UNIVERSITAS COMENIANA
ACTA MATHEMATICA UNIVERSITATIS COMENIANAE
LII—LIII—1987

CHARACTERIZATION OF POTENTIALLY MINIMAL PERIODIC ORBITS OF CONTINUOUS MAPPINGS OF AN INTERVAL

ĽUBOMÍR SNOHA, Banská Bystrica

1 Introduction

L. Alsedà, J. Llibre and R. Serra in [1] have characterized periodic orbits of even periods which are potentially minimal (in our terminology). Another but similar problem is solved in [3]. In the present paper which has arisen independently of [1] and [3], the potentially minimal periodic orbits of even periods are characterized, too. Our results are proved without using computers and Straffin's graphs. We think that our proofs are shorter and simpler than those in [1].

Let I be a compact interval on the real line and $C^0(I, I)$ the space of continuous maps from I into itself. Let N denote the set of positive integers. For any $n \in N$ and $f \in C^0(I, I)$ we define f^n inductively by $f^1 = f$ and $f^n = f \circ f^{n-1}$, where the symbol $\circ$ denotes the composition of functions. Let f^0 denote the identity map of I. A point $x \in I$ is said to be a periodic point of f if $f^n(x) = x$ for some $n \in N$. In this case the smallest element of $\{n \in N: f^n(x) = x\}$ is called the period of x. We define the orbit of x to be $\{f^n(x): n \in \{0\} \cup N\}$. If x is a periodic point we say the orbit of x is a periodic orbit, and we define the period of the orbit to be the period of x. The symbol conv A denotes the convex hull of the set $A \subset R$, $f|A$ denotes the restriction of f to the set A and card A is the cardinality of A. All the functions in this paper are assumed to be continuous.

Let $f \in C^0(I, I)$. Consider the following ordering of the positive integers:

$$3 \prec 5 \prec 7 \prec \dots \prec 2.3 \prec 2.5 \prec 2.7 \prec \dots \prec 4.3 \prec 4.5 \prec \dots$$
$$\dots \prec 8.3 \prec \dots \prec \dots \prec 8 \prec 4 \prec 2 \prec 1 .$$

1980 Mathematics subject Classification. Primary 54H20, 26A18.

A. N. Šarkovskii (see [6] or [7]) has proved that if f has a periodic orbit P_n of period n, then for every $n \prec m$ the function f has a periodic orbit P_m of period m with conv $P_n \supset$ conv P_m. This result holds also for continuous real functions defined on an arbitrary connected set on the real line. (Similarly, all results of our paper remain valid for such functions though they are formulated for functions belonging to $C^0(I, I)$.)

It is known that for every n there exists a function f such that f has a periodic orbit of period m if and only if m is not less (in the Šarkovskii sense) than n. Similarly, there exists a function f such that f has a periodic orbit of period m if and only if m is a power of 2 ($1 = 2^0$ is also a power of 2).

P. Štefan in [7] has defined the periodic orbit of an odd period ≥ 3 to be minimal. We extend his definition for all periods.

Definition. A periodic orbit P of $f \in C^0(I, I)$ of period n is said to be a minimal periodic orbit of f iff f has no periodic orbits of periods less (in the Šarkovskii sense) than n.

Suppose that P is a periodic orbit both of g and h, $g|P = h|P$ and P is a minimal periodic orbit of g. These assumptions do not imply that P is a minimal periodic orbit of h. Therefore the following notion seems to be more suitable.

Definition. A periodic orbit P of $f \in C^0(I, I)$ is said to be a potentially minimal periodic orbit ifff there exists a function $F \in C^0(I, I)$ such that $f|P = F|P$ and P is a minimal periodic orbit of F.

It is easy to see that the condition $F \in C^0(I, I)$ can be equivalently replaced by $F \in C^0$ (conv P, conv P). (We will use this fact in the proofs of Theorems 4,5 and 6.)

We will shortly write "PMPO" instead of "potentially minimal periodic orbit".

If P is a periodic orbit of f which is not a PMPO of f, then f must have a periodic orbit Q with period less (in the Šarkovskii sense) than the period of P. Note that f can have such an orbit Q also in the case that P is a PMPO of f. (But, of course, f need not have such an orbit Q in this case.).

The aim of the present paper is to characterize periodic orbits which are potentially minimal. Periodic orbits of periods 1, 2 or 3 are not interesting because they are always potentially minimal. The potentially minimal periodic orbits of odd periods were recently characterized by P. Štefan. He has shown that there are only two "types" of such orbits. (See [7], pp. 243 and 245.)

Theorem 1. (P. Štefan [7].) Let p be a positive integer and let P be a periodic orbit of $f \in C^0(I, I)$ of period $2p + 1$. Then P is potentially minimal if and only if there is a point $b_1 \in P$ such that

$$b_{2p+1} < b_{2p-1} < \ldots < b_3 < b_1 < b_2 < \ldots < b_{2p-2} < b_{2p}$$

or

$$b_{2p} < b_{2p-2} < \dots < b_2 < b_1 < b_3 < \dots < b_{2p-1} < b_{2p+1},$$

where for each i with $1 \leq i \leq 2p$, $f(b_i) = b_{i+1}$, and $f(b_{2p+1}) = b_1$.

The point b_1 or the points b_{2p}, b_{2p+1} will be called the middle point of P or the endpoints of P, respectively. Clearly, b_{2p} and b_{2p+1} are the endpoints of the set conv P.

In the present paper we give a characterization of potentially minimal periodic orbits of even periods. This notion was introduced by L. Block in [2] for periodic orbits of period a power of 2. Now we extend his definition for periodic orbits of all even periods. (A slightly different but equivalent formulation can be found in [4]. For other equivalent formulations see our Lemma 4.)

