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ON LORENTZ—ORLICZ SPACES

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Introduction

Let X be a linear space. A functional ϱ: $X \to \langle 0, +\infty \rangle$ is said to be convex modular on X if

1. $\varrho(x) = 0$ if and only if $x = 0$;
2. $\varrho(-x) = \varrho(x)$ for $x \in X$;
3. $\varrho(\alpha x + \beta y) \leqq \alpha\varrho(x) + \beta\varrho(y)$ for each $x, y \in X$ and $\alpha \geqq 0$, $\beta \geqq 0$, $\alpha + \beta = 1$ (cf. [5], p. 5)

The set $X_\varrho = \{x \in X\colon \exists_{t > 0}\, \varrho(tx) < +\infty\}$ is a linear subspace of X. We can define on X_ϱ the norm

$$\|x\| = \inf\left\{t > 0\colon \varrho\left(\frac{x}{t}\right) \leqq 1\right\}$$

The space $(X_\varrho, \| \ \|)$ is called the modular space determined by the modular ϱ (cf. [5], p.6).

An Orlicz function is a continuous non-decreasing and convex function f: $\langle 0, +\infty \rangle \to \langle 0, +\infty \rangle$ with $f(0) = 0, f(+\infty) = +\infty$ and $\lim_{t \to \infty} f(t) = +\infty$.

If $f(t) = 0$ for some $t > 0$, then f is called a degenerate Orlicz function (cf. [4], p. 137).

An Orlicz function f satisfies the Δ_2-condition for small t if there exist $K > 0$, $t_0 > 0$ such that $f(2t) \leqq Kf(t)$ for each $t \in \langle 0, t_0 \rangle$ (cf. [4]).

Let f be a non-degenerate Orlicz function whose right derivative P satisfies $P(0) = 0$ and $\lim_{t \to \infty} P(t) = +\infty$. Put $Q(u) = \sup\{t\colon P(t) \leqq u\}$ $(u \geqq 0)$ and $f^*(t) = \int_0^t Q(u)\,du$ $(t \geqq 0)$. Then f^* is also a non-degenerate Orlicz function and it is called the function complementary to f.

For any $u \geqq 0$, $v \geqq 0$ the following Young's inequality holds:

$$u \cdot v \leqq f(u) + f^*(v)$$

(cf. [4], p. 147).

In what follows s stands for the linear space of all real sequences. Further we put

$$e_n = \underbrace{0, 0, \ldots, 0}_{n-1}, 1, 0, \ldots \qquad (n \geqq 1)$$

Let π be a permutation of the set N of all positive integers and let $x = \{\xi_n\}_{n=1}^{\infty} \in s$. Then we shall denote by x_π the sequence $\{\xi_{\pi(n)}\}_{n=1}^{\infty}$.

Let $a = \{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers with $1 = a_1 \geqq a_2 \geqq \ldots \geqq \geqq a_n \geqq a_{n+1} \geqq \ldots a_n \to 0$, $\sum_{n=1}^{\infty} a_n = +\infty$. Let f be a non-degenerate Orlicz function. For $x = \{\xi_n\}_{n=1}^{\infty} \in s$ we put

$$\varrho(x) = \sup_{\pi} \sum_{n=1}^{\infty} f(|\xi_n|) a_{\pi(n)} = \sup_{\pi} \sum_{n=1}^{\infty} f(|\xi_{\pi(n)}|) a_n$$

where the supremum is taken over all permutations π of the set N. Then ϱ is a convex modular on s (cf. [2]) and

$$d(a, f) = \{x \in s: \underset{t>0}{\exists}\ \varrho(tx) < +\infty\}$$

is a modular space called a Lorentz—Orlicz space (cf. [2]).

In what follows l_∞ stands for the linear space of all bounded real sequences, c_0 denotes the linear space of all real sequences converging to 0.

1. Applications of the category method in the theory of Lorentz—Orlicz spaces

In what follows we shall use the following result from [2]:

Theorem A.

