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THE MAXIMAL ADDITIVE AND MULTIPLICATIVE FAMILIES FOR FUNCTIONS WITH CLOSED GRAPH

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In the present paper we shall deal with the functions whose domain is a topological space and the range is the set of real numbers R and which have the closed graph in $X \times R$. Let G_f denote the graph of a function f, $U(X, R)$ the family of all functions with closed graph and $C(X, R)$ the family of all continuous functions.

A. M. Bruckner in the monograph [1] has defined the maximal additive (multiplicative) family.

Definition 1. [1] Let $\mathscr{D}$ be a family of real functions defined on a topological space X. A subfamily $\mathscr{F}$ of $\mathscr{D}$ is called the maximal additive (multiplicative) family for $\mathscr{D}$ provided $\mathscr{F}$ is the set of all functions f in $\mathscr{D}$ such that $f + g \in \mathscr{D}$ ($f \cdot g \in \mathscr{D}$) whenever $f \in \mathscr{F}$ and $g \in \mathscr{D}$.

Lemma 1. Let $f \in U(X, R)$ be a function discontinuous at a point $x \in X$. Then there is a net $\{x_\gamma, \gamma \in \Gamma\}$ which converges to the point x such that a net $\{f(x_\gamma), \gamma \in \Gamma\}$ diverges to $+\infty$ or $-\infty$.

Proof. The function f is discontinuous at a point x, hence there is a net $\{x_\gamma, \gamma \in \Gamma\}$ which converges to the point x such that the net $\{f(x_\gamma), \gamma \in \Gamma\}$ does not converge to $f(x)$. Since $f \in U(X, R)$, the net $\{f(x_\gamma), \gamma \in \Gamma\}$ is not bounded. Let $\mathscr{A}$ be the family of sets $V \times (n, \infty)$ and $\mathscr{B}$ be the family of sets $V \times (-\infty, -n)$, where V is an arbitrary neighborhood of x, $n = 1, 2, \dots$. The net $\{(x_\gamma, f(x_\gamma)), \gamma \in \Gamma\}$ is frequently in each member of $\mathscr{A}$ or in each member of $\mathscr{B}$. The families $\mathscr{A}$ and $\mathscr{B}$ satisfy the conditions of Lemma 2.5 [2] and consequently there is a subnet $\{(x_{\gamma_i}, f(x_{\gamma_i})), i \in I\}$ which is eventually in each member of some of families $\mathscr{A}$ or $\mathscr{B}$. Then obviously the net $\{f(x_{\gamma_i}), i \in I\}$ diverges to $+\infty$ or $-\infty$.

Theorem 1. $C(X, R)$ is the maximal additive family for $U(X, R)$.

Proof. According to Theorem 3 of paper [3] $C(X, R) \subset U(X, R)$ holds.

Suppose f is an arbitrary member of $U(X, R)$ and g belongs to $C(X, R)$. We shall show that $(f + g) \in U(X, R)$. Let the net $\{(x_\gamma, (f + g)(x_\gamma)), \gamma \in \Gamma\}$ converge to a point (x, z). The function g is continuous and therefore the net $\{f + g)(x_\gamma)$,

$\gamma \in \Gamma\}$ converges to z if and only if the net $\{f(x_\gamma),\ \gamma \in \Gamma\}$ converge to $z - g(x)$. From the assumption $f \in U(X, R)$ it follows $f(x) + g(x) = z$, i.e. $(f + g) \in \in U(X, R)$.

Let $f \in U(X, R)$ be discontinuous at a point $\tilde{x}$. We shall show that there is a function $g \in U(X, R)$ such that $(f + g) \notin U(X, R)$. Without loss of generality [see Lemma 1] we can assume that there is a net $\{x_\gamma,\ \gamma \in \Gamma\}$, $x_\gamma \to \tilde{x}$, such that $f(x_\gamma)$ diverges to $+\infty$. Choose $c > f(\tilde{x})$ and let $A = \{x \in X, f(x) \geqslant c + 1\}$,

$$B = \{x \in X, f(x) \leqslant c\},$$
$$C = \{x \in X,\ c \leqslant f(x) \leqslant c + 1\}.$$

Define the function g in the following way:

$$g(x) = \begin{cases} f(x) - 1 & \text{if } x \in A \\ c & \text{if } x \in C \\ f(x) & \text{if } x \in B. \end{cases}$$

