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ON d-VARIATION AND d-SEMIVARIATION OF SET FUNKTIONS

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1 Introduction

Variation of set-functions taking values in normed spaces has been used extensively in the generations of measure, or for that matter, outer measure functions. Dinculeanu [2] used non-negative extended real valued functions as well as functions having values in an arbitrary normed linear space for the purpose. Pal [3] used non-negative extended real valued function to define what he called a d-variation of μ, namely $\bar{\mu}$, to obtain measure extension. Unlike Dinculeanu, Pal [3] obtained $\bar{\mu}$ on $\mathscr{P}(\mathscr{C})$ — a class of sets containing the domain $\mathscr{C}$ of μ by a method which is different from that used in [2]. Subsequently, Biswas [1] utilised Pal's construction to give some properties of $\bar{\mu}$.

In this note we take advantage of Pal's construction to define variations of set function taking values in normed spaces. We term this variation, following Pal, d-variation. We have also made a comparative study of d-variation and variation of set function as in [1]. Taking N, to be a Banach lattice we obtain some properties.

We have further utilised the construction of Pal [3] to define a real-valued function with the aid of a class of operator valued set functions. We call this function d-semivariation. This function is found to be dominated by semivariation of operator valued set function as in ([2], Chapter I, § 4). On $\mathscr{P}(\mathscr{C})$, we claim this construction to be new. We have not come across similar construction elsewhere.

2 d-variation of set functions

Definition 2.1. Let R be the set of real numbers. Then $R^+ = \{x/x \geqslant 0\}$ is the positive cone of R.

Definition 2.2 [4]. A vector space N over R, endowed with an order relation

'$\leq$', is called an ordered vector space if the following axioms are satisfied:

$$\text{I)}\ x \leq y \Rightarrow x + z \leq y + z \text{ for all } x, y, z \in N,$$

$$\text{II)}\ x \leq y \Rightarrow \lambda x \leq \lambda y \text{ for all } x, y \in N, \lambda \in R^{+}.$$

A vector lattice N is an ordered vector space (over R) such that $x \vee y = \sup\{x, y\}$ and $x \wedge y = \inf\{x, y\}$ exist for all $x, y \in N$.

Definition 2.3 [4]. Let N be a vector lattice.
A function $\|\cdot\| : N \to R^{+}$ is a seminorm iff

(i) $\|x + y\| \leq \|x\| + \|y\|$
and (ii) $\|\lambda x\| = |\lambda|\ \|x\|$

for all $x, y \in N$ and $\lambda \in R$; $\|\cdot\|$ is a norm iff, in addition, $\|x\| = 0$ implies $x = 0$.

Definition 2.4 [4]. Let N be a vector lattice. A seminorm (norm) $\|\cdot\|$ on N is called a lattice seminorm (norm) if $|x| \leq |y|$ implies $\|x\| \leq \|y\|$ for all $x, y \in N$.

If $\|\cdot\|$ is a lattice norm on N, the pair $(N, \|\cdot\|)$ is called a normed (vector)lattice; if, in addition, $(N, \|\cdot\|)$ is norm complete, it is called a Banach lattice.

Definition 2.5 [4]. A lattice norm $x \to \|x\|$ on a vector latice N is called an L-norm if it satisfies the axiom $\|x + y\| = \|x\| + \|y\|$ for $x, y \in N^{+}$ — the positive cone of N.

$(N, \|\cdot\|)$ is called an L-normed space, and an L-normed Banach lattice is called an abstract L-space (briefly, AL-space).

Let S be a nonvoid set, $\mathscr{C}$ be an arbitrary class of subsets of S with $\phi \in \mathscr{C}$ and N — a normed space. Let $m\colon \mathscr{C} \to N$ be a set function with $m(\varphi) = 0$.

Definition 2.6 [2]. For every set $E \subset S$, the variation $\bar{m}$ of m is defined by

$$\bar{m}(E) = \sup \sum_{i \in I} \|m(A_i)\|,$$

where the supremum is taken for all finite families $\{A_i\}_{i \in I}$ of disjoint sets of $\mathscr{C}$ such that $\bigcup_{i \in I} A_i \subset E$.

