UNIVERSITAS COMENIANA
ACTA MATHEMATICA UNIVERSITATIS COMENIANAE
LII—LIII—1987

Φ-SUMMING OPERATORS IN BANACH SPACES

R. KHALIL*) — W. DEEB, Kuwait

0 Introduction

Let Φ: $[0, \infty) \to [0, \infty)$ be a continuous function. Then function Φ is called a modulus function if

(i) $\Phi(x + y) \leq \Phi(x) + \Phi(y)$
(ii) $\Phi(0) = 0$
(iii) Φ is strictly increasing.

The functions $\Phi(x) = x^p$, $0 < p \leq 1$ and $\Phi(x) = ln(1 + x)$ are examples of modulus functions.

For Banach spaces E and F, a bounded linear operator A: $E \to F$ is called p-summing, $0 < p < \infty$, if there exists $\lambda > 0$ such that

$$\sum_{i=1}^{n} \|Ax_i\|^p \leq \lambda \sup_{\|x^*\| \leq 1} \sum_{i=1}^{n} |\langle x_i, x^* \rangle|^p,$$

for all sequences $\{x_1, \ldots, x_n\} \subseteq E$. For $p = 1$, this definition is due to Grothendieck [3], and for $p \neq 1$, the definition was given by Pietsch [6]. If $\Pi^p(E, F)$ is the space of all p-summing operators from E to F, then it is well known [3, p. 293] that $\Pi^p(E, F) = \Pi^q(E, F)$ for $0 < p, q \leq 1$. If E and F are Hilbert spaces then $\Pi^p(E, F) = \Pi^q(E, F)$ for $0 < p < q < \infty$, [6, p. 302].

The object of this paper is to introduce Φ-summing operators for modulus functions Φ. The basic properties of these operators are studied. We, further, prove that Φ-summing operators are p-summing for $0 < p \leq 1$, in case of Banach spaces having the metric approximation property.

Throughout this paper, $L(E, F)$ denotes the space of all bounded linear operators from E to F. The dual of E is E^*. The compact elements in $L(E, F)$

---

*) The first author would like to thank Prof. J. Jacod for helpful discussions.

will be denoted by $K(E, F)$. The unit sphere of a Banach space E is denoted by $S(E)$. The set of complex numbers is denoted by $\emptyset$.

I Π^{Φ}(E, F)

Let E and F be two Banach spaces and Φ be a modulus function on $[0, \infty)$. Consider the following two spaces:

(i) $$l^{\Phi}\langle E\rangle = \left\{(x_n) \colon \sup_{\|x^*\| \leq 1} \sum_n \Phi|\langle x_n, x^*\rangle| < \infty\,,\ x_n \in E\right\}.$$

(ii) $$l^{\Phi}(F) = \left\{(x_n) \colon \sum_n \phi\|x_n\| < \infty\,,\ x_n \in E\right\}.$$

For $x = (x_n) \in l^{\Phi}\langle E\rangle$, we define

$$\|x\|_{\varepsilon} = \sup_{\|x^*\| \leq 1} \sum_n \Phi|\langle x_n, x^*\rangle|\,,$$

and for $y = (y_n) \in l^{\Phi}(F)$ we define

$$\|y\|_{\Pi} = \sum_n \Phi\|y_n\|\,.$$

It is a routine matter to verify the following result:

Theorem 1.1. The spaces $(l^{\Phi}\langle E\rangle, \|\ \|_{\varepsilon})$ and $(l^{\Phi}(F), \|\ \|_{\pi})$ are complete metric linear spaces

Remark 1.2. The spaces $l^{\Phi}\langle E\rangle$ and $l^{\Phi}(E)$ are generalizations of the spaces $l^p\langle E\rangle$ and $l^p(E)$ for $0 < p < 1$. We refer to [6, Chap. 16] and [1] for a discussion of such spaces.

