $W \subset U$ such that $W \subset F^+(Y - \bar{V}_1) \subset F^+(Y - V_1)$ (5).

Since (5) is in contradiction with (3), (4) holds.

By (1) there exists $x \in U$ such that $x \in F^+(Y - V)$. By (2) $Y - V \subset Y - \bar{V}_2$. Upper almost continuity of F at x implies that there exists an open set G in X such that $x \in G$, $G \subset U$ and $G \subset \overline{F^+(Y - \bar{V}_2)}$ (6).

Let $z \in G \cap K(F)$. By (4) $z \in F^-(\bar{V}_1)$. Thus $z \in F^-(V_2)$. Since F is lower quasicontinuous at z, there exists an open set H in X such that $H \neq \emptyset$, $H \subset G$ and $H \subset F^-(V_2)$ (7).

(7) implies $F(x) \cap V_2 \neq \emptyset$ for any $x \in H$. Thus for any $x \in H$, $x \notin \overline{F^+(Y - V_2)}$, i.e. $x \notin \overline{F^+(Y - \bar{V}_2)}$. That is in contradiction with (6).

Now we prove the second part. The proof is similar.

Let F be a compact-valued multifunction. Suppose there exists $x_0 \in X$ such that F is not upper semicontinuous at x_0. There exists an open set V in Y such that $x_0 \in F^+(V)$, but $F^+(V)$ is not a neighbourhood of x_0 (1′).

Let V_1, V_2 be open sets in Y such that $x_0 \in F^+(V_1)$ and $V_1 \subset \bar{V}_1 \subset V_2 \subset \bar{V}_2 \subset$ $\subset V$ (2′).

Upper almost continuity of F at x_0 implies that there exists an open set U in X such that $x_0 \in U$ and $U \subset \overline{F^+(V_1)}$ (3′).

If $s \in U \cap K(F)$, then $s \in F^+(\bar{V}_1)$ (4′).

Suppose (4′) does not hold. There exists $s \in U \cap K(F)$ such that $s \in F^-(Y - \bar{V}_1)$. Lower quasicontinuity of F at s implies that there exists a non-empty open set $W \subset U$ such that $W \subset F^-(Y - \bar{V}_1) \subset F^-(Y - V_1)$ (5′).

But since (5′) is in contradiction with (3′), (4′) holds.

By (1′) there exists $x \in U$ such that $x \notin F^+(V)$. By (2′) $F^-(Y - V) \subset \subset F^-(Y - \bar{V}_2)$ and thus $x \in F^-(Y - \bar{V}_2)$. Lower almost continuity of F at x implies that there exists an open set G such that $x \in G$, $G \subset U$ and $G \subset \subset \overline{F^-(Y - \bar{V}_2)}$ (6′).

Let $z \in G \cap K(F)$. By (4′) $z \in F^+(\bar{V}_1) \subset F^+(V_2)$. Upper quasicontinuity of F at z implies that there exists a non-empty open set $H \subset G$ such that $H \subset F^+(V_2)$ (7′).

(7′) implies $x \in F^+(\bar{V}_2)$ for any $x \in H$. That is contradiction to (6′). The theorem is proved.

Corollary 1.2. Let Y be a regular space and $f\colon X \to Y$ be a single-valued almost continuous and $\overline{K(f)} = X$. Then f is continuous.

Corollary 1.3. (Theorem 2.1. [1]) Let Y be a regular space and $f\colon X \to Y$ be a single-valued mapping. Let f be almost continuous and $\overline{C(f)} = X$. Then f is continuous.

It is easy to see that Theorem 1.1. is valid if F is an almost continuous closed-valued multifunction and Y is a normal space.

We verify that the assumptions in Theorem 1.1. are essential. The following example shows that the regularity of Y is essential.

Example 1.4. Let $X = [0, 1]$ with the usual topology. Put $Y = [0, 1]$, where the topology of Y consists of the usual topology and, moreover, of all the sets $G - \left\{1, \frac{1}{2}, \dots \frac{1}{n} \dots\right\}$ where G is open in the usual topology. The spacce Y is not regular. The identity I from X onto Y is quasicontinuous and almost continuous but I is not continuous at 0, since the set $V = [0, 1] - \left\{1, \frac{1}{2}, \dots \frac{1}{n}, \dots\right\}$ is not the neighbourhood of 0 in the usual topology.