(i) The space $d(a, f)$ is a Banach space;

(ii) We have $d(a, f) \subset c_0$;

(iii) If $x \in d(a, f)$, then $x_\pi \in d(a, f)$ for each permutation π of the set N and $\varrho(x) = \varrho(x_\pi)$;

(iv) We have

$$\{x \in d(a, f): \varrho(x) \leqq 1\} = \{x \in d(a, f): \|x\| \leqq 1\}$$

(v) If the function f satisfies the Δ_2-condition for small t, then each $x = \{\xi_n\}_{n=1}^{\infty} \in d(a, f)$ has the form $x = \sum_{n=1}^{\infty} \xi_n e_n$ and $d(a, f) = \{x \in s: \underset{t>0}{\forall}\ \varrho(tx) <$

$< +\infty\}$. If $y = \{\eta_n\}_{n=1}^{\infty} \in d(a, f^*)$ (f^* being the complementary function to f), then we get from the Young's inequality:

$$\sup_{\pi} \sum_{n=1}^{\infty} |\xi_n \eta_{\pi(n)}| a_n < +\infty \tag{1}$$

for each $x = \{\xi_n\}_{n=1}^{\infty} \in d(a, f)$.

Theorem 1.1. Let a non-degenerate Orlicz function f satisfy the Δ_2-condition for small t. Let $y = \{\eta_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers such that there exists a $c = \{\gamma_n\}_{n=1}^{\infty} \in d(a, f)$ with

$$\sup_{\pi} \sum_{j=1}^{\infty} |\gamma_j \eta_{\pi(j)}| a_j = +\infty. \tag{2}$$

Then the set

$$M = \left\{x = \{\xi_j\}_{j=1}^{\infty} \in d(a, f) \colon \sup_{\pi} \sum_{n=1}^{\infty} |\xi_n \eta_{\pi(n)}| a_n < +\infty\right\}$$

is an F_σ — set of the first Baire category in $d(a, f)$.

Remark 1.1. It follows from (1) and (2) that $y = \{\eta_j\}_{j=1}^{\infty} \notin d(a, f^*)$ and $\eta_n \neq 0$ for an infinite number of $n's$.

Proof. Put

$$C_k = \bigcup_{n=1}^{\infty} \left\{x = \{\xi_j\}_{j=1}^{\infty} \in d(a, f) \colon \sup_{\pi} \sum_{j=1}^{n} |\xi_j \eta_{\pi(j)}| a_j > k\right\}.$$

We shall show that $C_k \neq \emptyset$ ($k = 1, 2, \ldots$).

Since $\sum_{j=1}^{\infty} a_j = +\infty$, there exists an n_0 such that for each $n \geqq n_0$ we have

$$\sum_{j=1}^{n} a_j > k. \tag{3}$$

Since $\eta_n \neq 0$ for infinitely many n' s we can choose $j_1 < j_2 < \ldots < j_n$ such that $\eta_{j_i} \neq 0$ ($i = 1, 2, \ldots, n$). Let us put

$$\xi_{j_i} = \eta_{j_i}^{-1} \qquad (i = 1, 2, \ldots, n)$$

$$\xi_m = 0 \quad \text{for} \quad m \neq j_i \ (i = 1, 2, \ldots)$$

and $x = \{\xi_j\}_{j=1}^{\infty}$. Then on account of (3) the sequence x belongs to C_k. Hence $C_k \neq \emptyset$.

We shal show that C_k ($k = 1, 2, \ldots$) is an open set in $d(a, f)$.

Let $x^0 = \{\xi_j^0\}_{j=1}^{\infty} \in C_k$. Then there is an n such that

$$\sup_{\pi} \sum_{j=1}^{n} |\xi_j^0 \eta_{\pi(j)}| a_j > k.$$

Choose $\varepsilon > 0$ and $\delta > 0$ in such a way that

(4)
$$\sup_{\pi} \sum_{j=1}^{n} |\xi_j^0 \eta_{\pi(j)}| a_j - \varepsilon > k ,$$
$$\delta \sup_{\pi} \sum_{j=1}^{n} |\eta_{\pi(j)}| a_j < \varepsilon$$

and $0 < \delta_1 = f(\delta) < 1$.