From the first part of proof it follows that $f - 1 \in U(X, R)$ and consequently the sets $G_{g/A} = G_{f-1} \cap (X \times [c, \infty))$, $G_{g/B} = G_f \cap (X \times (-\infty, c])$ are closed in $X \times R$. The set $G_{g/C} = C \times \{c\}$ is closed if and only if C is closed in X. Let a net $\{x_j, j \in J\}$, $x_j \in C$ converge to a point x_1. Then the net $\{f(x_j),\ j \in J\}$ is contained in the compact $[c,\ c + 1]$ and there is a convergent subnet $\{f(x_{j_i}),\ i \in I\}$, $f(x_{j_i}) \to \to y \in [c, c + 1]$, of the net $\{f(x_j), j \in J\}$. The net $\{(x_{j_i}, f(x_{j_i})),\ i \in I\}$ converges to a point $(x_1,\ y) \in G_f$, hence $y = f(x_1)$, $x_1 \in C$ and C is closed. The graph $G_g = = G_{g/A} \cup G_{g/B} \cup G_{g/C}$ is a closed set in $X \times R$, i.e. $g \in U(X, R)$. It is easy to verify that $-g \in U(X, R)$ and $f + (-g) \notin U(X, R)$, since the net $\{(x_\gamma,\ (f - g)(x_\gamma)),\ \gamma \in \Gamma\}$ converges to the point $(\tilde{x},\ 1)$ and $(f - g)(\tilde{x}) = 0$.

In the next text we denote $N_f = \{x \in X, f(x) = 0\}$.

Lemma 2. Let $f \in C(X, R)$. Then the function g defined by

$$g(x) = \begin{cases} \dfrac{1}{f(x)} & \text{if } x \in X - N_f \\ 0 & \text{if } x \in N_f \end{cases}$$

belongs to the family $U(x, R)$.

Proof. Let a net $\{(x_\gamma,\ g(x_\gamma)),\ \gamma \in \Gamma\}$ converge to a point $(x,\ y)$. If $x \in X - N_f$, then the function g is continuous at the point x and $g(x) = y$.

If $x \in N_f$, then $y = 0$ and consequently $g(x) = y$. The statement will be proved by contradiction.

Let $y \neq 0$. Choose $n > |y|$, then there is a $\gamma_0 \in \Gamma$ such that for every $\gamma > \gamma_0$ x_γ belongs to $f^{-1}\left(\left(-\frac{1}{n}, \frac{1}{n}\right)\right)$. From the definition of the function g it follows that $g(x_\gamma) = 0$ if $x_\gamma \in N_f$ or $|g(x_\gamma)| > n$ if $x_\gamma \in X - N_f$. It is a contradiction to the assumption $g(x_\gamma) \to y$.

Definition 2. [2] A topological space is normal if and only if for each disjoint pair of closed sets, A and B, there are disjoint open sets U and V such that $A \subset U$ and $B \subset V$.

Theorem 2. Let X be a locally compact normal topological space. Then the family of functions

$$M(X, R) = \{f \in C(X, R),\ N_f \text{ is open}\}$$

is the maximal multiplicative family for $U(X, R)$.

Proof. It is evident that $M(X, R) \subset C(X, R) \subset U(X, R)$. We shall prove the theorem in three parts.

Let $f \in U(X, R)$ be a discontinuous function. We show that f does not belong to the maximal multiplicative family for $U(X, R)$.

Let $\tilde{x}$ be a discontinuity point of the function f and let V be its compact neighborhood. From Lemma 1 it follows that the function f is not bounded in any neighborhood of $\tilde{x}$. Choose $b_1 \in V$ such that $|f(b_1)| > \max\{1, |f(\tilde{x})|\}$ and put $V_1 = X$. The sets $f^{-1}(f(b_1))$ and $f^{-1}(f(\tilde{x}))$ are closed (see Theorem 1 [3]) and disjoint. From normality of the topological space it follows that there exists a closed neighborhood V_2 of the set $f^{-1}(f(\tilde{x}))$ for which $V_2 \cap f^{-1}(f(b_1)) = \emptyset$. Choose $b_2 \in V_2 \cap V$ such that $|f(b_2)| > \max\{2, |f(\tilde{x})|\}$. Since the sets $\bigcup_{i=1,2} f^{-1}(f(b_i))$ and $f^{-1}(f(\tilde{x}))$ are closed and disjoint, there is a closed neighborhood V_3 of the set $f^{-1}(f(\tilde{x}))$ such that $V_3 \subset V_2$ and $V_3 \cap f^{-1}(f(b_2)) = \emptyset$We could continue in this way and construct a sequence of closed neighborhoods $V_1 \supset V_2 \supset \ldots$ of the point $\tilde{x}$ and a sequence of the points $b_n \in V_n \cap V$, $n = 1, 2, \ldots$ such that $f^{-1}(f(b_n)) \cap V_{n+1} = \emptyset$ and $0 < |f(b_n)| \to +\infty$. Designate A the closure of the set $\{b_n, n = 1, 2, \ldots\}$. Evidently $A = \left[\bigcup_{n=1}^{\infty} \{\overline{b_n}\}\right] \cup \{x, x \text{ is an accumulation}$ point of the sequence$\}$ and these two sets are disjoint. Define the function g in the following way

$$g(x) = \begin{cases} 0 & \text{if } x \text{ is an accumulation point of the sequence} \\ \dfrac{1}{f(x)} & \text{if } x \in \bigcup_{n=1}^{\infty} \{\overline{b_n}\}. \end{cases}$$