Definition. Let P be a periodic orbit of $f \in C^0(I, I)$ of period $2^k \cdot (2p+1)$, where $k \in N$ and $p \in \{0\} \cup N$. We say P is simple iff for every positive integers n, r with the property $2^k \cdot (2p+1) = n \cdot r$ and $r = 2^s$ for some $s \in \{0, 1, 2, \dots, k-1\}$, and for every periodic orbit of f^r of the form $\{q_1 < q_2 < \dots < q_n\} \subset P$ we have

$$f^r(\{q_1, \dots, q_{\frac{n}{2}}\}) = \{q_{\frac{n}{2}+1}, \dots, q_n\}.$$

L. Block has proved the following results.

Theorem 2. (L. Block [2].) Let $f \in C^0(I, I)$. Then f has a periodic point whose period is not a power of 2 if and only if f has a periodic orbit of period a power of 2 which is not simple.

Theorem 3. (L. Block [2].) Let $f \in C^0(I, I)$. Suppose f has a periodic orbit of period 2^k for some $k \geq 2$ which is not simple. Then f has a periodic point of period 3.2^{k-2}.

We have immediately the following corollary from this Block's result.

Corollary. Let P be a periodic orbit of $f \in C^0(I, I)$ of period a power of 2. If P is potentially minimal, then P is simple. (This is true also for period 2^1.)

Now we are mitivated for characterizing potentially minimal periodic orbits of even periods. We start with some preliminaries. Our main results are those in Theorems 4, 5, 6.

2 Preliminary results

Let P be a periodic orbit of f containing at least two points. Let $P_{\min}$ and $P_{\max}$ denote the smallest and the largest element of P, respectively. Let

$$U(f) = \{x \in I : f(x) > x\} \text{ and } D(f) = \{x \in I : f(x) < x\}.$$

Let $P_U(f)$ denote the largest element of $P \cap U(f)$ and $P_D(f)$ the smallest element of $P \cap D(f)$.

We will use the following three lemmas proved e.g. by P. Štefan in [7] (see (17), (9) and (10) in [7]).

Lemma 1. (see [7]) Let $f \in C^0(I, I)$ and let f^2 have a periodic orbit of period $n \geq 2$. Then f has a periodic orbit of period $2n$.

Lemma 2. (see [7] or [2]) Let $f \in C^0(I, I)$ and let P be a periodic orbit of f. If f has a fixed point between $P_{\min}$ and $P_U(f)$ (or between $P_D(f)$ and $P_{\max}$), then f has periodic orbits of every period.

Lemma 3. (see [7] or [2]) Let $f \in C^0(I, I)$ and let P be a periodic orbit of f. If $P_D(f) < P_U(f)$, then f has periodic orbits of every period.

We will use the following notation. Let $P = \{a_1 < a_2 < \dots < a_m\}$ be a periodic orbit of f of period m. Let n divide m. For $k = 1, 2, \dots, \frac{m}{n}$ we write

$$P(n, k) = \{a_i\colon i = (k-1)n + 1, (k-1)n + 2, \dots, kn\}.$$

Now let a, b be real numbers and let A, B be subsets of the real line. We denote $f(a) = b$ or $f(A) = B$ by $a \xrightarrow{f} b$ or $A \xrightarrow{f} B$, respectively. Similarly, $A \overset{f}{\leftrightarrow} B$ means $f(A) = B$ and $f(B) = A$. Further let A_i, $i = 1, 2, \dots, r$ be subsets of the real line. We will write

(a) $f \updownarrow \langle A_1, \dots, A_r \rangle$
(b) $f \updownarrow\updownarrow \langle A_1, \dots, A_r \rangle$
(c) $f \uparrow\uparrow \langle A_1, \dots, A_r \rangle$

iff there exists a permutation $(\alpha(1), \alpha(2), \dots, \alpha(r))$ or the set $\{1, 2, \dots, r\}$ such that

(a) $A_{\alpha(1)} \xrightarrow{f} A_{\alpha(2)} \xrightarrow{f} \dots \xrightarrow{f} A_{\alpha(r)}$
(b) $A_{\alpha(1)} \xrightarrow{f} A_{\alpha(2)} \xrightarrow{f} \dots \xrightarrow{f} A_{\alpha(r)} \xrightarrow{f} A_{\alpha(1)}$
(c) $f(A_{\alpha(i)}) \supset A_{\alpha(i+1)}$ for $i = 1, 2, \dots, r-1$ and $f(A_{\alpha(r)}) \supset A_{\alpha(1)}$

respectively. Instead of $\langle A_1, \dots, A_r \rangle$ we will write also $\langle A_i \colon i = 1, 2, \dots, r \rangle$. If $E = \{e_1, e_2, \dots, e_r\}$, then we will write $f \updownarrow E$ instead of $f \updownarrow \langle \{e_1\}, \dots, \{e_r\} \rangle$. Hence there is a difference between $f \updownarrow \langle E \rangle$ and $f \updownarrow E$.

Further if $\{A_1, \dots, A_r\}$ is a family of nonempty pairwise disjoint sets, then every set S consisting of exactly one element from each A_i is called a choice set for this family of sets. Let x, y, z be real numbers. We say x lies between y and z if x lies in the open interval with endpoints y and z.