If we require only lower or upper almost continuity instead of almost continuity, Theorem 1.1. need not apply.

Example 1.5. Let $X = Y = [0, 1]$ with the usual topology. We define $F\colon X \to Y$ as follows: $F(x) = \{0, 1\}$ for $x \neq \frac{1}{n}$, $n = 1, 2, \dots$ $F(x) = \{0\}$ for $x = \frac{1}{n}$, $n = 1, 2, \dots$

Then F is lower almost continuous, but F is not almost continuous. $\overline{C(F)} = X$. F is not lower semicontinuous at 0.

Example 1.6. Let $X = Y = [0, 1]$ with the usual topology. We define $F\colon X \to Y$ as follows: $F(x) = \{0\}$ for $x \neq \frac{1}{n}$, $n = 1, 2, \dots$ $F(x) = \{0, 1\}$ for $x = \frac{1}{n}$, $n =$

$= 1, 2 \ldots$ Then F is upper almost continuous, but F is not almost continuous. $\overline{C(F)} = X$, since $C(F) = X - \left\{0, 1, \frac{1}{2}, \ldots \frac{1}{n} \ldots\right\}$. F is not upper semicontinuous at 0.

The assumption of compactness in the second part of Theorem 1.1. is essential.

Example 1.7. Let $X = Y = R$ with the usual topology. We define $F: X \to Y$ as follows: $F(x) = [0, 1]$ for x rational and $F(x) = (0, 1]$ for x irrational.

Then F is almost continuous at every $x \in X$ and F is continuous at every rational $x \in X$. Thus $\overline{C(F)} = X$. However, F is not upper semicontinuous at any irational $x \in X$.

Definition 2.1. A topological space Y is said to be c-locally compact (c-locally countably compact) if each point of Y has a basis of compact (countably compact) neighbourhoods. A topological space Y is said to be strongly locally compact [6] if each point of Y has a basis of closed compact neighbourhoods.

Every locally compact regular and locally compact Hausdorff space are strongly locally compact, every locally countably compact regular space is c-locally compact and every strongly locally compact space is a c-locally compact space.

There exists a c-locally compact space which is not strongly locally compact.

Example 2.2. Let Y be the set of all positive integers. Let $\mathscr{G} = \{\{1, n\}: n \in N\}$ be the base for topology of Y. The space Y is a c-locally compact space, but Y is not strongly locally compact, because if H is a closed set containing 1, then $H = Y$.

Thus Theorem 7. in [6] is slightly extended as follows:

Proposition 2.3. Let $F: X \to Y$ be lower (upper) almost continuous. Let Y be a c-locally compact space. Let $F^-(K)$ $(F^+(K))$ be a closed set for every compact $K \subset Y$. Then F is lower (upper) semicontinuous.

Proof: We prove the case of lower almost continuity, the other case is similar.

Let $x_0 \in X$. Let V be an open set in Y such that $x_0 \in F^-(V)$. Let $y_0 \in F(x_0) \cap V$. Then there exists a compact neighbourhood W of y_0 such that $W \subset V$. Lower almost continuity of F at x_0 implies that $x_0 \in \operatorname{Int} \overline{F^-(\operatorname{Int} W)}$. Thus we have $x_0 \in \operatorname{Int} F^-(V)$, because $\overline{F^-(\operatorname{Int} W)} \subset \overline{F^-(W)} = F^-(W) \subset F^-(V)$.

The proof of the following Proposition 2.4. is similar.

Proposition 2.4. Let $F: X \to Y$ be lower (upper) almost continuous. Let Y be a c-locally countably compact space. Let $F^-(K)$ $(F^+(K))$ be a closed set for every countably compact set $K \subset Y$. Then F is lower (upper) semicontinuous.

Recall that the graph $G(F)$ of a multifunction $F: X \to Y$ is the set $G(F) = \{(x, y): y \in F(x)\}$. It is said to be closed if the set $G(F)$ is closed in $X \times Y$.

Corollary 2.5. Let $F: X \to Y$ be a lower almost continuous multifunction with a closed graph. Let Y be a c-locally compact space. Then F is lower semicontinuous.