The following results are known ([2], Chapter I, § 3).

Theorem A. (i) $\bar{m}(\varphi) = 0$;
(ii) $0 \leq \bar{m}(A) \leq \infty$, $A \subset S$;
(iii) $\|m(A)\| \leq \bar{m}(A)$;
(iv) $\bar{m}$ is increasing;
and (v) $\bar{m}$ is superadditive.

Definition 2.7 [3]. Let $\mathscr{P}(\mathscr{C})$ be the class of sets $E \subset S$ such that $E - A \in \mathscr{C}$ for every $A \in \mathscr{C}$, $A \neq \phi$.

The following results are evident:

(i) If $A - B \in \mathscr{C}$ for every A, $B \in \mathscr{C}$, then $\mathscr{C} \subset \mathscr{P}(\mathscr{C})$;

(ii) if $E \in \mathscr{P}(\mathscr{C})$ is disjoint from some set $A \in \mathscr{C}$ then, $E \in \mathscr{C}$ and

(iii) if $\mathscr{C}$ is a ring (σ-ring) then $\mathscr{P}(\mathscr{C})$ is a ring (σ-ring) containing $\mathscr{C}$.

Definition 2.8. For every $E \in \mathscr{P}(\mathscr{C})$, we define

$$\bar{m}_d(E) = \sup \|m(E - A)\| ,$$

the supremum being taken for all $A \in \mathscr{C}$, $A \subset E$ and $A \neq \phi$ if $E \notin \mathscr{C}$. If there is no $A \in \mathscr{C}$, $A \neq \phi$ when $E \neq \mathscr{C}$, then we put $\bar{m}_d(e) = 0$. The function $\bar{m}_d$ is called the d-variation of m (cf. [3]).

Theorem 2.1. $\bar{m}_d$ has the following properties:

(i) For $E \in \mathscr{P}(\mathscr{C})$, $\bar{m}_d(E) \leq \bar{m}(E)$;

(ii) $\bar{m}_d(\phi) = 0$;

(iii) $\bar{m}_d$ is the smallest of all non-negative set functions μ defined on $\mathscr{P}(\mathscr{C})$ which are non-decreasing and satisfy the inequality

$$\|m(E - A)\| \leq \mu(E - A) \text{ for every } E \in \mathscr{P}(\mathscr{C}) \text{ and } A \in \mathscr{C},\ A \subset E;$$

(iv) if m and μ be two set functions defined on $\mathscr{C}$ and α be a scalar, then

$$\overline{(m + \mu)}_d \leq \bar{m}_d + \bar{\mu}_d \text{ and } (\overline{\alpha m})_d = |\alpha| \bar{m}_d;$$

and (v) $\bar{m}_d$ is monotone.

Proof. (i) For $E \in \mathscr{P}(\mathscr{C})$ and $A \in \mathscr{C}$ such that $A \subset E$, we have $\|m(E - A)\| \leq \leq \bar{m}(E)$. Now, taking supremum for all $A \in \mathscr{C}$, $A \subset E$ we get $\bar{m}_d(E) \leq \bar{m}(E)$.

(ii) We know $\bar{m}(\phi) = 0$ (cf. Th. A) and $\bar{m}_d(\phi) \leq \bar{m}(\phi)$ by (i). Hence $m_d(\phi) = 0$.

(iii) We have $\|m(E - A)\| \leq \mu(E - A) \leq \mu(E)$ for $E \in \mathscr{P}(\mathscr{C})$ and $A \in \mathscr{C}$ such that $A \subset E$. Taking supremum for all $A \in \mathscr{C}$, $A \subset E$ we have $\bar{m}_d(E) \leq \mu(E)$. Hence the result.