The function g is continuous on the set A and (according to Tietze's theorem) there is a continuous extension g^* of the function g on the space X. The function $g^* \in C(X, R) \subset U(X, R)$ but $f \cdot g^*$ does not belong to $U(X, R)$. There is a convergent subnet $\{b_{n_i}, i \in I\}$, $b_{n_i} \to b \in V$ for which the net $\{(b_{n_i}, (f \cdot g^*)(b_{n_i})), i \in I\}$ converges to the point $(b, 1)$ but $(f \cdot g^*)(b) = 0$.

In the second part we shall assume that the function $f \in C(X, R)$ and that the set N_f is not open. We show that f does not belong to the maximal multiplicative family for $U(X, R)$.

Since the set N_f is not open, there is $\tilde{x} \in N_f$ and the net $\{x_\gamma, \gamma \in \Gamma\}$, $x_\gamma \in X - N_f$, which converges to the point $\tilde{x}$. We have $f(\tilde{x}) = 0$ and $f(x_\gamma) \neq 0$ for every $\gamma \in \Gamma$. Define the function g by

$$g(x) = \begin{cases} \dfrac{1}{f(x)} & \text{if } x \in X - N_f \\ 0 & \text{if } x \in N_f. \end{cases}$$

According to Lemma 2 $g \in U(X, R)$ but $(g \cdot f) \notin U(X, R)$, because the net $\{(x_\gamma, (f \cdot g)(x_\gamma)), \gamma \in \Gamma\}$ converges to the point $(\tilde{x}, 1)$ and $(g \cdot f)(\tilde{x}) = 0$.

In the last part suppose $f \in M(X, R)$ and $g \in U(X, R)$. It is sufficient to prove $(f \cdot g) \in U(X, R)$.

Let the net $\{(x_\gamma, (f \cdot g)(x_\gamma)), \gamma \in \Gamma\}$ converge to a point $(\tilde{x}, z)$. If $\tilde{x} \in N_f$, then there is $\gamma_0 \in \Gamma$ such that $f(x_\gamma) = 0$ for every $\gamma > \gamma_0$. Then $(f \cdot g)(x_\gamma) \to 0$ and $z = (f \cdot g)(\tilde{x})$.

If $\tilde{x} \in X - N_f$, then from the continuity of the function f it follows that the net $\{(f \cdot g)(x_\gamma), \gamma \in \Gamma\}$ converges to z if and only if the net $\{g(x_\gamma), \gamma \in \Gamma\}$ converges to $\dfrac{z}{f(\tilde{x})}$. Since $g \in U(X, R)$, it is easy to see that $z = (f \cdot g)(\tilde{x})$. Hence $(f \cdot g) \in U(X, R)$.

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SÚHRN

MAXIMÁLNA ADITÍVNA A MULTIPLIKATÍVNA TRIEDA FUNKCIÍ S UZAVRETÝM GRAFOM

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V článku je daná charakterizácia maximálnej aditívnej a multiplikatívnej triedy v triede funkcií definovaných na topologickom priestore X s oborom funkčných hodnôt v množine reálnych čísel, ktoré majú uzavretý graf v $X \times R$. V článku sú dokázané nasledujúce vety:

Veta 1. Množina všetkých spojitých funkcií je maximálna aditívna trieda v triede funkcií s uzavretým grafom.

Veta 2. Nech X je lokálne kompaktný, normálny topologický priestor. Množina všetkých spojitých funkcií f, pre ktoré je $N_f = \{x \in X, f(x) = 0\}$ otvorená množina, je maximálna multiplikatívna trieda v triede funkcií s uzavretým grafom.

РЕЗЮМЕ

МАКСИМАЛЬНЫЙ АДДИТИВНЫЙ И МУЛЬТИПЛИКАТИВНЫЙ КЛАСС В КЛАССЕ ОТОБРАЖЕНИЙ С ЗАМКНУТЫМ ГРАФИКОМ

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В статье характеризуются максимальный аддитивный и мультипликативный классы в классе отображений определеных на топологическом пространстве X со значениями в множестве вещественных чисел R, графики которых являются замкнутыми подмножествами топологическово пространства $X \times R$. В работе доказаны следующие теоремы:

Теорема 1. Множество всех непрерывных отображений является максимальным аддитивным классом в классе отображений с замкнутым графиком.

Теорема 2. Пусть X будет локално-компактное нормальное топологическое пространство. Множество всех непрерывных отображений f, для которых $N_f = \{x \in X, f(x) = 0\}$ открытое множество в X является максимальным мультипликативным классом в классе отображений с замкнутым графиком.