A linear operator $T\colon l^{\Phi}\langle E\rangle \to l^{\Phi}(F)$ will be callled **metrically bounded** if there is a $\lambda > 0$ such that

$$\|Tx\|_{\pi} \leq \lambda\|x\|_{\varepsilon}$$

for all $x = (x_n) \in l^{\Phi}\langle E\rangle$. Clearly, every metrically bounded operator is continuous. We let $L^{\Phi}(E, F)$ denote the space of all metrically bounded operator from $l^{\Phi}\langle E\rangle$ into $l^{\Phi}(F)$. For $T \in L^{\Phi}(E, F)$, we set $\|T\|_{\Phi} = \inf\{\lambda\colon\ \|Tx\|_{\pi} \leq \leq \lambda\|x\|_{\varepsilon},\ x\, l^{\Phi}\langle E\rangle\}$. The proof of the following result is similar to the proof in case of Banach spaces, [7, p. 185], and it will be omitted.

Theorem 1.3. The space $(L^{\Phi}(E, F), \|\ \|_{\Phi})$ is a complete metric linear space.

Definition 1.4. Let E and F be two Banach spaces. Then, a bounded linear operator $T\colon E \to F$ is called Φ-summing if there is $\lambda > 0$ such that

$$\sum_1^N \Phi\|Tx_n\| \leq \lambda \sup_{\|x^*\| \leq 1} \sum_1^N \Phi|\langle x_n, x^*\rangle| \tag{*}$$

for all sequences $\{x_1, \ldots, x_n\} \subseteq E$.

The definition is a generalization of the definition of p-summing operators for $0 < p \leq 1$. We refer to [6] for a full study of p-summing operators $0 < p < \infty$.

Let $\Pi^{\Phi}(E, F)$ be the set of all Φ-summing operators from E to F. Every $T \in \Pi^{\Phi}(E, F)$ defines an element $\hat{T} \in L^{\Phi}(E, F)$ via:

$$\hat{T}: l^{\Phi}\langle E \rangle \to l^{\Phi}(E)$$

$$\hat{T}((x_n)) = ((Tx_n)).$$

For $T \in \Pi^{\Phi}(E, F)$ we define the Φ-summing metric of T as: $\|T\|_{\Phi} = \|\hat{T}\|_{\Phi}$. Hence $\|T\|_{\Phi} = \inf\{\lambda: * \text{ holds}\}$. The definition of Φ-summing operators together with Theorem 1.2 implies:

Theorem 1.5. $(\Pi^{\Phi}(E, F), \| \|_{\Phi})$ is a complete metric linear space.

Theorem 1.6. Let $A \in \Pi^{\Phi}(E, F)$, $B \in L(G, E)$ and $D \in L(F, H)$. Then $AB \in \Pi^{\Phi}(G, E)$ and $DA \in \Pi^{\Phi}(E, H)$. Further: $\|AB\|_{\Phi} \leq (\|B\| + 1)\|A\|_{\Phi}$ and $\|DA\|_{\Phi} \leq (\|D\| + 1)\|A\|_{\Phi}$.

Proof: The proof follows from the fact that for all $a > 0$ $\Phi(at) \leq (a + 1)\Phi(t)$, which is a consequence of the monotonocity and subadditivity of Φ.

Q.E.D.

Let $B_1(E^*)$ be the unit ball of E^* equipped with the w^*-topology, and M be the space of all regular Borel measures on $B_1(E^*)$. The unit sphere of M is denoted by $S(M)$.

Theorem 1.6. Let $A \in L(E, F)$. The followings are equivalent:

(i) $A \in \Pi^{\Phi}(E, F)$.

(ii) Tthere exists $\lambda > 0$ and $\upsilon \in S(M)$ such that

$$\Phi\|Ax\| \leq \lambda \int_{B_1(E^*)} \Phi|\langle x, x^* \rangle| \, d\upsilon(x^*).$$

Proof. (ii) $\to$ (i). This is evident.

(i) $\to$ (ii). Let $A \in \Pi^{\Phi}(E, F)$ and $\lambda\|A\|_{\Phi}$.