(iv) Let $E \in \mathscr{P}(\mathscr{C})$, $E \notin \mathscr{C}$ if $A \in \mathscr{C}$, $A \subset E$; we have

$$\begin{aligned} \|(m + \mu)(E - A\| &= \|m(E - A) + \mu(E - A)\| \leq \\ &\leq \|m(E - A)\| + \|\mu(E - A)\| \leq \\ &\leq \bar{m}_d(E) + \bar{\mu}_d(E). \end{aligned}$$

Taking supremum for all $A \in \mathscr{C}$, $A \subset E$, $A \neq \phi$ we have

$$\overline{(m + \mu)}_d(E) \leq \bar{m}_d(E) + \bar{\mu}_d(E).$$

Next,

$$(\overline{\alpha m})_d(E) = \sup_{\substack{A \in \mathscr{C} \\ A \subsetneq E \\ A \neq \phi \text{ when } E \notin \mathscr{C}}} \|\alpha m(E - A)\| =$$

$$= |\alpha| \sup_{\substack{A \in \mathscr{C} \\ A \subset E \\ A \neq \phi \text{ when } E \notin \mathscr{C}}} \|m(E - A)\| =$$

$$= |\alpha| \bar{m}_d(E).$$

(v) Let E, $F \in \mathscr{P}(\mathscr{C})$, $E \subset F$; for every $A \in \mathscr{C}$, $A \subset E$, we have

$$E - A = F - (F - (E - A),$$

where

$$E - A \in \mathscr{C}, \quad F - (E - A) \in \mathscr{C}.$$

Accordingly,

$$\bar{m}_d(E) = \sup_{\substack{A_a \in \mathscr{C} \\ A_a \subset E}} \|m(E - A_a)\| \leq \sup_{\substack{F_a \in \mathscr{C} \\ F_a \subset F}} \|(F - F_a)\| = \bar{m}_d(F).$$

Theorem 2.2. Let N be an L-normed Banach lattice, $m: \mathscr{C} \to N^+$ an additive set function. Then $\bar{m}_d$ is superadditive on $\mathscr{P}(\mathscr{C})$.

Proof: For $\varepsilon(>0)$, we can find A, $B \in \mathscr{C}$, $A \subset E$ and $B \subset F$ such that

$$\bar{m}_d(E) - \varepsilon/2 < \|m(E - A)\|, \bar{m}_d(F) - \varepsilon/2 < \|m(F - B)\|.$$

$$\bar{m}_d(E) + \bar{m}_d(F) - \varepsilon < \|m(E - A)\| + \|m(F - B)\| = \|m(E - A) + m(F - B)\|,$$

(since N is an AI — space)

$$= \|m((E - A) \cup (F - B))\| = \|m((E \cup F) - (A \cup B))\| \leq \bar{m}_d(E \cup F).$$

$\varepsilon(\succ 0)$ being arbitrary, $\bar{m}_d(E \cup F) \geqslant \bar{m}_d(E) + \bar{m}_d(F)$. Hence, for every finite family $\{E_i\}_{i \in J}$ of disjoint sets we deduce that

$$\bar{m}_d\left(\bigcup_{i \in J} E_i\right) \geqslant \sum_{i \in J} \bar{m}_d(E_i).$$

If $\{E_i\}_{i \in I}$ be a sequence of disjoint sets of $\mathscr{P}(\mathscr{C})$ with $\bigcup_{i \in I} E_i$ and $\bigcup_{i \in J} E_i \in \mathscr{P}(\mathscr{C})$ for every finite subset $J \subset I$ we have

$$\sum_{i \in J} \bar{m}_d(E_i) \leq \bar{m}_d\left(\bigcup_{i \in J} E_i\right) \leq \bar{m}_d\left(\bigcup_{i \in I} E_i\right), \bar{m}_d$$

is monotone and hence

$$\sum_{i \in I} \bar{m}_d(E_i) \leq \bar{m}_d\left(\bigcup_{i \in I} E_i\right).$$

This proves the theorem.

Theorem 2.3. Let N be an L — normed Banach lattice; $\mathscr{C}$ be a σ-ring of sets, and $m: \mathscr{C} \to N^+$ be a σ-additive function. Then $\bar{m}_d$ is σ-additive on $\mathscr{P}(\mathscr{C})$.

