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ON THE DEGREE OF APPROXIMATION OF FUNCTIONS OF GENERALIZED BOUNDED VARIATION

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1. Let f be a periodic function with period 2π and Lebesgue integrable on $[-\pi, \pi]$. Let

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx). \tag{1.1}$$

We assume that

$$f(x) = \frac{1}{2}\{f(x+0) + f(x-0)\}$$

and write

$$g_x(t) = f(x+t) + f(x-t) - 2f(x).$$

Let h a real-valued function on an interval $[a, b]$, $\Lambda = \{\lambda_n\}$ a non-decreasing sequence of positive numbers such that $\sum_1^{\infty} \frac{1}{\lambda_n} = \infty$. The function h is said to be of Λ-bounded variation ($f \in \Lambda BV$) [3] on $[a, b]$ if there exists a positive constant M such that

$$\sum_{k=1}^{n} |h(b_k) - h(a_k)| / \lambda_k \leq M$$

for every sequence of non-overlapping intervals $[a_k, b_k]$, $k = 1, 2, \ldots, n$ contained in $[a, b]$. If $h \in \Lambda BV[a, b]$, the Λ-variation of h is defined as

$$V_\Lambda(h, [a, b]) = \sup \left\{ \sum_{k=1}^{n} |h(b_k) - h(a_k)| / \lambda_k ; \bigcup_{k=1}^{n} [a_k, b_k] \subseteq [a, b] \right\}.$$

When $\Lambda = \{n^\beta\}$, we write $h \in \beta BV$.

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In what follows we assume that $\frac{\lambda_k}{k^{1+\alpha}}$ is non-increasing and, for fixed n, $H(t)$ is a continuous non-increasing function on $(0, \pi]$ such that

$$H(t) = \frac{\lambda_k}{t^{1+\alpha}} \text{ for } t = \frac{k\pi}{n+1}, \quad k = 1, 2, \dots, n+1, \ |\alpha| < 1 .$$

We use C_1, C_2, ... etc. to denote positive constants.

First we prove the following general theorem.

Theorem. Let $k_n(x)$ be even, 2π periodic and Lebesgue integrable on $[-\pi, \pi]$ satisfying the following conditions:

$$\frac{2}{\pi}\int_0^\pi k_n(t)\, dt = 1 , \tag{1.2}$$

$$|k_n(t)| = C_1 n , \quad 0 < t < \pi , \tag{1.3}$$

$$\int_{\frac{k\pi}{n+1}}^{\frac{(k+1)\pi}{n+1}} |k_n(t)|\, dt \le \frac{C_2}{k^{1+\alpha}}, \quad k = 1, 2, \dots, n, \ |\alpha| < 1 , \tag{1.4}$$

and

$$\left|\int_t^\pi k_n(u)\, du\right| \le \frac{C_3}{(nt)^{1+\alpha}}, \quad 0 < t \leqslant \pi . \tag{1.5}$$

Let $f \in \Lambda BV$ and let $V(t)$ denote the Λ-variation of $g_x(t)$ on $[0, t]$, then

$$|K_n(f, x) - f(x)| \le \frac{C_4}{(n+1)^{1+\alpha}}\left\{\lambda_{n+1} V(\pi) + \pi^{1+\alpha} \sum_{v=0}^{n-1} V(a_v)(H(_{v+1}) - H(a_v))\right\},$$

where

$$C_4 = C_1 + 2^{1+\alpha}\frac{C_2}{\pi} + 2^{1+\alpha}\frac{C_3}{\pi^{2+\alpha}},$$

$$\frac{\pi}{n+1} = a_n < a_{n-1} < \dots < a_0 = \pi \text{ and } K_n(f, x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)k_n(t)\, dt .$$

2. The following lemma is required for the proof of our theorem.

Lemma [2]. Let f be a bounded, measarable and 2π-periodic function such that

$$\lim_{t \to 0} (f \pm t) = f(x \pm 0)$$

exist and $f(x) = \frac{1}{2}\{f(x+0) + f(x-0)\}$. Let $k_n(x)$ satisfy the conditions of our theorem.

Then

$$|K_n(f, x) - f(x)| \leq C_4 \sum_{k=0}^{n} \frac{1}{(k+1)^{1+\alpha}} \text{osc.}\,(g_x, I_{k,n}),$$

where

$$g_x(t) = f(x+t) + f(x-t) - 2f(x),\ I_{k,n} = \left[\frac{k\pi}{n+1}, \frac{(k+1)\pi}{n+1}\right] \quad k = 0, 1, 2, \ldots, n$$

and

$$\text{osc.}\,(g_x, [a, b]) = \sup\{|g_x(t) - g_x(t')|;\ t, t' \in [a, b]\}.$$

3. Proof of the teorem. Let

$$M_k = \sum_{j=0}^{k} \frac{1}{\lambda_{j+1}} \text{osc.}\,(g_x, I_{j,n}).$$

We write $M(t) = M_{\left[\frac{(n+1)t}{\pi}\right]-1}$ for $\frac{\pi}{n+1} \leq t < \pi$, then clearly

$$M(t) = M_k \ \text{for}\ \frac{(k+1)\pi}{n+1} \leq t < \frac{(k+2)\pi}{n+1}, \quad k = 0, 1, \ldots n-1.$$