Let $x = \{\xi_j\}_{j=1}^{\infty} \in d(a, f)$ satisfy the condition $\|x - x^0\| < \delta_1 < 1$. According to Theorem A (iv) we have $\varrho\left(\frac{x - x^0}{\delta_1}\right) \leqq 1$. From this we get

$$\sup_{\pi} \sum_{j=1}^{\infty} f(|\xi_j - \xi_j^0|)\, a_{\pi(j)} \leqq \delta_1 = f(\delta).$$

Hence for each j we have $f(|\xi_j - \xi_j^0|) < f(\delta)$ and so $|\xi_j - \xi_j^0| < \delta\, (j = 1, 2, \ldots)$.

So for every permutation π of the set N the following inequalities hold:

$$\sum_{j=1}^{n} |\xi_j \eta_{\pi(j)}| a_j \geqq \sum_{j=1}^{n} |\xi_j^0 \eta_{\pi(j)}| a_j - \sum_{j=1}^{n} |\xi_j - \xi_j^0|\, |\eta_{\pi(j)}| a_j > \sum_{j=1}^{n} |\xi_j^0 \eta_{\pi(j)}| a_j - \varepsilon .$$

According to (4) this gives

$$\sup_{\pi} \sum_{j=1}^{n} |\xi_j \eta_{\pi(j)}| a_j > k ,$$

i.e. $\{x \in d(a, f)\colon \|x - x^0\| < \delta_1\} < C_k$. Hence C_k $(k = 1, 2, \ldots)$ is an open set.

According to our assumption the sequence $c = \{\gamma_n\}_{n=1}^{\infty}$ belongs to the set $C = \bigcap_{k=1}^{\infty} C_k$.

Let $\varepsilon > 0$. According to Theorem A (v) there exists a $p \in N$ such that

(5)
$$\left\| \sum_{j=p+1}^{\infty} \xi_j e_j \right\| < \frac{\varepsilon}{2}, \quad \left\| \sum_{j=p+1}^{\infty} \gamma_j e_j \right\| < \frac{\varepsilon}{2}.$$

Put $\beta_j = \xi_j$ for $j \leqq p$ and $\beta_j = \gamma_j$ for $j > p$. Then it can be easily checked that $w = \{\beta_j\}_{j=1}^{\infty} \in C$ and using (5) we see that

$$\|x - w\| = \left\| \sum_{j=p+1}^{\infty} (\xi_j - \beta_j) e_j \right\| < \varepsilon .$$

Thus C is a dense G_δ set in $d(a, f)$. Hence C is a residual set in $d(a, f)$ (see Theorem A (i)). Since $M = d(a, f) \setminus C$, the assertion follows.

Let f be a non-degenerate Orlicz function, let f^* be the complementary function to f. Denote by ϱ and ϱ^* the modulars determined by f and f^*, respectively. Similarly denote by $\| \ \|$ and $\| \ \|^*$ the norms in $d(a, f)$ and $d(a, f^*)$, respectively.

Theorem 1.2. Let both of f and f^* satisfy the Δ_2-condition for small t. Then

a) $|||x||| = \sup\left\{\sum_{j=1}^{\infty} |\xi_j\eta_j| a_j: \varrho^*(y) \leqq 1\right\}$ is a norm on $d(a, f)$;

b) For each $x = \{\xi_j\}_{j=1}^{\infty} \in d(a, f)$ and $y = \{\eta_j\}_{j=1}^{\infty} \in d(a, f^*)$ the following inequalities are satisfied:

$$\sum_{n=1}^{\infty} |\xi_n\eta_n| a_n \leqq |||x||| \cdot \|y\|^*,$$

$$\sum_{n=1}^{\infty} |\xi_n\eta_n| a_n \leqq \|x\| \cdot |||y|||^*,$$

where $|||\ |||^*$ is a norm on $d(a, f^*)$ defined by the same manner as $|||\ |||$.