Lemma 4. Let P be a periodic orbit of f of period $2^k \cdot (2p + 1)$, where $k \in N$ and $p \in \{0\} \cup N$. Then the following three conditions are equivalent:

(i) P is simple

(ii) for every positive integers n, r with the property $2^k \cdot (2p + 1) = n \cdot r$ and $r = 2^t$ for some $t \in \{0, 1, \dots, k\}$ there is $f \updownarrow\updownarrow \langle P(n, 1), \dots, P(n, r) \rangle$

(iii) for every positive integers n, r with the same property as in (ii) the sets $P(n, 1), \ldots, P(n, r)$ are periodic orbits of f^r of period n.

Proof. The implication (ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i). Let our periodic orbit P fulfil the condition (iii) and let $2^k \cdot (2p + 1) = n \cdot r$, where $r = 2^s$ for some $s \in \{0, 1, \ldots, k - 1\}$. Let $Q = \{q_1 < q_2 < \ldots < q_n\} \subset P$ be a periodic orbit of f^r. We are going to prove that $f^r\left(Q\left(\frac{n}{2}, 1\right)\right) = Q\left(\frac{n}{2}, 2\right)$. The set Q is one of the sets $P(n, 1), \ldots, P(n, r)$ because these sets are also periodic orbits of f^r and their union is the whole set P. Further, the integer n is even, since $s < k$. Hence the sets $Q\left(\frac{n}{2}, 1\right)$ and $Q\left(\frac{n}{2}, 2\right)$ are two from the sets $P\left(\frac{n}{2}, 1\right), \ldots, P\left(\frac{n}{2}, 2r\right)$. According to (iii) they are periodic orbits of f^{2r} of period $\frac{n}{2}$. We have the following situation. The set Q is a periodic orbit of f^r and the sets $Q\left(\frac{n}{2}, 1\right)$, $Q\left(\frac{n}{2}, 2\right)$ are periodic orbits of f^{2r}. Now it is easy to see that $f^r\left(Q\left(\frac{n}{2}, 1\right)\right) = Q\left(\frac{n}{2}, 2\right)$. Thus P is simple.

(i) $\Rightarrow$ (ii). Let P be simple. We are going to show that for each $t \in \{0, 1, \ldots, k\}$ the integers $r = 2^t$ and $n = 2^{k-t} \cdot (2p + 1)$ satisfy the condition $f \updownarrow\updownarrow \langle P(n, 1), \ldots, P(n, r)\rangle$. If $t = 0$, then $r = 1$, $P(n, 1) = P$ and clearly $f \updownarrow\updownarrow \langle P \rangle$. Now suppose by induction that the relation

$$f \updownarrow\updownarrow \langle P(n, 1), \ldots, P(n, r)\rangle \tag{1}$$

holds for some $t \in \{0, 1, \ldots, k - 1\}$ and corresponding r, n. Let us take $t^* = t + 1$ and r^*, n^* with $r^* = 2^{t^*}$, $2^k \cdot (2p + 1) = n^* \cdot r^*$. Clearly, $r^* = 2r$, $n^* = \frac{n}{2}$ and n^* is an integer. It suffices to prove

$$f \updownarrow\updownarrow \left\langle P\left(\frac{n}{2}, 1\right), \ldots, P\left(\frac{n}{2}, 2r\right)\right\rangle. \tag{2}$$

From (1) we have that each of the sets $P(n, 1), \ldots, P(n, r)$ is a periodic orbit of f^r of period n. Further, for each $j \in \{1, 2, \ldots, r\}$ there is $P(n, j) = P\left(\frac{n}{2}, 2j - 1\right) \cup P\left(\frac{n}{2}, 2j\right)$. Since P is simple we have

$$P\left(\frac{n}{2}, 2j - 1\right) \overset{f^r}{\leftrightarrow} P\left(\frac{n}{2}, 2j\right), \qquad j = 1, 2, \ldots, r. \tag{3}$$

Now (2) follows from (1) and (3). In fact, (1) implies that for every $x \in P\left(\frac{n}{2}, 1\right) \subset$ $\subset P(n, 1)$ the set $A = \{f^m(x): m = 0, 1, \ldots, r-1\}$ is a choice set for the family of sets $\{P(n, j): j = 1, 2, \ldots, r\}$. For every $j \in \{1, 2, \ldots, r\}$ let $\alpha(j) \in \{2j-1, 2j\}$ be chosen such that $P\left(\frac{n}{2}, \alpha(j)\right) \cap A$ is nonempty. Then the set A is a choice set for the family of sets $\left\{P\left(\frac{n}{2}, \alpha(j)\right): j = 1, 2, \ldots, r\right\}$. By (3), the set $B = \{f^m(x):$ $m = 0, 1, \ldots, 2r-1\}$ is a choice set for the family of sets $\left\{P\left(\frac{n}{2}, i\right): i = 1, 2, \ldots, 2r\right\}$ and $f^{2r}(x) \in P\left(\frac{n}{2}, 1\right)$. Since x was an arbitrary point belonging to $P\left(\frac{n}{2}, 1\right)$, (2) is true Q.E.D.

Lemma 5. Let p be a positive integer and let $P = \{a_1 < a_2 < \ldots < a_{2p+1}\}$ be a PMPO of h of period $2p+1$. Then there exists such a permutation $(\varepsilon(1), \varepsilon(2))$ of the set $\{a_1, a_{2p+1}\}$ that

$$\varepsilon(1) \xrightarrow{h} \varepsilon(2) \xrightarrow{h} a_{p+1}.$$

Moreover, if $p \geq 2$, then the point $h(a_{p+1})$ lies between $\varepsilon(1)$ and a_{p+1}.

Proof. See Theorem 1.