For every finite sequence $\{x_1, \ldots, x_N\} \subseteq E$, define the map:

$$Q: S(M) \to \mathbb{C}$$

$$Q(\mu) = \sum_{n=1}^{N} \Phi\|Ax_n\| - \lambda\sum_{1}^{N} \int_{b_1(E^*)} \Phi|\langle x_n, x^* \rangle| \, d\mu \ldots \qquad (**)$$

Clearly, the function Q is convex. Further, there is a point $\mu_0 \in S(M)$ such that $Q(\mu_0) < 0$. Indeed, choose $\mu_0 =$ the dirac measure at x_0^*, where

$$\sum_{1}^{N} \Phi|\langle x_n, x_0^* \rangle| = \sup_{\|x^*\| \leq 1} \sum_{1}^{N} \Phi|\langle x_n, x^* \rangle|.$$

Further: If $\{Q_1, \ldots, Q_r\}$ is a collection of such functions defined by (**), then for any $a_1, \ldots, a_r$, $\sum_1^r a_k = 1$, there is Q defined in a similar way, such that $\sum_1^r a_k Q_k(\mu) \leq Q(\mu)$ for all $\mu \in S(M)$. Hence the collection of functions on $S(M)$ defined by (**) satisfies Fan's Lemma [6, p. 40]. Consequently there is a measure υ in $S(M)$ such that $Q(\upsilon) \leq 0$ for all Q defined by (**). In particular, if Q is defined by (**) with an associated sequence $\{x\}$, $x \in E$, we get

$$\Phi\| Ax \| \leq \lambda \int_{B_1(E^*)} \Phi|\langle x, x^* \rangle| \, d\upsilon,$$

This completes the proof. Q.E.D.

Remark 1.7. The proof of Theorem 1.6 is similar to the proof of Theorem 17.3.2 in [6], where $\Phi(t) = t^p \; 0 < p \leq 1$. We included the detailed proof here for completeness and to include modulus functions.

II $\Pi^{\Phi}(H, H) = \Pi^p(H, H)$, $0 \leq p \leq 1$

Let m be the Lebesgue measure on $I = [0, 1]$. For the modulus function Φ, set L^{Φ} to denote the space of all measurable functions f on $[0, 1]$ for which $\int_0^1 \Phi|f(t)| \, dm(t) < \infty$. For $f \in L^{\Phi}$ we define $\|f\|_{\Phi} = \Phi^{-1} \int_0^1 \Phi|f(t)| \, dm(t)$. The function $\| \; \|_{\Phi}$ is not a metric on L^{Φ}. However, we can define a topology via: $f_n \to f$ in L^{Φ} if $\Phi^{-1} \int \Phi|f_n - f| \, dm(t) \to 0$. It is not difficult to prove that such a topology makes L^{Φ} a topological vector space. In case $\Phi(t) = t^p \; 0 < p \leq 1$, L^{Φ} is a quasi-normed space, [4, p, 159]. If $\Phi(t) = \dfrac{t}{1+t}$, we write L^0 for L^{Φ}.

The concept of Φ-summing operators is still valid for operators $T: E \to L^{\Phi}$, where E is a Banach space.

Definition 2.1. Let E be a Banach space. A linear map $T: E \to L^{\Phi}$ is called Φ-decomposable if there is a function ψ: $[0, 1] \to E^*$ such that

(i) The function $\langle x, \psi(t) \rangle$ is m-measurable and $(Tx)(t) = \langle x, \psi(t) \rangle$ a.e.m for all $x \in E$.

(ii) There exists $f \in L^1$ such that $\| \psi(t) \| \leq f(t)$ a.e.m.

This definition is due to Kwapien [5] for $\Phi(t) = t^p$. In [5], the function f in (ii) is assumed to belong to L^p.

Since $L^{\Phi} \subseteq L^0$ for all modulus functions Φ, the following lemma is immediate:

Lemma 2.2. Every Φ-decomposable map $T: E \to L^{\Phi}$ is 0-decomposable.