Proof. a) If

$$x = \{\xi_j\}_{j=1}^{\infty} \in d(a, f), \ y = \{\eta_j\}_{n=1}^{\infty} \in d(a, f^*)$$

and $\varrho^*(y) \leqq 1$, then

$$\sum_{j=1}^{\infty} |\xi_j\eta_j| a_j \leqq \varrho(x) + 1$$

Hence $|||x||| < +\infty$ for each $x \in d(a, f)$ and obviously $|||\ |||$ is a norm on $d(a, f)$.

b) Let

$$x = \{\xi_n\}_{n=1}^{\infty} \in d(a, f), \ y = \{\eta_n\}_{n=1}^{\infty} \in d(a, f^*)$$

and $\varepsilon > 0$. Then

$$\varrho^*\left(\frac{y}{\|y\|^* + \varepsilon}\right) \leqq 1$$

and

$$\sum_{n=1}^{\infty} \frac{|\xi_n\eta_n|}{\|y\|^* + \varepsilon} a_n \leqq |||x|||$$

by the definition of the norm $|||\ |||$. It implies

$$\sum_{n=1}^{\infty} |\xi_n\eta_n| a_n \leqq |||x||| \cdot \|y\|^*$$

The proof of the second inequality is analogous. The proof is finished.

2. Some bitopological properties of Lorentz—Orlicz spaces

A set X with two topologies T_1 and T_2, i.e. the triplet (X, T_1, T_2) is called a bitopological space (cf. [3]). A bitopological space (X, T_1, T_2) is pairwise Hausdorff ([3], [6]) if for each distinct points $x, y \in X$ there exist disjoint sets $U \in T_i$, $V \in T_j$, $i \neq j$, $i, j = 1, 2$, such that $x \in U$, $y \in V$.

In (X, T_1, T_2) the topology T_1 is said to be — regular with respect to T_2 (cf. [3], [6], [7]) if for every $U \in T_1$ and $x \in U$ there exists a set $V \in T_1$ such that $x \in V \subset \bar{V}^{(2)} \subset U$, where $\bar{V}^{(2)}$ denotes the T_2 — closure of V;

— perfect with respect to T_2 (cf. [1], [6]) if every T_1 — open set $U \subset X$ is an F_σ — set in (X, T_2).

A bitopological space is said to be:

— pairwise regular (pairwise perfect) if T_i is regular (perfect) with respect to T_j for $i \neq j$, $i, j = 1, 2$ (cf. [3], [6], [7]);

— pairwise normal (cf. [1], [3]) if for every T_i — closed set A and T_j — closed set B such that $A \cap B = \emptyset$ there exist $U \in T_j$, $V \in T_i$ with $A \subset U$, $B \subset V$, $U \cap V = \emptyset$ for $i, j = 1, 2$, $i \neq j$;

— pairwise perfect normal if it is paiwise perfect and pairwise normal.

The following result is known (cf. [1], Lema 2.4):

Theorem B. In (X, T_1, T_2) the following conditions are equivalent:

(i) (X, T_1, T_2) is pairwise perfect normal;

(ii) For each T_i — open set W there exists a sequence $\{W_n\}_{n=1}^{\infty}$ of T_i — open sets such that $W = \bigcup_{n=1}^{\infty} W_n$

and

$$\bar{W}_n^{(j)} \subset W_{n+1} \quad (n = 1, 2, \dots; i, j = 1, 2, i \neq j).$$

Let f, g be non-degenerate Orlicz functions, let $a = \{a_n\}_{n=1}^{\infty}$, $b = \{b_n\}_{n=1}^{\infty}$ be two sequences from $c_0 \setminus l^1$ such that

$$1 = a_1 \geqq a_2 \geqq \dots \geqq a_n \geqq a_{n+1} \geqq \dots$$

$$1 = b_1 \geqq b_2 \geqq \dots b_n \geqq b_{n+1} \geqq \dots$$

It is easy to verify that $l^1 \subset d(a, f) \cap d(b, g)$. So $d(a, f) \cap d(b, g)$ is a non-trivial linear space.

Let

$$\varrho_1(x) = \sup_{\pi} \sum_{j=1}^{\infty} f(|\xi_{\pi(j)}|) a_j,$$

$$\varrho_2(x) = \sup_{\pi} \sum_{j=1}^{\infty} g(|\xi_{\pi(j)}|) b_j,$$

where $x = \{\xi_j\}_{j=1}^{\infty} \in d(a, f) \cap d(b, g)$.