Lemma 6. Let the assumptions of Lemma 5 be satisfied (p is any positive integer). Then each point $z \in \{a_2, a_3, \ldots, a_{2p}\}$ lies between $h(z)$ and $h^2(z)$.

Proof. See Theorem 1.

Lemma 7. Let n be a positive integer and let $P = \{a_1 < a_2 < \ldots < a_{2n}\}$ be a periodic orbit of $f \in C^0(I, I)$ of period $2n$.

(a) If P is a PMPO of f, then

$$P(n, 1) \overset{f}{\leftrightarrow} P(n, 2). \tag{4}$$

(b) If (4) does not hold, then f has a periodic point of period s, where $3 \leq s \leq 2n-1$, and s divides $2n-1$ (consequently, s is odd).

Proof. It suffices to prove (b), since (a) is a consequence of (b). Suppose that (4) does not hold. There are two possibilities.

Case 1. $\{a_1, a_2, \ldots, a_n\} \not\subset U(f)$. Let k be the smallest element of $\{1, 2, \ldots, n\}$ such that $a_k \in D(f)$. If for some $m \in \{k+1, k+2, \ldots, 2n\}$, $a_m \in U(f)$, then f has periodic points of every period (see Lemma 3). Now let $\{a_{k+1}, a_{k+2}, \ldots, a_{2n}\} \subset$ $\subset D(f)$. Since the cardinality of $\{a_1, a_2, \ldots, a_{k-1}\}$ is less than the cardinality of $\{a_{n+1}, \ldots, a_{2n}\}$, there exist $b, \beta \in \{a_{n+1}, \ldots, a_{2n}\}$, $b < \beta$ such that $f(\beta) = b$. Further, there obviously exists $\alpha \in \{a_1, \ldots, a_{k-1}\}$ with $f(\alpha) = a_{2n}$. Thus we have $f^{2n-1}(a_{2n}) = \alpha < a_{2n}$ and $f^{2n-1}(b) = \beta > b$. Hence f^{2n-1} has a fixed point between b and a_{2n}. This point is a periodic point of f of period s, where s divides $2n-1$. If $s=1$, f has periodic points of every period (see Lemma 2). If $s>1$, we have $3 \leq s \leq 2n-1$.

Case 2. $\{a_1, a_2, \ldots, a_n\} \subset U(f)$. In this case, by our assumption that (4) does not hold, there is a point $x \in \{a_1, \ldots, a_n\}$ such that $x < z = f(x) \in \{a_1, \ldots, a_n\}$. Then $f^{2n-1}(a_1) > a_1$ and $f^{2n-1}(z) = x < z$. Hence f^{2n-1} has a fixed point between a_1 and z. Now the proof can be finished in a similar way as in Case 1. Q.E.D.

Remark. We see that our Lemma 7 is stronger than Proposition 9 in [2]. Further, Lemma 10 in [2] is a special case of our Lemma 7.

Lemma 8. Let n be a positive integer and let the set P be a PMPO of $f \in C^0(I, I)$ of period $2n$. Then the sets $P(n, 1)$ and $P(n, 2)$ are potentially minimal periodic orbits of f^2 of period n.

Proof. There exists a function $F \in C^0(I, I)$ such that $f|P = F|P$ and F has no periodic orbit of period less (in the Šarkovskii sense) than $2n$. Further, by Lemma 7, the sets $P(n, 1)$ and $P(n, 2)$ are periodic orbits of f^2 (and also of F^2) of period n. We show that they are potentially minimal. Assume, on the contrary, that e.g. the set $P(n, 1)$ is not a PMPO of f^2. Hence the function F^2 has a periodic orbit of period $m \prec n$ (in the Šarkovskii sense). But then the function F has a periodic orbit of period $2m \prec 2n$ (see Lemma 1). A contradiction. Q.E.D.

We will use the following notation. Let $k \in N$, $p \in \{0\} \cup N$ and let P be a simple periodic orbit of $f \in C^0(I, I)$ of period $2^k \cdot (2p+1)$. For $n \in \{0, 1, 2, \ldots, k-1\}$, $i \in \{1, 2, 3, \ldots, 2^n\}$ we define open intervals

$$G(2^n, i) =]\max P(2^{k-n-1} \cdot (2p+1), 2i-1), \min P(2^{k-n-1} \cdot (2p+1), 2i)[$$

where, as usual, max A denotes the maximum of the set A. Similarly min A.

Lemma 9. Let $k \in N$, $p \in \{0\} \cup N$ and let $P = \{a_1 < a_2 < \ldots < a_w\}$, where $w = 2^k \cdot (2p+1)$, be a simple periodic orbit of $f \in C^0(I, I)$ of period $2^k \cdot (2p+1)$. Let $g \in C^0$ (conv P, conv P) be such a function that $f|P = g|P$ and g is linear on every interval whose endpoints are two neighbouring points from P. Then all periodic points of g lying in the $\bigcup_{n=0}^{k-1} \bigcup_{i=1}^{2^n} G(2^n, i)$ have periods from the set $\{2^j : j = 0, 1, 2, \ldots, k\}$.