Proof. Let $\{E_i\}$ be a disjoint sequence of sets in $\mathscr{P}(\mathscr{C})$. Then $\bigcup_{i=1}^{\infty} E_i \in \mathscr{P}(\mathscr{C})$. For $A \subset \bigcup_{i=1}^{\infty} E_i$ and $A \in \mathscr{C}$, $A \neq \Phi$, we have

$$\left\| m\left(\left(\bigcup_{i=1}^{\infty} E_i\right) - A\right)\right\| = \left\| m\left(\bigcup_{i=1}^{\infty} (E_i - A)\right)\right\| = \left\| \sum_{i=1}^{\infty} m(E_i - A)\right\| \leq$$

$$\leq \sum_{i=1}^{\infty} \|m(E_i - A)\| \leq \sum_{i=1}^{\infty} \bar{m}_d(E_i).$$

Accordingly, taking supremum for all $A \in \mathscr{C}$, $A \subset \bigcup_{i=1}^{\infty} E_i$, $A \neq \phi$ we have

$$\bar{m}_d\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \bar{m}_d(E_i). \tag{2.4.1}$$

By the preceding theorem $\bar{m}_d$ is superadditive, so

$$\bar{m}_d\left(\bigcup_{i=1}^{\infty} E_i\right) \geqslant \sum_{i=1}^{\infty} \bar{m}_d(E_i). \tag{2.4.2}$$

From (2.4.1) and (2.4.2), $\bar{m}_d\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \bar{m}_d(E_i)$.

Remark: $\bar{m}_d$ is a measure on $\mathscr{P}(\mathscr{C})$ if m is σ-additive on the σ-ring $\mathscr{C}$.

3 Semi *d*-variation

Let S, $\mathscr{C}$, $\mathscr{P}(\mathscr{C})$ be defined as in §2. Let x, y be two normed spaces, $\mathscr{B}(x, y)$ be the Banach space of all bounded linear operator $f: x \to y$ and $m: \mathscr{C} \to \mathscr{B}(x, y)$ be a set function such that $m(\phi) = 0$.

Definition 3.1 [1]. For every $E \subset S$, the semivariation $\tilde{m}$ of m is defined by

$$\tilde{m}(E) = \sup \left\| \sum_{i \in I} m(A_i) x_i \right\|,$$

where the supremum is taken for all finite families $\{A_i\}_{i \in I}$ of disjoint sets of $\mathscr{C}$ contained in E, and for all finite families $\{x_i\}_{i \in I}$ of elements of X such that $\|x_i\| \leq 1$ for each $i \in I$.

The following results are known ([2] Chapter 1, §4).

Theorem B.

(i) $0 \leq \tilde{m}(E) \leq \infty$, $E \subset S$.

(ii) $\|m(E)\| \leq \tilde{m}(E)$, $\leq \bar{m}(E)$,for every $E \in \mathscr{C}$.

Definition 3.2. For every $E \in \mathscr{P}(\mathscr{C})$ we define $\tilde{m}_d(E) = \sup \|m(E - A)x\|$, where the supremum is taken for all $A \in \mathscr{C}$, $A \subset E$ and $A \neq \phi$ when $E \notin \mathscr{C}$, and for all $x \in X$ such that $\|x\| \leq 1$.

We call $\tilde{m}_d(E)$ the d-semivariation of m.

