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REMARK ON THE FORMULA FOR m-th DERIVATIVE OF THE COMPOSITION OF TWO BANACH-VALUED FUNCTIONS

MAREK W. MICHALSKI, Poland

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Let I be an open subset of R, E_1 and E_2 — Banach spaces, U an open subset of E_1, and m — a positive integer. Suppose there are two functions $F: I \to E_1$ and $g: U \supset f(I) \to E_2$ which have derivative up to order m on their domains respectively. The composition $g \circ f$ of the functions f and g has derivatives up to order m (cp. [1], Th. 19, p. 256) and, moreover.

Theorem. For every $m \in N$, the following formula holds:

$$(1) \qquad (g \circ f)^{(m)}(x) = \sum_{l=1}^{m} \sum_{\substack{|\mathbf{r}_{(l)}| = m \\ r_1, \ldots, r_l \geq 1}} C_{\mathbf{r}_{(l)}}(g^{(l)} \circ f(x) \cdot B_{\mathbf{r}_{(l)}}(x)$$

($x \in I$; $\mathbf{r}_{(l)} = (r_1, r_2, \ldots, r_l)$ multiindex; $|\mathbf{r}_{(l)}| = r_1 + r_2 + \ldots + r_l$; $1 \leqslant l \leqslant m$), where

$$(2) \qquad B_{\mathbf{r}_{(l)}}(x) = (f^{(r_1)}(x), f^{(r_2)}(x), \ldots, f^{(r_l)}(x));$$

$$(3) \qquad C_{\mathbf{r}_{(l)}} = \prod_{j=1}^{l} \binom{|\mathbf{r}_{(j)}| - 1}{|\mathbf{r}_{(j)}| - r_j}$$

and $(g^{l} \circ f(x)) \cdot B_{\mathbf{r}_{(l)}}(x)$ means the value of l-linear function $g^{(l)} \circ f(x) = g^{(l)}(f)|_{f=f(x)}$ on the vector $B_{\mathbf{r}_{(l)}}(x) \in E_1^l$ is valid.

To prove the Theorem we need a lemma. Due to the property of the derivative of bilinear continuous function and the principle of mathematical induction one can easy obtain

Lemma (sp. [1], Th. 18, p. 252). Let $T: I \to L(E_1; E_2)$ and $z: I \to E_1$ be m times differentiable. Then function

$$(4) \qquad I \ni x \to T(x) \cdot z(x)$$

is m times differentiable and the formula

(5) $$(T(x) \cdot z(x))^{(m)} = \sum_{j=0}^{m} \binom{m}{j} T^{(j)}(x) \cdot z^{(m-j)}(x)$$

$(x \in I)$ holds.

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Proof of the Theorem. For $m = 1$ formula (1) holds (sp. [1], Th. 11, p. 214), and assume its truthfulness for $m = 1, 2, \ldots, s$, where s is a fixed positive integer. Due to formula (1) for $m = 1$, Lemma and inductive assumption (with the replacement of g and E_2 by g' and $L(E_1; E_2)$ respectively) one can write (cp. also [1], pp. 245, 246)

$$(g \circ f)^{(s+1)}(x) = (g' \circ f(x)) \cdot f^{(s+1)}(x) + \tag{6}$$
$$+ \sum_{j=1}^{s} \binom{s}{j} \left\{ \sum_{l=1}^{j} \sum_{\substack{|\mathbf{r}_{(l)}| = j \\ r_1, \ldots, r_l \geqslant 1}} C_{\mathbf{r}_{(l)}}(g^{(l+1)} \circ f(x)) \cdot B_{\mathbf{r}_{(l)}}(x) \right\} \cdot f^{(s+1-j)}(x) =$$
$$= (g' \circ f(x)) \cdot f^{(s+1)}(x) +$$
$$+ \sum_{j=1}^{s} \sum_{l=1}^{s} \sum_{\substack{|\mathbf{r}_{(l)}| = 1 \\ r_1, \ldots, r_l \geqq 1}} \binom{s}{j} C_{\mathbf{r}_{(l)}}(g^{(l+1)} \circ f(x)) \cdot (B_{\mathbf{r}_{(l)}}(x), f^{(s+1-j)}(x))$$