Proof. Let $0 \leq m \leq k-1$, $1 \leq t \leq 2^m$ *and* $x \in G(2^m, t)$. We prove that if x is a periodic point of g, then its period is 2^j for some $0 \leq j \leq k$. We have

$$\operatorname{conv} P(2^{k-m} \cdot (2p+1), i) = \operatorname{conv} P(2^{k-m-1} \cdot (2p+1), 2i-1) \cup G(2^m, i) \cup$$
$$\cup \operatorname{conv} P(2^{k-m-1} \cdot (2p+1), 2i),$$

where the three sets on the right side are mutally disjoint. By Lemma 4 and by the definintion of g there is

$$g \updownarrow\updownarrow \langle \operatorname{conv} P(2^{k-m} . (2p+1), i): \quad i = 1, 2, 3, \ldots, 2^m \rangle \tag{5}$$

$$g \updownarrow\updownarrow \langle \operatorname{conv} P(2^{k-m-1}.(2p+1), i) \qquad i = 1, 2, 3, \ldots, 2^{m+1}\rangle. \tag{6}$$

It follows

$$g \uparrow\uparrow \langle G(2^m, i): \qquad i = 1, 2, 3, \ldots, 2^m\rangle. \tag{7}$$

Let us define

$$A = \bigcup_{i=1}^{2^m} \operatorname{conv} P(2^{k-m} \cdot (2p+1), i)$$

$$B = \bigcup_{i=1}^{2^{m+1}} P(2^{k-m-1} \cdot (2p+1), i)$$

$$C = \bigcup_{i=1}^{2^m} G(2^m, i)$$

We have

$$x \in C \subset A = B \cup C, \quad B \cap C = \emptyset. \tag{8}$$

There are two possibilities.

Case 1. There exists such an $s \in N$ that $g^s(x) \notin C$. Then by (5) and (8) we have $g^s(x) \in B$ and by (6) and (8) the point x is not a periodic point of g.

Case 2. For every $s \in N$ there is $g^s(x) \in C$. Put $S = \{y \in G(2^m, t)$: for every $s \in N$, $g^s(y) \in C\}$. By (7), (8) and (5), if $x \in S$ is a periodic point of g, then its period is $2^m \cdot q$ for some positive integer q. Then x is a periodic point of the function g^{2^m} of period q. But this function is linear on S (because of (7) and linearity of g on each $G(2^m, i)$), $x \in S$ and $g^{2^m}(S) \subset S$. Hence $q \in \{1, 2\}$. We have shown that if $x \in G(2^m, t)$ is a periodic point of g, then its period is 2^m or 2^{m+1}. Consequently, all the periodic points of g lying in the set $\bigcup_{n=0}^{k-1} \bigcup_{i=1}^{2^n} G(2^n, i)$ have periods from $\{2^j$: $j = 0, 1, 2, \ldots, k\}$. Q.E.D.

Remark. We were interested in those periodic points of the function g (see Lemma 9) which lie in the set $\bigcup_{n=0}^{k-1} \bigcup_{i=1}^{2^n} G(2^n, i)$. It would not be too difficult to find all such points but we did not need to do it for our purposes.

Lemma 10. Let P be a periodic orbit of $f \in C^0(I, I)$ of period $2^k \cdot (2p+1)$, where $k \in N$ and $p \in \{0\} \cup N$. Let P be potentially minimal. Then P is simple.

Proof. Let the assumptions be satisfied. There exists a function $F \in C^0(I, I)$ such that $f|P = F|P$ and F has no periodic orbits with a period less (in the Šarkovskii sense) than $2^k \cdot (2p+1)$. We are going to prove that P is a simple periodic orbit of f or equivalently, of F.

So let $2^k \cdot (2p+1) = n \cdot r$, where $r = 2^s$ for some $s \in \{0, 1, \ldots, k-1\}$ and let $Q = \{q_1 < q_2 < \ldots < q_n\} \subset P$ be a periodic orbit of F^r(see the definition of simple periodic orbits). We claim that $F^r\left(Q\left(\frac{n}{2}, 1\right)\right) = Q\left(\frac{n}{2}, 2\right)$. Suppose the claim is false. Then by Lemma 7 (note that our integer n is even) the function F^r, $r = 2^s$ has a periodic orbit of odd period $z \geq 3$. But then by Lemma 1 the function F has a periodic orbit of period $2^s \cdot z$. Since for $p = 0$ and also for $p > 0$ the period $2^s \cdot z$ is less (in the Šarkovskii sense) than $2^k \cdot (2p + 1)$, we have a contradiction. Q.E.D.

3 Main results

In the following three theorems we give a full characterization of potentially minimal periodic orbits of even periods.

Theorem 4. Let P be a periodic orbit of $f \in C^0(I, I)$ of period 2^k, where $k \in N$. Then P is potentially minimal if and only if P is simple.

Proof. According to Lemma 10 (or to the corollary of Theorem 3) it suffices to prove the "if" part of the theorem. Suppose that P is simple and define the function $g \in C^0$ (conv P, conv P) such that $f|P = g|P$ and g is linear on every interval whose endpoints are two neighbouring points from P. It suffices to show that g has no periodic point of period 2^{k+1}. Let us define the sets $G(2^n, i)$ analogously as in Lemma 9 (we have $p = 0$). Then

$$\operatorname{conv} P = P \cup \bigcup_{n=0}^{k-1} \bigcup_{i=1}^{2^n} G(2^n, i) \tag{9}$$

and, by Lemma 9, it suffices to prove that no point from P is a periodic point of g of period 2^{k+1}. But it is clear, since P is a periodic orbit of g of period 2^k. Q.E.D.

Theorem 5. Let P be a periodic orbit of $f \in C^0(I, I)$ of period 3.2^k, where $k \in N$. Then P is potentially minimal if and only if P is simple.