Theorem 3.1. Let $\tilde{m}_d$ be the d-semivariation of m: $\mathscr{C} \to \mathscr{B}(x, y)$. The $\tilde{m}_d$ has the following properties:

(i) For every $E \in \mathscr{P}(\mathscr{C})$, $0 \leq \tilde{m}_d(E) \leq \infty$ and $\tilde{m}_d(E) \leq \tilde{m}(E)$;

(ii) $\tilde{m}_d(\phi) = 0$;

(iii) if m, and n: $\mathscr{C} \to \mathscr{B}(x, y)$ be two set functions with $m(\phi) = n(\phi) = 0$, and 'a' be a scalar, then $(\tilde{m} + n)_d \leq \tilde{m}_d + \tilde{n}_d$ and $(a\tilde{m})_d = |a|\tilde{m}_d$;

(iv) if $m_{\mathscr{A}}$ be the restriction of m to a subclass $\mathscr{A} \subset \mathscr{C}$ with $\phi \in \mathscr{A}$, then $(\tilde{m}_{\mathscr{A}})_d \leq \tilde{m}_d$;

(v) $\tilde{m}_d(E) \leq \tilde{m}_d(F)$, $E \subset F$, E, $F \in \mathscr{P}(\mathscr{C})$;

(vi) $\tilde{m}_d = \tilde{m}_d(E)$ for every $E \in \mathscr{P}(\mathscr{C})$;

(vii) $\|m(E)\| \leq \tilde{m}_d(E)$, $E \in \mathscr{C}$;

and (viii) $\|m(E)\| \leq \tilde{m}_d(E) \leq \tilde{m}(E) \leq \bar{m}(E) = |m|(E)$, for every $E \in \mathscr{C} \subset \mathscr{P}(\mathscr{C})$.

Proof.

The first part of (i) is clear from the definition.

Let $E \in \mathscr{P}(\mathscr{C})$, $A \in \mathscr{C}$ such that $A \subset E$. Then for every $x \in X$, $\|x\| \leq 1$ we have

$$\|m(E - A)x\| \leq \sup_{i \in I} \left\| \sum_i m(A_i)x_i \right\| = \tilde{m}(E),$$

where the supremum is taken for all finite disjoint sequences $\{A_i\}_{i \in I}$ of sets of $\mathscr{C}$ such that $\bigcup_{i \in I} A_i \subset E$ and for all finite sequence $\{x_i\}_{i \in I}$ of elements of X with $\|x_i\| \leq 1$.

Taking supremum for all $A \in \mathscr{C}$, $A \subset E$ and for all $x \in X$, $\|x\| \leq 1$, we have, $\tilde{m}_d(E) \leq \tilde{m}(E)$.

(ii) $\tilde{m}_d(\phi) \leq \tilde{m}(\phi) \leq \bar{m}(\phi) = 0$, [Th. B (i)]. So, by (i) $\tilde{m}_d(\phi) = 0$.

(iii) Let $E \in \mathscr{P}(\mathscr{C})$, $A \in \mathscr{C}$, $A \subset E$. Then for $x \in X$, $\|x\| \leq 1$, we have

$$\begin{aligned} \|(m + n)(E - A)x\| &= \|m(E - A)x + n(E - A)x\| \leq \\ &\leq \|m(E - A)x\| + \|n(E - A)x\| \leq \\ &\leq \tilde{m}_d(E) + \tilde{n}_d(E). \end{aligned}$$

Taking supremum for all $A \in \mathscr{C}$, $A \subset E$ and for all $x \in X$, $\|x\| \leq 1$, we have

$$(\tilde{m} + n)_d(E) \leq \tilde{m}_d(E) + \tilde{n}_d(E).$$

Next,

$$(a\tilde{m})_d(E) = \sup_{\substack{A \in \mathscr{C} \\ A \subset E \\ A \neq \phi, \text{ when } E \notin \mathscr{C} \\ \|x\| \leq 1}} \|am(E - A)\chi\| =$$

$$= |\alpha| \sup_{\substack{A \in \mathscr{C},\, A \subset E \\ a \neq \phi, \text{ when } E \notin \mathscr{C} \\ \|x\| \leq 1}} \|m(E - A)x\| = |\alpha| \tilde{m}_d(E).$$

(iv) For $E \in \mathscr{P}(\mathscr{C})$ and for any $x \in X$, $\|x\| \leq 1$ we have, since $\mathscr{A} \subset \mathscr{C}$,

$$\sup_{\substack{A \in \mathscr{A} \\ A \subset E}} \|m(E - A)x\| \leq \sup_{\substack{A \in \mathscr{C} \\ A \subset E}} \|m(E - A)x\| \leq \tilde{m}_d(E).$$