$(x \in I)$. Alternating two first sums and then replacing $l + 1$ by l, we have

(7) $$(g \circ f)^{(s+1)}(x) = (g' \circ f(x)) \cdot f^{(s+1)}(x) +$$
$$+ \sum_{l=2}^{s+1} \sum_{j=l-1}^{s} \sum_{\substack{|\mathbf{r}_{(l)}| = s+1 \\ r_1, \ldots, r_{l-1} \geqslant 1;\, r_l = s+1-j}} \binom{s}{j} C_{\mathbf{r}_{(l-1)}}(g^{(l)} \circ f(x)) \cdot B_{\mathbf{r}_{(l)}}(x)$$

$(x \in I)$. Hence

(8) $$(g \circ f)^{(s+1)}(x) = g' \circ f(x) = (g' \circ f(x)) \cdot f^{(s+1)}(x) +$$
$$+ \sum_{l=2}^{s+1} \sum_{\substack{|\mathbf{r}_{(l)}| = s+1 \\ r_1, \ldots, r_l \geqslant 1}} \binom{s}{s+1-r_1} C_{\mathbf{r}_{(l-1)}}(g^{(l)} \circ f(x)) \cdot B_{\mathbf{r}_{(l)}}(x)$$

$(x \in I)$. Finally, in virtue of (3), we get formula (1) for $m = s + 1$.

Application of the principle of mathematical induction ends the proof.

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Remark 1. If $E_1 = R$, then formula (1) has the form

$$(g \circ f)^{(m)}(x) = \sum_{l=1}^{m} A_l^m(x) g^{(l)} \circ f(x) \tag{9}$$

$(x \in I)$, where

$$A_l^m(x) = \sum_{\substack{|\mathbf{r}_{(l)}| = m \\ r_1, \ldots, r_l \geq 1}} \prod_{j=1}^{l} \binom{|\mathbf{r}_{(j)}| - 1}{|\mathbf{r}_{(j)}| - r_j} f^{(r_j)}(x) . \tag{10}$$

Remark 2 (cp. [1], Th. 21ter, p. 262). In the case $E_1 = E_2 = R$, application of the Taylor formula yields

$$(g \circ f)^{(m)}(x) = \tag{11}$$

$$= \sum_{r_1 + 2r_2 + \ldots + mr_m = m} \frac{m!}{\vec{r}_{(m)}!} \prod_{j=1}^{m} \left[\frac{1}{j!} f^{(j)}(x)\right]^{r_j} g^{(|\vec{r}_{(m)}|)} \circ f(\dot{x})$$

$$(x \in I;\ \mathbf{r}_{(m)} = (r_1, r_2, \ldots, r_m);\ \mathbf{r}_{(m)}! = r_1! r_2! \ldots r_m!;$$
$$|\mathbf{r}_{(m)}| = r_1 + r_2 + \ldots + r_m;\ m \in N) .$$

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Author's address:
Marek Michalski
15-007 Towarowa 4m, 127
Warszawa
Poland

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SÚHRN

POZNÁMKA O VZORCI PRE *m*—tú DERIVÁCIU KOMPOZÍCIE DVOCH FUNKCIÍ S HODNOTAMI V BANACHOVÝCH PRIESTOROCH

M. W. Michalski, Poľsko

V práci sa dokazuje zovšeobecnenie Leibnizovej formuly pre funkcie s hodnotami v Banachových priestoroch.

РЕЗЮМЕ

ЗАМЕЧАНИЕ О ФОРМУЛЕ ДЛЯ m-той ПРОИЗВОДНОЙ КОМПОЗИЦИИ ДВУЦХ ФУНКЦИЙ СО ЗНАЧЕНИЯМИ В ПРОСТРАНСТВАХ БАНАХА

М. В. Михалски, Польша

В работе доказывается обобщение формулы Лейбница для функций со значениями в пространствах Банаха.