Proof is similar to that of Theorem 4. It suffices to prove the "if" part. Let P be simple and let the function g be defined in the same way as in the proof of Theorem 4. We are going to show that g has no periodic point of period $2^{k-1} \cdot (2p + 1)$, $p \in N$. Now instead of (9) we have

$$\operatorname{conv} P = \bigcup_{j=1}^{2^k} \operatorname{conv} P(3, j) \cup \bigcup_{n=0}^{k-1} \bigcup_{i=1}^{2^n} G(2^n, i).$$

If $z \in \operatorname{conv} P$ is a periodic point of g of period $m \neq 2^q$, $q = 0, 1, 2, \ldots$, then by Lemma 9, $z \in \operatorname{conv} P(3, r)$ for some $r \in \{1, 2, \ldots, 2^k\}$. Lemma 4 and the definition of g imply $g \updownarrow\updownarrow \langle \operatorname{conv} P(3,1), \ldots, \operatorname{conv} P(3,2^k)\rangle$. Hence m must be divisible by 2^k, and thus $m \neq 2^{k-1} \cdot (2p + 1)$, $p \in N$. Q.E.D.

In Theorem 6 we use the following notation. Let P be a periodic orbit of $f \in C^0(I, I)$ of period $2^k \cdot (2p + 1)$, where $k \in N$, $p \in N$. Then E denotes the set

$$\bigcup_{j=1}^{2^k} \{\min P(2p+1, j), \max P(2p+1, j)\}$$

of all endpoints of the intervals conv $P(2p+1, j)$, $j = 1, 2, 3, \ldots, 2^k$.

Theorem 6. Let P be a periodic orbit of $f \in C^0(I, I)$ of period $2^k \cdot (2p+1)$ where $k \in N$, $p \in N$ and $p \geq 2$. Consider the following four conditions:

(C1) P is simple

(C2) the sets $P(2p+1, j)$, $j = 1, 2, 3, \ldots, 2^k$ are potentially minimal periodic orbits of f^{2^k} (see Theorem 1)

(C3-a) $f \updownarrow E$

(C3-b) f is monotonic on each of the sets $P(2p+1, j)$, $j = 1, 2, 3, \ldots, 2^k$ exept for one of them.

Then the following three conditions are equivalent:

(i) P is potentially minimal

(ii) (C1) and (C2) and (C3-a)

(iii) (C1) and (C2) and (C3-b).

Proof. (i) $\Rightarrow$ (ii). Let P be a PMPO of f. Then Lemma 10 implies (C1) and Lemma 8 (used k-times) implies (C2). Thus it suffices to prove that (i) implies (C3-a). Actually, we will show that this implication is true under more general assumptions, namely, that P is a PMPO of f of period $2^k \cdot (2p+1)$, where $k \in \{0\} \cup N$, $p \in N$, $p \geq 2$. Fix some p. If $k = 0$, then $f \updownarrow E$, i.e. (C3-a) is true, cf. Lemma 5. Now suppose by induction that $f \updownarrow E$ for $k = n-1$. Let P be a PMPO of f of period $2^n \cdot (2p+1)$. We prove that $f \updownarrow E$. Take a function $F \in C^0(I, I)$ such that $f|P = F|P$ and F has no periodic orbit with period less (in the Šarkovskii sense) than $2^n \cdot (2p+1)$. Clearly, it suffices to show that $F \updownarrow E$. Denote $P_i = P(2^{n-1} \cdot (2p+1), i)$ and $E_i = E \cap P_i$ for $i = 1, 2$. By Lemma 8 and induction hypothesis we have $F^2 \updownarrow E_1$ and $F^2 \updownarrow E_2$. Hence there exist permutations $(\alpha(1), \alpha(2), \ldots, \alpha(2^n))$ and $(\beta(1), \beta(2), \ldots, \beta(2^n))$ of the sets E_1 and E_2, respectively, such that

$$\alpha(1) \xrightarrow{F^2} \alpha(2) \xrightarrow{F^2} \ldots \xrightarrow{F^2} \alpha(2^n) \tag{10}$$

$$\beta(1) \xrightarrow{F^2} \beta(2) \xrightarrow{F^2} \ldots \xrightarrow{F^2} \beta(2^n). \tag{11}$$

Furher, by Lemma 7,

$$P_1 \overset{F}{\leftrightarrow} P_2 \tag{12}$$

Now we claim that

$$F(\alpha(1)) \in E_2 \text{ and } F(\beta(1) \in E_1 \tag{13}$$

simultaneously. Suppose for a moment the claim is true (the proof will be given later). We show that then either $F(\alpha(1)) = \beta(1)$ or $F(\beta(1)) = \alpha(1)$, which by (10) and (11) immediately implies $F \updownarrow E$.

Thus assume on the contrary that $F(\alpha(1)) = \beta(1+i)$ and $F(\beta(1)) = \alpha(1+j)$, where $i, j > 0$. By (10), $F^2(\alpha(1)) = F(\beta(1+i)) = \alpha(2)$, and using again (10) and (11) we have

$$\alpha(1+j) = F^{2\cdot(j-1)}(\alpha(2)) = F^{1+2\cdot(j-1)}(\beta(1+i)) =$$
$$= F^{1+2\cdot(j-1)+2i}(\beta(1)) = F^{2\cdot(j-1)+2i}(\alpha(1+j)).$$

Consequently, the period $2^n \cdot (2p+1)$ of $\alpha(1+j)$ is less than or equal to $2\cdot(j-1)+2i$. But $2p+1 \geq 5$ and $2\cdot(j-1)+2i < 4.2^n$, since $i, j < 2^n$. We have a contradiction.