Taking supremum for all $x \in X$, $\|x\| \leq 1$ we get

$$(\tilde{m}_{\mathscr{A}})_d(E) \leq \tilde{m}_d(E).$$

(v) Let E, $F \in \mathscr{P}(\mathscr{C})$ and $E \subset F$. For $A \in \mathscr{C}$, $A \subset E \subset F$ we have$E - A = F - - (F - (E - A))$, $F - (F - (E - A)) \in \mathscr{C}$ and so for $x \in X$, $\|x\| \leq 1$ we have

$$\|m(E - A)x\| \leq \sup_{\substack{F_a \in \mathscr{C} \\ F_a \subset E \\ F_a \neq \phi, \text{ when } E \notin \mathscr{C}}} \|m(F - F_a)x\| \leq \tilde{m}_d(F).$$

Taking supremum for all $A \in \mathscr{C}$, $A \subset E$ and for all $x \in X$, $\|x\| \leq 1$ we get

$$\tilde{m}_d(E) \leq \tilde{m}_d(F).$$

(vi) For $E \in \mathscr{P}(\mathscr{C})$ and $A \in \mathscr{C}$, $A \subset E$ we have

$$\|m(E - A)\| = \sup_{\substack{x \in X \\ \|x\| \leq 1}} \|m(E - A)x\| \leq \tilde{m}_d(E).$$

Now, taking supremum for all $A \subset E$, $A \in \mathscr{C}$ we have

$$\bar{m}_d(E) \leq \tilde{m}_d(E). \tag{3.1.1}$$

Also for any $x \in X$, $\|x\| \leq 1$ and $A \subset E$, $A \in \mathscr{C}$, we have

$$\|m(E - A)x\| \leq \|m(E - A)\| \leq \bar{m}_d(E).$$

Taking supremum for all $A \in \mathscr{C}$, $A \subset E$ and $x \in X$, $\|x\| \leq 1$,

$$\tilde{m}_d(E) \le \bar{m}_d(E). \tag{3.1.2}$$

From (3.1.1) and (3.1.2), $\tilde{m}_d(E) = \bar{m}_d(E)$.

(vii) For any $E \in \mathscr{C}$ we have

$$\|m(E)\| = \sup_{\substack{\|x\| \le 1 \\ x \in X}} \|m(E)x\|, \tag{3.1.3}$$

Also for $E \in \mathscr{C}$ and for a $x \in X$, $\|x\| \le 1$,

$$\|m(E)x\| \le \sup_{\substack{A \subset E \\ A \in \mathscr{C}}} \|m(E - A)x\| \le \tilde{m}_d(E). \tag{3.1.4}$$

Taking supremum over $x \in X$, $\|x\| \le 1$ we have from (3.1.3) and (3.1.4), $\|m(E)\| \le \tilde{m}_d(E)$.

(viii). We have for $E \in \mathscr{P}(\mathscr{C})$, $\tilde{m}_d(E) \le \tilde{m}(E)$, by (i) and $\tilde{m}_d(E) = \bar{m}_d(E)$, by (vi).

Hence $\tilde{m}_d(E) = \bar{m}_d(E) \le \tilde{m}(E)$ for $E \in \mathscr{P}(\mathscr{C})$, (3.1.5)

and, a fortiori, for $E \in \mathscr{C}$.

But for $E \in \mathscr{C}$ we have $\|m(E)\| \le \tilde{m}_d(E)$, [by (vii)], (3.1.6)

and $\tilde{m}(E) \le \bar{m}(E) = |m|(E)$, (by Th. B). (3.1.7).

Hence from (3.1.5), (3.1.6) and (3.1.7) we have

$$\|m(E)\| \le \tilde{m}_d(E) = \bar{m}_d(E) \le \tilde{m}(E) \le \bar{m}(E) = |m|(E)$$

for every $E \in \mathscr{C}$. This proves the theorem.