By $\| \ \|_1$ and $\| \ \|_2$ we denote the norms determined by modulars ϱ_1 and ϱ_2, respectively. We shall use the symbol $K_i(x, r)$ to denote the T_i — open ball ($i = 1, 2$) with center x and radius $r > 0$ in $d(a, f) \cap d(b, g)$. Moreover, let T_1 and T_2 be the topologies on $d(a,f) \cap d(a,g)$ induced by norms $\| \ \|_1$ and $\| \ \|_2$, respectively. Thus $(d(a,f) \cap d(b,g), T_1, T_2)$ can be cinsidered as a bitopological space.

Theorem 2.1 Let f and g be non-degenerate Orlicz functions, let f and g satisfy the Δ_2 — condition for small t. Then the following assertions hold:

a) The bitopological space $(d(a,f) \cap d(b,g), T_1, T_2)$ is pairwise Hausdorff;

b) For each $x \in d(a,f) \cap d(b,g)$ and $r > 0$ the set $\bar{K}_i^{(j)}(x, r)$ is T_j— closed for $i, j = 1, 2$;

c) The space $(d(a,f) \cap d(b,g), T_1, T_2)$ is pairwise regular;

d) The space $(d(a,f) \cap d(b,g), T_1, T_2)$ is pairwise perfect normal.

Proof.

a) Let $x = \{\xi_j\}_{j=1}^{\infty}$ and $y = \{\beta_j\}_{j=1}^{\infty}$ be two distinct points from $d(a,f) \cap d(b,g)$. Because $x, y \in c_0$ (see Theorem A (ii)) we have

$$0 < r = \frac{1}{4} \sup_{j=1,2,\ldots} |\xi_j - \beta_j| < +\infty .$$

Put

$$U = \{z \in d(a,f) \cap d(b,g) \colon \|z - x\|_1 < r\},$$

$$V = \{z \in d(a,f) \cap d(b,g) \colon \|z - y\|_2 < r\}.$$

Then U is a T_1 — neighbourhood of x and V is a T_2 — neighbourhood of y. Let us suppose that $z = \{\alpha_j\}_{j=1}^{\infty} \in U \cap V$. Then for every $\varepsilon > 0$ the following inequalities are satisfied:

$$\sup_{\pi} \sum_{n=1}^{\infty} f\left(\frac{|\alpha_n - \xi_n|}{\|z - x\|_1 + \varepsilon}\right) a_{\pi(n)} \leqq 1,$$

$$\sup_{\pi} \sum_{n=1}^{\infty} g\left(\frac{|\alpha_n - \beta_n|}{\|z - y\|_2 + \varepsilon}\right) b_{\pi(n)} \leqq 1 .$$

If we choose $0 < \varepsilon < \frac{r}{3}$, then we have

$$|\alpha_n - \xi_n| \leqq \|z - x\|_1 + \varepsilon < \frac{4r}{3} = \frac{1}{3} \sup_{j=1,2,\ldots} |\xi_j - \beta_j| ,$$

$$|\alpha_n - \beta_n| \leqq \|z - y\|_2 + \varepsilon < \frac{4r}{3} = \frac{1}{3} \sup_{j=1,2,\ldots} |\xi_j - \beta_j| .$$

These inequalities imply

$$|\xi_n - \beta_n| < \frac{2}{3} \sup_{j=1,2,\ldots} |\xi_j - \beta_j|$$

for every $n = 1, 2, \dots$, which is impossible. Thus we have shown that $U \cap V = \emptyset$.

b) For any $\alpha > 0$, $k \geqq 1$ let

$$B_k(\alpha) = \left\{ x = \{\xi_j\}_{j=1}^{\infty} \in d(a,f) \cap d(b, g): \sup_{\pi} \sum_{i=1}^{k} f(\alpha|\xi_i|) a_{\pi(i)} \leqq 1 \right\}.$$

We show that $B_k(\alpha)$ is a T_2 — closed set. Let $y = \{\eta_i\}_{i=1}^{\infty} \in \bar{B}_k^{(2)}(\alpha)$. Then $y = T_2 - \lim_{i \to \infty} y_i$, where $y_i = \{\eta_j^{(i)}\}_{j=1}^{\infty} \in B_k(\alpha)$ $(i = 1, 2, \dots)$.