It remains to prove (13). Assume, on the contrary, that (13) does not hold. Then, by (12), there exists $x \in \{\alpha(1), \beta(1)\}$ such that $F(x) \notin$ E. Denote $y = F^{2^n}(x)$, $z = F^{2^n}(y)$ and $G = F^{2^n-1}$. We have

$$x \xrightarrow{F} F(x) \xrightarrow{G} y \xrightarrow{F} F(y) \xrightarrow{G} z \xrightarrow{F} F(z). \tag{14}$$

By Lemma 4 (note that P is a simple periodic orbit of F), every set $P(2p+1, i)$ is a periodic orbit of F^{2^n} and $F \updownarrow\updownarrow < P(2p+1, 1), \ldots, P(2p+1, 2^n)\rangle$. Hence there exist $u, v \in \{1, 2, 3, \ldots, 2^n\}$, $u \neq v$ such that $\{x, y, z\} \subset P(2p+1, u)$ and $\{F(x), F(y), F(z)\} \subset P(2p+1, v)$. Since $P(2p+1, u)$ and $P(2p+1, v)$ are periodic orbits of $F^{2^n}$of period $2p+1 \geq 5$, all the points in (14) are mutually different. The point x belongs to E, since $x \in \{\alpha(1), \beta(1)\}$. We claim that also $y \in E$. In fact, if we denote $F^2 = H$, we have $y = H^{2^{n-1}}(x)$. But in each of the relations (10) and (11) there 2^{n-1} arrows and $2^{n-1} \leq 2^n - 1$ for $n \geq 1$. Hence the points x and y must be endpoints of conv $P(2p+1, u)$. Since we have proved (i) $\Rightarrow$ (C2), the set $P(2p+1, u)$ is a PMPO of F^{2^n}. By Lemma 5, z is the middle point of $P(2p+1, u)$ and $F^{2^n}(z)$ lies between x and z. On the other hand, $F(x) \notin$ $\notin E$. By Lemma 6, $F(x)$ lies between $F(y) = F^{2^n}(F(x))$ and $F(z) = F^{2^n} \circ F^{2^n}(F(x))$. Thus there is a point w between y and z with $f(w) = F(x)$ and, consequently, $F^{2^n}(w) = y$. Therefore the function F^{2^n} has a fixed point between y and w. Clearly, this fixed point lies between y and z. Further, if $y < z$ or $y > z$, then both y and z belong to $U(F^{2^n})$ and $D(F^{2^n})$, respectively. In each of these cases F^{2^n} has a periodic orbit of period 3 (see Lemma 2). But then, by Lemma 1, the function F has a periodic orbit of period $2^n \cdot 3$, a contradiction. (The function F was chosen such that it has no periodic orbit with a period less (in the Šarkovskii sense) than $2^n \cdot (2p+1)$, where $p > 1$.) The proof of (i) $\Rightarrow$ (ii) is finished.

(ii) $\Rightarrow$ (iii). Let (ii) be satisfied. We need to prove (C3-b). There exist a permutation $(e(1), e(2), \ldots, e(2^{k+1}))$ of the set E and $t \in \{1, 2, 3, \ldots, 2^k\}$ such that

$$e(1) \xrightarrow{f} e(2) \xrightarrow{f} \ldots \xrightarrow{f} e(2^{k+1}) \in P(2p+1, t).$$

Note that $f(e(2^{k+1})) \notin E$, but $f(E \cap P(2p+1, i)) \subset E$ for each $i \in \{1, 2, 3, \ldots, 2^k\}$ except for $i = t$. Now it suffices to prove that f is monotonic on every such set

$P(2p+1, i)$.So let $r \in \{1, 2, 3, \ldots, 2^k\} \setminus \{t\}$. We have $f(E \cap P(2p+1, r)) \subset E$. Moreover, by Lemma 4, $f(P(2p+1, r)) = P(2p+1, s)$ for some $s \in \{1, 2, 3, \ldots, 2^k\}$, $s \neq r$ and, consequently,

$$f(E \cap P(2p+1, r)) = E \cap P(2p+1, s). \tag{15}$$

By (C2), the sets $P(2p+1, r)$ and $P(2p+1, s)$ are potentially minimal periodic orbits of f^{2^k}. By Lemma 5,

$$f \updownarrow E \cap P(2p+1, r) \quad \text{and} \quad f \updownarrow E \cap P(2p+1, s). \tag{16}$$

Since we have (15) and (16), it is easy to see that there exist permutations $(\alpha(1), \alpha(2), \ldots, \alpha(2p+1))$ and $(\beta(1), \beta(2), \ldots, \beta(2p+1))$ of the sets $P(2p+1, r)$ and $P(2p+1, s)$ respectively, such that the following four conditions are fulfilled simultaneously:

$$\alpha(1) < \alpha(2) < \ldots < \alpha(2p+1) \quad \text{or} \quad \alpha(1) > \alpha(2) > \ldots > \alpha(2p+1) \tag{17}$$

$$f^{2^k}(\alpha(1)) = \alpha(2p+1) \tag{18}$$

$$f(\alpha(1)) = \beta(1) \quad \text{and} \quad f(\alpha(2p+1)) = \beta(2p+1) \tag{19}$$

$$\beta(1) < \beta(2) < \ldots < \beta(2p+1) \quad \text{or} \quad \beta(1) > \beta(2) > \ldots > \beta(2p+1). \tag{20}$$

It follows

$$f^{2^k}(\beta(1)) = f^{2^k+1}(\alpha(1)) = f(\alpha(2p+1)) = \beta(2p+1). \tag{21}$$

The relations (18), (21) and Theorem 1 imply that for each $j \in \{1, 2, 3, \ldots, 2p+1\}$ there exists a number $n(j) \in \{0, 1, 2, \ldots, 2p\}$ such that $f^{2^k \cdot n(j)}(\alpha(1)) = \alpha(j)$ and $f^{2^k \cdot n(j)}(\beta(1)) = \beta(j)$ simultaneously. Then for each $j \in \{1, 2, \ldots, 2p+1\}$

$$f(\alpha(j)) = f^{1+2^k \cdot n(j)}(\alpha(1)) = f^{2^k \cdot n(j)}(\beta(1)) = \beta(j).$$