Corollary 2.6. Let F be a lower almost continuous multifunction with a closed graph. Let Y be a c-locally countably compact space and X a first countable space. Then F is lower semicontinuous.

Proof: The assumptions imply that $F^-(K)$ is a closed set for every countably compact set $K \subset Y$.

Corollary 2.7. (See [6].) Let $f: X \to Y$ be a single-valued almost continuous function with a closed graph. Let Y be a strongly locally compact space. Then f is continuous.

Corollary 2.8. (See [1].) Let $f: X \to Y$ be a single-valued function. Let Y be a locally countably compact regular space and X be a Frechet space (i.e. if $p \in X$ is a limit point of a set $C \subset X$, then there is a sequence of points from C converging to p).

Let f be an almost continuous function with a closed graph. Then f is continuous.

Proof: If X is a Frechet space and f is a function with a closed graph, then the set $f^-(K) = \{x \in X: f(x) \in K\} = \{x \in X: F(x) \cap K \neq \emptyset\}$ where $F(x) = \{f(x)\}$ for every $x \in X$, is a closed set for every countably compact set $K \subset Y$.

The c-local compactness in Proposition 2.9. and c-local countable compactness in Proposition 2.3. are essential as shown in the following example.

Example 2.9. Put $X = [0, 1]$ with the usual topology. Let Y be the set of all rational numbers with the usual topology. It is obvious that Y is not a c-locally countably compact space and thus Y is not a c-locally compact space.

Define a multifunction $F: X \to Y$ as follows: $F(x) = \{1\}$ for x irational and $F(x) = \{x, 1\}$ for x rational. The multifunction F is almost continuous and the graph of F is closed, thus $F^-(K)$ is a closed set for every compact set $K \subset Y$ and $F^-(K)$ is a closed set for every countably compact set $K \subset Y$ too. However, F is not lower semicontinuous at rational numbers except for $x = 1$.

The main theorem of [1] states that if $f: X \to Y$ is almost continuous with a closed graph (closed in $X \times Y$) and X and Y are complete metric spaces, then f is continuous.

The following counterexample shows that this theorem does not hold for a multifunction.

Example 3.1. Let $X = R$ with the usual metric. Let $Y = Q'$ with the usual metric d, wher Q' is the set of irational numbers. Then (Y, d) is topologically complete [9], that is, there exists a metric ϱ such that ϱ and d are topologically equivalent, and (Y, ϱ) is complete.

We define $F: X \to Y$ as follows: $F(x) = \{e\}$ for x rational and $F(x) = \{x, e\}$ for x irational. F is almost continuous with a closed graph. (Let $(x, y) \notin (G(F)$.

Then $y \neq e$. There exists open sets V_1, V_2 in X such that $x \in V_1$, $y \in V_2$, $e \notin V_2$ and $V_1 \cap V_2 = \emptyset$. The set $(V_1 \times (V_2 \cap Y)) \cap G(F) = \emptyset$, that means that $G(F)$ is closed in $X \times Y$.) But F is not continuous, since F is not lower semicontinuous at irational numbers different from e and F is not upper semicontinuous at rational numbers.

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SÚHRN

NEJAKÉ PODMIENKY, KTORÉ IMPLIKUJÚ SPOJITOSŤ SKORO SPOJITÝCH MULTIFUNKCIÍ

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Hlavným výsledkom tohto článku je tvrdenie, že keď $F: X \to Y$ je skoro spojitá kompaktne hodnotová multifunkcia, množina bodov kvázispojitosti je hustá v X a Y je regulárny priestor, tak F je spojitá.

Ďalej článok obsahuje zovšeobecnenie niektorých výsledkov z [1] pre multifunkcie.

РЕЗЮМЕ

НЕКОТОРЫЕ УСЛОВИЯ, ИЗ КОТОРЫХ СЛЕДУЕТ НЕПРЕРЫВНОСТЬ ПОЧТИ НЕПРЕРЫВНЫХ ОТНОШЕНИЙ

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Главным результатом этой статьи является утверждение: Если $F: X \to Y$ почти непрерывное отношение, значения которого являются бикомпактнымы множествами если множество точек квазинепрерывности плотное в X, и Y-регулярное пространство, тогда F представляет собой непрерывное отношение. Далее, здесь приводятся некоторые обобщения результатов из [1] для отношений.