Theorem 2.3. Let E be any Banach space. If a linear map $T: E \to L^{\Phi}$ is Φ-decomopsable, then T is Φ-summing.

Proof: Let ψ: $[0, 1] \to E^*$ be as in Definition 2.1, and $\{x_1, \dots, x_N\} \subseteq E$. Then:

$$\sum_1^N \Phi \| Tx_n \|_\Phi = \sum_1^N \Phi \left[\Phi^{-1} \int_0^1 \Phi | \langle x_n, \psi(t) \rangle | \, \mathrm{d}m(t) \right.$$

$$\leq \sum_1^N \int_0^1 (\| \psi(t) \| + 1) \Phi \left| \left\langle x_n, \frac{\psi(t)}{\| \psi(t) \|} \right\rangle \right| \mathrm{d}m(t)$$

$$\leq (\|f\|_1 + 1) \sup_{\|x^*\| \leq 1} \sum_1^N \Phi | \langle x_n, x^* \rangle | .$$

Q.E.D.

Before we state the next theoerem, we should remark that the topology on L^Φ generated by the gauget $\|f\|_\Phi = \Phi^{-1} \int \Phi |f| \, \mathrm{d}m$, is equivalent to the topology generated by the metric $|||f|||_\Phi = \int \Phi |f| \, dm$. Consequently, the bounded sets in both topologies coincide.

Theorem 2.4. Let $T \in L(E, F)$ such that $T^* \in \Pi^\Phi(F^*, E^*)$. If F has the metric approximation property, then for any continuous linear map γ: $F \to L^\Phi$, the map γT is Φ-decomposable.

Proof: First, we claim that there exists an $M > 0$ such that for all $x_1, x_2, \dots$ $\dots, x_n \in E$, $\|x_i\| \leq 1$ and for all measurable disjoint sets $A_1, \dots, A_n$ in $[0, 1]$ we have

$$\sum_{i=1}^n \int_{A_i} \Phi | \gamma T(x_i)(t) | \, \mathrm{d}t \leq M . \qquad (*)$$

By the remark preceeding the theorem and the assumption that F has the metric approximation property, it is enough to prove (*) for operators $\gamma = \sum_{i=1}^k y_i' \otimes 1_{B_i}$, $y_i' \in F^*$ and B_i measurable in $[0, 1]$. One can take B_i to be disjoint of equal length and $\bigcup_{i=1}^k B_i = [0, 1]$.

Let $\gamma = \sum_{j=1}^k y_i' \otimes 1_{B_i}$, B_i disjoint in $[0,1]$ and $m(B_i) = \frac{1}{k}$, $y_i' \in F^*$, for $i = 1, \dots, k$. If $x_1, \dots, x_n \in E$, with $\|x_i\| \leq 1$ and if $A_1, \dots, A_n$ are disjoint measurable subsets in $[0, 1]$, then:

$$\sum_{i=1}^n \int_{A_i} \Phi | \gamma T(x_i)(t) | \, \mathrm{d}m(t)$$

$$= \sum_{i=1}^n \int_{A_i} \Phi \left| \sum_{j=1}^k \langle Tx_i, y_j \rangle 1_{B_j}(t) \right| \mathrm{d}m(t)$$

$$\leq \sum_{i=1}^n \sum_{j=1}^k \Phi | \langle Tx_i, y_j \rangle | \, m(B_j \cap A_i) \text{ (since } \Phi \text{ is subadditive)}$$

$$\leq \sum_{j=1}^{k} \Phi \| T^* y_j' \| \cdot \sum_i m(B_j \cap A_i)$$

$$\leq \sum_{j=1}^{k} \frac{1}{k} \Phi \| T^* y_j' \| \quad \text{(since } A_i^{'s} \text{ are disjoint)}$$

$$\leq \lambda \sup_{\substack{\|x^*\| \leq 1 \\ x^* \in F}} \sum_{j=1}^{k} \Phi |\langle y_j', x^* \rangle| \, m(B_j) \quad \text{since } T^* \in \Pi^{\Phi}(F^*, E^*))$$

$$= \lambda \sup_{\|x^*\| \leq 1} \int \Phi |\gamma x^*(t)| \, \mathrm{d}m(t).$$

Since γ is continuous, by the remark preceeding the theorem we get $\sup\limits_{\|x^*\| \leq 1} \int \Phi |\gamma x^*(t)\, \mathrm{d}m(t) \leq M$ for some $M > 0$, and (*) is proved.