Hence we have permutations $(\alpha(1), \ldots, \alpha(2p+1))$ and $(\beta(1), \ldots, \beta(2p+1)$ of the sets $P(2p+1, r)$ and $P(2p+1, s)$ respectively such that they satisfy (17), (20) and $f(\alpha(j)) = \beta(j)$ for each $j \in \{1, 2, \ldots, 2p+1\}$. Now it is easy to see that if the inequalities in (17) are of the same or opposite sense as those in (20), then f restricted to the set $P(2p+1, r)$ is increasing or decreasing, respectively, which finishes the proof of (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (i). Let (iii) be satisfied. Let f be monotonic on each of the sets $P(2p+1, i)$ for $i \in \{1, 2, 3, \ldots, 2^k\}$, $i \neq t$. Take the function $g \in C^0$ (conv P, conv P) such that $f|P = g|P$ and g is linear on every interval whose endpoints are two neighbouring points from P. It suffices to show that g has no periodic point of period $2^k \cdot (2p-1)$. Suppose, on the contrary, that g has a periodic orbit Q of period $2^k \cdot (2p-1)$. By Lemma 9, $Q \subset \bigcup_{i=1}^{2^k} \operatorname{conv} P(2p+1, i)$. Further, Lemma 4 implies

$$g \updownarrow\updownarrow \langle \operatorname{conv} P(2p+1, 1), \ldots, \operatorname{conv} P(2p+1, 2^k) \rangle . \qquad (22)$$

Consequently, each of the sets conv $P(2p + 1, i)$ contains $2p - 1$ points of Q which form a periodic orbit of g^{2^k} of period $2p - 1$. On the other hand, denote by c the positive integer for which $g(P(2p + 1, t)) = P(2p + 1, c)$. We show that the interval conv $P(2p + 1, c)$ contains no periodic orbit of g^{2^k} of period $2p - 1$. By (C2), the set $P(2p + 1, c)$ is a PMPO of g^{2^k} of period $2p + 1$. Moreover, if J is an arbitrary interval whose endpoints are two neighbouring points from $P(2p + 1, c)$, then by (22) and definitions of g and $P(2p + 1, c)$, $g^{2^k - 1}(J)$ is an interval whose endpoints are two neighbouring points from $P(2p + 1, t)$. Since $g^{2^k - 1}$ is linear on J and g is linear on $g^{2^k - 1}(J)$, the function g^{2^k} is linear on J. We have shown that $P(2p + 1, c)$ is a PMPO of g^{2^k} of period $2p + 1$ and g^{2^k} is linear on each interval whose endpoints are two neighbouring points from $P(2p + 1, c)$. Hence there exists no periodic orbit of g^{2^k} of period $2p - 1$ lying in the set conv $P(2p + 1, c)$ (see [7], p. 245). A contradiction. Q.E.D.

Remark. It can be shown that f is not monotonic on $P(2p + 1, t)$ (see the proof of (ii) $\Rightarrow$ (iii)).

4 Remarks

(A) A. N. Šarkovskii in [6] has proved that his theorem remains valid if we assume that f is a continuous real function defined on an arbitrary connected set on the real line. In the same way it can be shown that Theorems 1, 2, 3, 4, 5, 6 imply analogous theorems for functions defined on such sets.

(B) Every function having some periodic orbit which is not potentially minimal has a minimal periodic orbit (see the Šarkovskii's ordering and Theorems 3, 4). The following problem seems to be interesting. For every integer $n \geq 4$, to characterize the set of periods of minimal periodic orbits of all functions belonging to $C^0(I, I)$ and having a periodic orbit of period n which is not potentially minimal. This problem is solved in [5].

Acknowledgement. I wish to thank Jaroslav Smítal for bringing the problem of minimal periodic orbits to my attention and for several very helpful discussions.

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Author's address.
Ľubomír Snoha
Katedra matematiky
Pedagogická fakulta
Banská Bystrica

Received: 19. 3. 1985

SÚHRN

CHARAKTERIZÁCIA POTENCIÁLNE MINIMÁLNYCH PERIODICKÝCH ORBÍT SPOJITÝCH ZOBRAZENÍ INTERVALU

Ľubomír Snoha, Banská Bystrica

R. 1977 P. Štefan zaviedol pojem minimálnej periodickej orbity spojitého zobrazenia intervalu. Ukázal, že pre každé nepárne $n \geq 3$ existujú len dva „typy" periodických orbít periódy n, ktoré môžu byť minimálnymi periodickými orbitami. Inak povedané, charakterizoval takzvané potenciálne minimálne periodické orbity nepárnych periód.

V práci je podaná charakterizácia potenciálne minimálnych periodických orbít párnych periód.

РЕЗЮМЕ

ХАРАКТЕРИЗАЦИЯ ПОТЕНЦИАЛЬНО МИНИМАЛЬНЫХ ПЕРИОДИЧЕСКИХ ОРБИТ НЕПРЕРЫВНЫХ ОТОБРАЖЕНИЙ ОТРЕЗКА

Любомир Снога, Банска Быстрица

В 1977 г. характеризовал П. Штефан потенциально минимальные периодические орбиты нечетных периодов непрерывных отображений замкнутого вещественного отрезка в себя.

Главный результат работы — характеризация потенциально минимальных периодических орбит четных периодов.