For any $\varepsilon > 0$ there exists an i_0 such that

$$\varrho_2\left(\frac{y_i - y}{\varepsilon}\right) \leqq 1,$$

i.e.

$$\sup_{\pi} \sum_{j=1}^{\infty} g_j\left(\frac{|\eta_j^{(i)} - \eta_j|}{\varepsilon}\right) b_{\pi(j)} \leqq 1$$

for $i \geqq i_0$.

From this it follows that

$$|\eta_j^{(i)} - \eta_j| \leqq \varepsilon g^{-1}(1) \ (i \geqq i_0; j = 1, 2, \dots).$$

So $\eta_j = \lim_{i \to \infty} \eta_j^{(i)}$ for each $j \geqq 1$.

The condition $y_m \in B_k(\alpha)$ implies

$$\sum_{j=1}^{k} f(\alpha|\eta_j^{(m)}|) a_{\pi(j)} \leqq 1$$

for each permutation π and $m \geqq 1$. Hence using the continuity of f we get

$$\sum_{j=1}^{k} f(\alpha|\eta_j|) a_{\pi(j)} \leqq 1$$

for every π and in the consequence

$$\sup_{\pi} \sum_{j=1}^{k} f(\alpha|\eta_j|) a_{\pi(j)} \leqq 1.$$

Thus $y \in B_k(\alpha)$, which means that $B_k(\alpha)$ is T_2 — closed.

According to Theorem A (iv) for any $r > 0$ we have

$$\bar{K}_1^{(1)}(0, r) = \left\{ x \in d(a, f) \cap d(b. g): \varrho_1\left(\frac{x}{r}\right) \leqq 1 \right\}.$$

So

$$\bar{K}_1^{(1)}(0, r) = \bigcap_{k=1}^{\infty} B_k\left(\frac{1}{r}\right).$$

Thus we have shown that $\bar{K}_1^{(1)}(0, r)$ is T_2 — closed. Moreover the equality $\bar{K}_1^{(1)}(x, r) = x + \bar{K}_1^{(1)}(0, r)$ implies that $\bar{K}_1^{(1)}(x, r)$ is T_2 — closed for every $x \in d(a, f) \cap d(b, g)$ and $r > 0$.

c) Let $U \in T_1$ and $x \in U$. Then $\bar{K}_1^{(1)}(x, r) \subset U$ for some $r > 0$. It follows from the part b) that $\bar{K}_1^{(2)}(x, r) \subset U$. So T_1 is regular with respect to T_2. In the same way we can show that T_2 is regular with respect to T_1. In the same way we can show that T_2 is regular with respect to T_1. Hence $(d(a, f) \cap d(b, g), T_1, T_2)$ is pairwise regular.

d) Let U be any T_1 — open set in $d(a, f) \cap d(b, g)$. Since $(d(a, f) \cap d(b, g), \| \ \|_1)$ is separable, we can write

$$U = \bigcup_{i,j=1}^{\infty} K_1(x_i, r_{ij}),$$

where $\bar{K}_1^{(1)}(x_i, r_{ij}) \subset K_1(x_i, r_{i,j+1})$ for $i, j \geqq 1$.
Put

$$W_m = \bigcup_{i=1}^{m} K_1(x_i, r_{i,m+1-i}).$$

Evidently, $W_m \in T_1$ and $U = \bigcup_{m=1}^{\infty} W_m$. Mereover,

$$\bar{W}_m^{(2)} = \bigcup_{i=1}^{m} \bar{K}_1^{(2)}(x_i, r_{i,m+1-i}) \subset \bigcup_{i=1}^{m} \bar{K}_1^{(1)}(x_i, r_{i,m+1-i}) \subset$$

$$\subset \bigcup_{i=1}^{m} K_1(x_i, r_{i,m+2-i}) \subset W_{m+1}.$$

Using analogous methods we can show that every T_2 — open set U' is of the form $U' = \bigcup_{m=1}^{\infty} V_m$, where V_m are T_2 — open sets such that $\bar{V}_m^{(1)} \subset V_{m+1}$.