It follows from (*) that the image of the unit ball of E under γT is bounded in the lattice L^{Φ}. If $g \in L^{\Phi}$ such that $\gamma T(x) \leq g$ for all $x \in E$, $\|x\| \leq 1$, then the function $\theta(t) = \gamma Tx(t)/g(t)$ if $g(t) \neq 0$ and $\theta(0) = 0$, is an element of L^{∞}. Consequently, the linear map

$$S\colon E \to L^{\infty},$$

$$S(x) = \gamma Tx \,|\, g$$

is continuous and $\|S\| \leq 1$. Hence, by the lifting theorem, there exists $Q\colon [0, 1] \Rightarrow (L^{\infty})^*$ such that the function $\langle Q(t), f \rangle$ is m-measurable a.e., for all $f \in L^{\infty}$, and $f(t) = \langle Q(t), f \rangle$ a.e. Further $\|Q(t)\| = 1$ for all $t \in [0, 1]$. Now, consider the function $\psi\colon [0, 1] \to E^*$ defined by $\psi(t) = g(t) \cdot S^*(Q(t))$. It is not difficult to see that ψ is the function needed for γT to be Φ-decomposable, noting that $g \in L^{\infty} \subseteq L^{\Phi}$. Q.E.D.

Before we prove the next result, we need the following two lemmas:

Lemma 2.5. Let $T\colon L^{\Phi} \to L^2$ be continuous linear operator. Then $\|Tf\| \leq \leq \lambda \int \Phi |f(t)| \, \mathrm{d}m(t)$ for all $f \in L^{\Phi}$ for which $\int \Phi |f(t)| \, \mathrm{d}m(t) = |||f|||_{\Phi} \leq 1$.

Proof: First we prove it for $f \in L^{\Phi}$, $|||f|||_{\Phi} = 1$. If the inequality $\|Tf\| \leq \leq \lambda |||f|||_{\Phi}$ is not true, then we can find a sequence (f_n) such that $|||f_n|||_{\Phi} = 1$ but $\|Tf_n\| > n|||f_n|||_{\Phi}$. Then the sequence $\frac{f_n}{n} \to 0$ in L^{Φ}, but $\left\| T\frac{f_n}{n} \right\| > 1$, which contradicts the continuity of T.

Now, let $f \in L^{\Phi}$. $|||f|||_{\Phi} < 1$. Then one can find an $\alpha > 1$ such that $|||\alpha f|||_{\Phi} = 1$. Hence

$$\|Tf\| = \frac{1}{\alpha} \|T\alpha f\|$$

$$\leq \frac{\lambda}{\alpha} |||\alpha f|||_\Phi$$

$$\leq \lambda \frac{\alpha+1}{\alpha} |||f|||_\Phi$$

$$\leq 2\lambda |||f|||_\Phi.$$

Q.E.D.

It should be remarked that for every $r > 0$ there exists $\lambda > 0$ such that $\| Tf \| \leq \leq \lambda |||f|||_\Phi$ for all $f \in L^\Phi$, $|||f|||_\Phi \leq r$.

Lemma 2.6. Let $T) \; L^2 \to L^\Phi$ be p-summing operator. Then ST: $L^2 \to L^2$ is p-summing for continuous operators S: $L^\Phi \to L^2$.

Proof: Using Lemma 2.5 and the arguement in the proof of Theorem 1.6, the result follows. Q.E.D.