Applying Abel's partial summation formula

$$\sum_{k=0}^{n} \frac{l}{(k+1)^{1+\alpha}} \text{osc.}\,(g_x, I_{k,n}) =$$

(3.1)

$$= \frac{\lambda_{n+1} M_n}{(n+1)^{1+\alpha}} + \sum_{k=0}^{n-1} M_k \cdot \left(\frac{\lambda_{k+1}}{(k+1)^{1+\alpha}} - \frac{\lambda_{k+2}}{(k+2)^{1+\alpha}}\right).$$

Now the last sum is

$$= -\left(\frac{\pi}{n+1}\right)^{1+\alpha} \sum_{k=0}^{n-1} \int_{I_{k+1,n}} M(t)\,\mathrm{d}H(t)$$

$$= -\left(\frac{\pi}{n+1}\right)^{1+\alpha} \int_{\frac{\pi}{n+1}}^{\pi} M(t)\,\mathrm{d}H(t)$$

$$= -\left(\frac{\pi}{n+1}\right)^{1+\alpha} \sum_{v=0}^{n-1} \int_{a_{v+1}}^{a_v} M(t)\,\mathrm{d}H(t)$$

$$\leq \left(\frac{\pi}{n+1}\right)^{1+\alpha} \sum_{v=0}^{n-1} M(a_v)(H(a_{v+1}) - H(a_v)).$$

Since $M_n \leq V(\pi)$ and $M(t) \leq V(t)$ it follows that

$$\sum_{k=0}^{n} \frac{1}{(k+1)^{1+\alpha}} \operatorname{osc.}(g_x, I_{k,n}) \leq$$

$$\leq \frac{\lambda_{n+1} V(\pi)}{(n+1)^{1+\alpha}} + \left(\frac{\pi}{n+1}\right)^{1+\alpha} \sum_{v=0}^{n-1} V(a_v)(H(a_{v+1}) - H(a_v))$$

and therefore in view of the above lemma

$$|K_n(f, x) - f(x)| \leq$$

$$\leq \frac{C_4}{(n+1)^{1+\alpha}} \left\{ V(\pi)\lambda_{n+1} + \pi^{1+\alpha} \sum_{v=0}^{n-1} V(a_v)(H(a_{v+1}) - H(a_v)) \right\}.$$

This proves our theorem.

4. For $\Lambda = \{n^\beta\}$, $0 < \beta < 1$,f the above result was obtained by the author [2]. Let $\sigma_n^\alpha(f, x)$ denote the nth (C, α) mean of the series (1.1), $-1 < \alpha \leq 0$, then

$$\sigma_n^\alpha(f, x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) k_n^\alpha(t)\,\mathrm{d}t,$$

where

$$k_n^\alpha(t) = \frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} D_k(t) \text{ and } D_k(t) = \frac{\sin\left(k + \frac{1}{2}\right)t}{2 \sin t/2}.$$

It can be easily shown (see [1], [2]) that $k_n^\alpha(t)$ satisfies the conditions (1.2)—(1.5), thus we decude the following result for (c, α) means:

Corollary. Let $f \in \Lambda BV$, then for $-1 < \alpha \leq 0$,

$$|\sigma_n^\alpha(f, x) - f(x)| \leq$$

$$\leq \frac{C_5}{(n+1)^{1+\alpha}} \left\{ V(\pi)\lambda_{n+1} + \pi^{1+\alpha} \sum_{v=0}^{n-1} V(a_v)(H(a_{v+1}) - H(a_v)) \right\}.$$

When $\alpha = 0$, the above corollary reduces to a result of Waterman[4].

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SÚHRN

RÁD APROXIMÁCIE FUNKCIÍ S OHRANIČENOU ZOBECNENOU VARIÁCIOU

S. M. Mazhar, Kuvajt

V práci je podaný odhad rádu aproximácie 2π-periodickej lebesgueovsky integrovateľnej funkcie f, ktorá má ohraničenú tzv. Λ-variáciu, s použitím postupnosti

$$K_n(f, x) = \pi^{-1} \int_{-\pi}^{\pi} f(x + t) k_n(t) \, dt ,$$

kde $k_n(t)$ má špeciálne vlastnosti.

РЕЗЮМЕ

ПОРЯДОК АППРОКСИМАЦИИ ФУНКЦИИ С ОБОБЩЕННОЙ ОГРАНИЧЕННОЙ ВАРИАЦИЕЙ

С. М. Мазхар, Кувайт

В работе получена оценка порядка аппроксимации 2π-периодической функции f, интегрируемой по Лебегу, которая имеет ограниченную Λ-вариацию. Использована последовательность

$$K_n(f, x) = \pi^{-1} \int_{-\pi}^{\pi} f(x + t) k_n(t) \, dt$$

где $k_n(t)$ имеет определённые свойства.