Thus from Theorem B it follows that the space $(d(a, f) \cap d(b, g), T_1, T_2)$ is pairwise perfect normal.

From Corollary 1.1 in [1] we have

Theorem C. Let X be a topological space and let (Y, T_1, T_2) be a bitopological space such that T_2 is second countable and T_2 is perfect with respect to T_1. If $\varphi: X \to Y$ is a T_1 — continuous mapping, then the set $D(\varphi, T_2)$ of all points at which φ is T_2 — discontinuous is a set of the first Baire category in X.

Theorem 2.2. Let f and g be non-degenerate Orlicz functions that satisfy the Δ_2 — condition for small t.

a) Every T_j — open set in $(d(a, f) \cap d(b, g), T_1, T_2)$ is of the form $U \cup B$, where $U \in T_i$ and B is of T_i — first Baire category $(i, j = 1, 2)$ (hence every set $A \in T_j$ has the T_i — Baire property).

b) The set M of all $y \in d(a, f) \cap d(b, g)$ for which there exists a sequence

$\{y_n\}_{n=1}^{\infty}$ of points from $d(a, f) \cap d(b, g)$ such that $\lim_{n\to\infty} \|y_n - y\|_i = 0$ and $\{\|y_n - y\|_j\}_{n=1}^{\infty}$ does not converge to 0, is of the first category in $(d(a,f) \cap d(b, g), T_i)$ $(i, j = 1, 2, i \neq j)$.

c) For each subset $A \subset d(a, f) \cap d(b, g)$ the set $\bar{A}^{(i)} \setminus \bar{A}^{(j)}$ is a set of the first Baire category in $(d(a, f) \cap d(b, g), T_i)(i, j = 1, 2)$.

Proof.

a) Let φ: $(d(a, f) \cap d(b, g), T_i) \to (d(a, f) \cap d(b, g), T_1, T_2)$ be the mapping given by $\varphi(y) = y$ for $y \in d(a, f) \cap d(b, g)$. According to Theorem 2.1 the bitopological space $(d(a,f) \cap d(b, g), T_1, T_2)$ is pairwise perfect. Moreover, both of T_1 and T_2 are second countables. So it follows from Theorem C that $D(\varphi, T_j)$ is a set of the first category in $(d(a, f) \cap d(b, g), T_i)$. But $D(\varphi, T_i) = \bigcup\{\varphi^{-1}(V) \setminus \setminus \mathrm{Int}_{(i)}\varphi^{-1}(V)\colon V \in T_j\} = \bigcup\{V \setminus \mathrm{Int}_{(i)} V\colon V \in T_j\}$,
where $\mathrm{Int}_{(i)}$ denotes the T_i — interior. Hence for each $V \in T_j$ the set $V \setminus \mathrm{Int}_{(i)} V$ is of the T_i — first category and $V = \mathrm{Int}_{(i)} V \cup (V \setminus \mathrm{Int}_{(i)} V)$ which completes the a). Parts b) and c) follow from a), since $M = D(\varphi, T_j)$ and $\bar{A}^{(i)} \setminus \bar{A}^{j} \subset M$. This ends the proof.

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SÚHRN

O LORENTZOVÝCH — ORLICZOVÝCH PRIESTOROCH

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Práca pozostáva z dvoch častí. V prvej sú dokázané isté výsledky o štruktúre Lorentzových—Orliczových priestorov z hľadiska Baireových kategorií množín. Druhá časť práce je venovaná štúdiu niektorých bitopologických vlastností Lorentzových—Orliczových priestorov.

РЕЗЮМЕ

О ПРОСТРАНСТВАХ ЛОРЕНЦА И ОРЛИЧА

Янина Эверт, Слупск — Тибор Шалат, Братислава

Работ состоит из двух частей. В первой части доказаны некоторые результаты относительно структуры пространств Лоренца и Орлича с точки зрения бэровских категорий множеств. Вторая часть работы посвящена исследованию некоторых свойств битопологических пространств Лоренца и Орлича.