Now we prove:

Theorem 2.7. Let ϕ be any modulus function. Then $\Pi^\Phi(L^2, L^2) \subseteq \Pi^2(L^2, L^2)$.

Proof: Let T: $L^2 \to L^2$ be Φ-summing operator. By Theorem 2.4: γT^*: $L^2 \to L^2 \to L^\Phi$ is Φ decomposable for all continuous linear operators γ: $L^2 \to L^\Phi$. In particular, we can choose $\gamma(f) = \int f(t) \, dx_t$, [2, 5], where (x_t) is a symmetric stable process on ([0, 1], m) with exponent 2. This makes γ an isomorphic embedding of L^2 into $L^\Phi$and also into L^0. Hence γT^*: $L^2 \to L^0$ is zero decomposable. Using Theorem 3 in [5], we get T^*: $L^2 \to L^2$ is zero summing. By Lemma 2.6, γT^*: $L^2 \to L^0$ is zero decomposable. By another application of Theorem 3 in [5], we get T: $L^2 \to L^2$ is zero-summing. However, every zero-summing map is 2-summing, [5]. Hence $T \in \Pi^2(L^2, L^2)$.

Theorem 2.8. For any modulus function Φ, $\Pi^2(L^2, L^2) \subseteq \Pi^\Phi(L^2, L^2)$.

Proof: Let T: $L^2 \to L^2$ be 2-summing operator. If γ is the isomorphic embedding of L^2 into L^Φ as in Theorem 2.7, then using Theorem 3 in [5], we get:

$$\gamma T: L^2 \to L^2 \to L^\Phi$$

is Φ-decomposable. By Theorem 2.3, γT is Φ-summing. Using Lema 2.5, we get T: $L^2 \to L^2$ is Φ-summing. Q.E.D.

REFERENCES

1. Apiola, H.: Duality between spaces of p-summable sequences, (p, q) summing operators, and characterization of nuclearity, Math. Ann. 219 (1976), 53—64.
2. Doob, J. L.: Stochastic processes, 1953.
3. Grothendieck, A.: Produits tensoriels topologiques et espaces nucléaires, Mem. Amer. Math. Soc. 16, Providence, 1955.
4. Köthe, G.: Topological vector spaces, Berlin—New York, 1969.

5. Kwapien, S.: On a theorem of L. Schwartz and its applications to absolutely summing operators. Studia Math. 38 (1970), 193—210.
6. Pietsch, A.: Operator ideals, North Holand Comp. 1980.
7. Royden, J. L.: Real Analysis, McMillan Comp. Toronto, 1968.

Authors' addresses:
Khalil R. and Deeb W.
Kuwait University
Department of mathematics
Kuwait

Received: 1. 10. 1985

SÚHRN

Φ-SUMAČNÉ OPERÁTORY V BANACHOVÝCH PRIESTOROCH

R. Khalil — W. Deeb

Nech E a F sú Banachove priestory. Nech Φ: $[0, \infty) \to [0, \infty)$ je spojitá, subaditívna a rastúca funkcia s vlastnosťou $\Phi(0) = 0$. K nim sa definuje Φ-sumačný operátor z E do F. Priestor všetkých takých operátorov značíme ako $\Pi^{\Phi}(E, F)$.

V práci sa študuje priestor $\Pi^{\Phi}(E, F)$.

РЕЗЮМЕ

Φ-СУММИРУЮЩИЙ ОПЕРАТОР В ПРОСТРАНСТВАХ БАНАХА

П. Калил — В. Диб

Пусть E и F — пространства Банаха. Пусть Φ: $[0, \infty) \to [0, \infty)$ непрерывное, субадитивное и возрастающее отображение удовлетворящее свойству $\Phi(0) = 0$. При помощи этих отображений определяется Φ-суммирующий оператор с пространства E до F. $\Pi^{\Phi}(E, F)$ обозначает пространство всех таких операторов.

В работе изучается пространство $\Pi^{\Phi}(E, F)$.