Theorem 3.2. Let $\mathscr{C}$ be a σ-ring of sets and m be σ-additive. Then $\tilde{m}_d$ is subadditive on $\mathscr{P}(\mathscr{C})$.

Proof. Let $\{E_i\}$ be a sequence of sets in $\mathscr{P}(\mathscr{C})$. If m is additive, we suppose that $E_i = \phi$ except for a finite number of indices. We put $E'_1 = E_1$ and $E'_n = E_n - \bigcup_{i=1}^{n-1} E_i$, $n = 2, 3, \ldots$. The sets $\{E'_i\}$ are mutually disjoint and belong to $\mathscr{P}(\mathscr{C})$ and $\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} E'_i$. Since $E'_i \subset E_i$ and is non-decreasing on $\mathscr{P}(\mathscr{C})$, we have $\tilde{m}_d(E'_i) \le \tilde{m}_d(E_i)$, for each i. Now for any $A \in \mathscr{C}$, $A \ne \phi$ and $A \subset E'_i$ we have

$$\left(\bigcup_{i=1}^{\infty} E'_i\right) - A = \bigcup_{i=1}^{\infty} (E'_i - A),$$

and hence $m\left(\left(\bigcup_{i=1}^{\infty} E'_i\right) - A\right) = \sum_{i=1}^{\infty} m(E'_i - A)$, since $E'_i - A \in \mathscr{C}$.

Therefore for any $x \in X$, $\|x\| \le 1$,

$$\left\| m\left(\left(\bigcup_{i=1}^{\infty} E'_i\right) - A\right)x \right\| = \left\| \left\{\sum_{i=1}^{\infty} m(E'_i - A)\right\} x \right\| = \left\| \sum_{i=1}^{\infty} m(E'_i - A)x \right\| \le$$

$$\leq \sum_{i=1}^{\infty} \|m(E_i' - A)x\| \leq \sum_{i=1}^{\infty} \tilde{m}_d(E_i').$$

Now taking supremum for all $A \in \mathscr{C}$, $A \subset \bigcup_{i=1}^{\infty} E_i'$ and for all $x \in X$, $\|x\| \leq 1$ we get

$$\tilde{m}_d\left(\bigcup_{i=1}^{\infty} E_i'\right) \leq \sum_{i=1}^{\infty} \tilde{m}_d(E_i').$$

Hence,

$$\tilde{m}_d\left(\bigcup_{i=1}^{\infty} E_i\right) = \tilde{m}_d\left(\bigcup_{i=1}^{\infty} E_i'\right) \leq \sum_{i=1}^{\infty} \tilde{m}_d(E_i') \leq \sum_{i=1}^{\infty} \tilde{m}_d(E_i).$$

Hence the theorem.

REFERENCES

1. Biswas, Arati.: A note on extension of measures, Ind. J. Mechanics and Mathematics. Vol. XIX, No. 2, pp. 1—6.
2. Dinculeanu, N.: Vector Measures, Pargamon Press, 1967.
3. Pal, S.: Extensions of measures, Universitas Comeniana Acta Facultatis Rerum Naturalium Universitatis Comenianae MATHEMATICA XXXIII (1977), pp. 59—61.
4. Schaefer, H. H.: Banach lattice and positive operators, Springer Verlag, 1974.

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SÚHRN

O d-VARIÁCIÁCH A d-POLOVARIÁCIÁCH MNOŽINOVÝCH FUNKCIÍ

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Systematicky sa študujú d-variácie a d-polovariácie množinových funkcií s hodnotami v normovanom priestore, resp. v L-normovanom Banachovom zväze.

РЕЗЮМЕ

О *D*-ВАРИАЦИЯЦХ И *d*-ПОЛУВАРИАЦИЯХ МНОЖЕСТВЕННЫХ ФУНКЦИЙ

С. К. Кунду — К. Н. Баумик, Индия

Систематически изучаются *d*-вариации и *d*-полувриации функций множеств со значениями в нормированном пространстве, и в частности, в *L*-нормированной решетке Банаха.